Laser second threshold: Its exact analytical dependence on detuning and relaxation rates

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(Received 18 November 1992; revised manuscript received 8 March 1993)

An exact analysis has been carried out for general analytical expressions for the second threshold of a single-mode homogeneously broadened laser and for the initial pulsation frequency at the second threshold, for arbitrary physical values of the relaxation rates, and at an arbitrary detuning between the cavity frequency and the atomic resonance frequency. These expressions also give correspondingly exact forms for asymptotic cases that have been previously studied with some approximations. Earlier approximate results are partly confirmed and partly improved by these more general expressions. The physical status of various expressions and approximations is reconsidered and specified more clearly, including an analysis of what reasonably can be attained in lasers or masers. A general analytical proof is given of the fact that, for a larger detuning of the laser cavity from resonance, a higher value of the laser excitation is required to destabilize the steady-state solution (the second threshold). We also present results for the minimum value of the second threshold at fixed detuning as a function of the other parameters of the system and for the dependence of the ratio of the second threshold to the first threshold as a function of detuning. Minima of the second threshold and of the threshold ratio occur only if the population relaxation rate is equal to zero. The minima of the threshold ratio are shown to be bounded from above as well as from below (as functions of the relaxation rates, so long as the second threshold exists). The upper bound on the minima is equal to 17. The variation of the second threshold in the semi-infinite parameter space of the decay rates is shown at various detunings in plots with a finite domain by normalizing the material relaxation rates to the cavity decay rate.

PACS number(s): 42.60.Lh, 42.60.Mi, 47.20.Ky, 05.45.+b

I. INTRODUCTION

A. Background

The problem of obtaining and analyzing an exact analytical expression for the second threshold (at which steady-state operation gives way to time-dependent behavior) in lasers has been and continues to be a challenge in nonlinear and quantum optics. Although the problem is exactly solvable, at least in principle, for the semiclassical equations for a homogeneously broadened single-mode laser, the solution requires both an enormous amount of calculations and further simplifications of the resulting equations. The calculations and the symbolic simplifications are so daunting that to our knowledge they have not been attempted by hand or by ordinary symbolic-manipulation computer programs.

For instance, a rigorous mathematical treatment of the problem in [1] met rather formidable algebraic relations which had to be restricted to a special limiting case for evaluation. Such technical complexity has been the main obstacle to a full treatment of the problem and it has limited previous work to only a few particular, though highly instructive, cases.

Our goal is to present a general study of the problem. This has become possible with the availability of MATHEMATICA, notable for its versatility in computer-based symbolic mathematics [2]. While we demonstrate this power for solving our particular problem, this can also be taken as an illustration of how this tool can be applied in the fields of nonlinear dynamics and nonlinear optics.

B. Outline

Our main purpose in this article is to investigate the influence of the detuning and the relaxation rates on the second threshold of the homogeneously broadened single-mode laser, especially the influence of the detuning on the relative accessibility of the second threshold as the pump parameter is varied.

We have completed this investigation without any restrictions (within physically accessible ranges) on the values of the parameters of the problem, and we present a purely analytical treatment.

The paper is structured as follows.

In this section we start from the original semiclassical equations of motion for the homogeneously broadened single-mode laser. These equations are variously known as the standard laser equations, the single-mode homogeneously broadened laser equations, the single-mode Maxwell-Bloch equations, or the laser-Lorenz or Haken-Lorenz equations. To the best of our knowledge, the stability of the steady-state lasing solutions of these equations (or their Maxwell-Schrödinger equivalents for lasers or other two-level systems) were first investigated by Gurtovnik [3] (for applications to masers) and then by Grasiuk and Oraevskiy [4,5] and Uspenskiy [6] and later by Korobkin and Uspenskiy [7] for the resonantly tuned laser, and by Uspenskiy [8] for the case of a detuned laser. An alternative development in the framework of studying the response of lasers to intrinsic noise through analysis of nonlinear Langevin equations for the resonantly tuned laser was given by Haken [9] and in the framework of the Fokker-Planck equations by Risken, Schmid, and Weidlich [10], who also set up the stability analysis of the detuned laser without computing instability thresholds. Using what are now more conventional methods of linear stability analysis of steady-state solutions, their earlier results were soon thereafter generalized to considerations of multilongitudinal-mode stability and instabilities as well as single-mode instabilities [11,12], although in these papers the single-mode studies were limited to the resonantly tuned situation.

Section II is devoted to an extensive discussion of the intersections, overlaps, and isomorphism of the standard laser equations and various versions of the equations known as the Lorenz equations (both real and complex).

We give a systematic derivation of Lorenz models from the standard laser equations and summarize, in the process, previous results on this topic which are a bit scattered in the literature. Taken together, these results give three levels of isomorphism between the standard laser equations and the "hydrodynamical" Lorenz models. The levels of this isomorphism are presented in Table I. The established isomorphisms allow us to continue our analysis in alternative sets of terms: either laser or "hydrodynamical."

After Sec. II, we present our calculations which have been done using the "hydrodynamical" form which is more convenient for the usage of MATHEMATICA. However, since we deal here with laser physics we will reformulate all intermediate results in laser terms.

In Sec. III, we give the rigorous solution for the second threshold of the homogeneously broadened single-mode laser. In order to do this, we prove that another analytically possible solution for the second threshold does not exist although MATHEMATICA successfully produces it. (It turns out that this alternative second threshold requires a negative intensity.) We also demonstrate that we start with the same expressions as in previous studies, which makes it possible to straightforwardly compare our general results with previous results which should be obtained from our expressions for particular choices of the parameter values.

In a very short Sec. IV the absolute minimum of the second threshold for the resonantly tuned laser will be made more precise than has been previously reported.

Since the evaluation of the pump value for the second threshold is closely related to the evaluation of the initial pulsation frequency of the Hopf-like bifurcation at the second threshold, we derive in Sec. V the general expression for this initial pulsation frequency and, on the basis of this, we present the values for the initial pulsation frequency for certain special cases.

In Sec. VI our general results are compared with previous results which have been derived for special cases, the dependence of the second threshold on all parameters is restored in comparison with those previous results, and special limits that can be reached only for either lasers or masers are distinguished. The importance of the order in which double parameter limits are taken is also discussed.

In Sec. VII we show that a perturbative approach to the problem of the position of the second threshold can evidently fail. Thus, to overcome this, we give a general analytical proof at arbitrary parameter values that increasing the detuning has the effect of increasing the second threshold. In this section we also assess how the ratio of the second threshold to the first threshold depends on the detuning.

In Sec. VIII, in order to satisfy the common temptation to look at the surface which represents the second threshold in the parameter space, we demonstrate that a new normalization of the relaxation rates of the laser system brings the domain of the existence of the second threshold from a semi-infinite half plane of the parameter space into a small triangle, allowing us to plot the second threshold surface easily at various values of the detuning. This will help to illustrate our general result on the influence of the detuning on the second threshold.

Section IX is devoted to the search for minima of the second threshold and of its ratio to the first laser threshold as functions of the detuning and the relaxation rates. We show that the minima of both occur when the population relaxation rate is equal to zero. We also find that the minima of the threshold ratio are bounded not only from below, but also from above. The minima of the threshold ratio are found to be limited between 9.0 and 17.0. Numerical results of this study are given in Table III.

We have summarized the results of this article in Sec. X, where we have also summarized the main points of our discussions.

In the Appendixes we have collected some details on points which are only mentioned in the main discussion of the physical results, but which are necessary for the rigorous justification of our statements.

C. How to check and compare other results with our results

Since we have used MATHEMATICA, it is probably not practical to check most of our results by hand because they are produced by computer. However, in order to provide interested readers with the opportunity to check our results, to compare them with other results, and/or to develop them further, we have created a number of special files which are available upon request by electronic mail from the first author. The choice of the file we will send depends on the purposes of the requester.

It should be also mentioned that working with symbolic programs requires some precautions, otherwise one can get explicit, though wrong results.

D. The basic dynamical system

Semiclassical equations for the homogeneously broadened single-mode laser have the following well known form [13–16]:

$$\frac{d}{dt}E(t) = -\kappa E(t) - i\omega E(t) - igP(t), \tag{1}$$

$$\frac{d}{dt}P(t) = -\gamma_\perp P(t) - i\varepsilon P(t) + ig^* E(t)S(t), \tag{2}$$

$$\frac{d}{dt}S(t) = -\gamma_\| (S(t) - Nd_0) + 2i(gE^*(t)P(t) - g^*E(t)P^*(t)). \tag{3}$$

In these equations, $E(t)$ is the electric field amplitude of the lasing mode of the optical field, $P(t)$ is the macroscopic polarization of the two-level medium inside the cavity of the laser in the rotating wave approximation, and $S(t)$ is one-half of the macroscopic population difference between upper and lower atomic levels.

Further, ω is the frequency of the cavity mode which is the nearest one to the atomic transition frequency ε, g is the atom-field coupling, $\gamma_\perp$ and $\gamma_\|$ are, respectively, the polarization and population relaxation rates of the two-level medium inside the cavity, and κ is the relaxation rate of the optical field mode. An incoherent pumping rate for the population inversion is described by the level of the unsaturated inversion per atom, d_0; N is the density of the atoms in the cavity.

Contributions to the analytical study of the threshold properties of the standard laser models have been made by many workers [3–25] where the contribution of Mandel and co-workers should be particularly noted.

II. COMPLEX LORENZ MODEL AND STANDARD LASER EQUATIONS

A. Alternatives for identification of the standard laser equations with the complex Lorenz model

The *complex* Lorenz model [26], which is a five-dimensional generalization of the famous (real) Lorenz model of three dimensions [27,28], has the form

$$\frac{d}{d\tau}x(\tau) = -\sigma x(\tau) + \sigma y(\tau), \tag{4}$$

$$\frac{d}{d\tau}y(\tau) = -ay(\tau) + rx(\tau) - x(\tau)z(\tau), \tag{5}$$

$$\frac{d}{d\tau}z(\tau) = -bz(\tau) + \tfrac{1}{2}[x^*(\tau)y(\tau) + x(\tau)y^*(\tau)], \tag{6}$$

where τ is the normalized time which is measured in the laser case in the units of $\gamma_\perp^{-1}$ (this will be shown later on), the parameters σ and b are real and positive, and the parameters a and r are complex:

$$a = 1 - ie; \quad r = r_1 + ir_2. \tag{7}$$

The connection of the basic equations (1)–(3) with the complex Lorenz model has been pointed out and described in the literature several times [18–23,29], after the basic work of Haken [17] on the connection of the resonantly tuned laser equations with the *real* Lorenz model and it is, therefore, well known. However, it seems to us that the discussions of the connection between Eqs. (1)–(3) and the complex Lorenz model, and of the physical meaning of the complex Lorenz model in laser physics, are somewhat scattered [20,22,23,29]; we find it useful to collect everything in one place.

There are two *different* ways to develop a correspondence between the standard laser equations and the complex Lorenz model. They both were described in the article by Fowler, Gibbon, and McGuinness [26].

The first possibility is to get the parameter r_2 identically equal to zero (making the pump parameter, r, real as in the laser equations) and to identify at the same time the parameter e with the normalized detuning. Fowler, Gibbon, and McGuinness preferred this strategy (Ref. [26], p. 140): "The complex Lorenz equations ... form the ... model for ... optical systems of two level atoms ... semiclassical equations are the set of damped Maxwell-Bloch equations. ... In this case, $r_2 = 0$, but $e \neq 0$." This approach has been used in almost all other work devoted to this subject [18–23].

The alternative approach is to keep r_2 always equal to $-e$. Although this approach was called by Fowler, Gibbon, and McGuinness a "...rather pathological possibility" (Ref. [26], p. 141), it has been developed and explored in recent work by Ning and Haken [29]. An important feature of the work by Ning and Haken [29] is that for the first time, we believe, the spatial dependence has been embedded into the connection between semiclassical equations for slowly varying amplitudes and complex Lorenz equations, thereby generalizing the Lorenz-laser analogy to apply as well to the multimode laser equations and providing a complex-variable version of this analogy which was first set forth for real variables by Graham [32]. This approach has proven helpful, albeit somewhat controversial, in discussions of geometrical phases in detuned laser systems [30,31].

Since we will not develop in this article the second approach, let us try to characterize briefly that possibility for the identification and reasons why we have chosen another one.

Fowler, Gibbon, and McGuinness [26] pointed out both ways for the identification, obtaining for the first time *formal* expressions for the second threshold (formula 2.43 in [26]) and the initial pulsation frequency (formula 2.38 in [26]) basing their consideration on the biquadratic equation they derived (formula 2.41 in [26]).

Eight years later Ning and Haken repeated many of these formal results: finding the expressions for the second threshold [formula (74) in [29]], the initial pulsation frequency [formula (75) in [29]], and the basic biquadratic equation [formula (70) in [29]]. In addition they formally included consideration of the spatial coordinate and analyzed numerically the supercritical bifurcation in the laser case.

Nevertheless, while it was relatively easy to write down the biquadratic equation and the *formal* solution for it, the main part of the work still lays ahead.

The following questions which were obstacles to converting those formal results to truly analytical ones had been left unanswered.

(i) Is one of the roots of the biquadratic equation always negative? If not, then one should compare each root and its prediction of a second threshold with the value of the first lasing threshold to make clear whether or not there are two branches for the second threshold.

(ii) Is another root of the biquadratic equation always greater than the first laser threshold under the bad-cavity condition or does a domain of parameters exist where the second threshold has no meaning at all? (If so, then there is an additional restriction on the parameter values for which the second threshold can occur, and the physical meaning of such a restriction should be understood.)

(iii) Does the second threshold increase with detuning or can it be decreased by varying the detuning and relaxation rates (a result that would be important for experimental observations of the phenomena)?

(iv) Even if the second threshold does increase with detuning, does the ratio of the second threshold to the first threshold decrease in some parameter domain?

(v) Generally, what are the extrema of the second threshold; what is the shape of the second threshold in a parameter space?

(vi) What are the asymptotes for the second threshold; what is the physical meaning for those asymptotes?

No exact analysis of these problems has yet been given making use of the beautiful formal expressions obtained in both works [26] and [29]. To complete our analysis of these expressions, to answer many of the questions we have identified, it was necessary to find many simplifications and substitutions in order to bring the expressions to a form which would make possible a subsequent analysis.

Further, one can see that the approach taken by Ning and Haken connects the commonly used bifurcation parameter r in (7) with the difference of the normalized pump and normalized detuning [[29], formulas (27) and (29) under the condition $r_2 = -e$]. Thus, two *physically different* dependences, on the pumping rate and on the detuning, are mixed in the single parameter in the Ning-Haken approach. This particular feature can sometimes be a disadvantage because it makes it difficult to separate the effects of detuning from the effects of variation of pumping intensity in the course of variation of bifurcation parameter r.

We present here the derivation of a particular form of the complex Lorenz model from (1)–(3), following the approaches taken in [18–23]. In this way, we collect in one article most of the various forms of the standard equations for a homogeneously broadened single-mode laser with nonzero detuning, and try to understand their overlaps and hierarchy.

B. Idea of transformation

We will establish this connection in two steps. In the first step, Eqs. (1)–(3) are brought to a form which is qualitatively similar to the complex Lorenz model, and in the second step we properly normalize those intermediate equations.

First of all, one can see from direct comparison between the standard laser equations (1)–(3) and the complex Lorenz model (4)–(6) that if the connection is possible, then it should be between $E(t)$ and $x(\tau)$ (because the differential equations for these quantities are the only linear equations in these two systems), between $S(t)$ and $z(\tau)$ (because these are the only real variables in these two systems) and, consequently, between $P(t)$ and $y(\tau)$.

However, Eq. (1) for $E(t)$ has an imaginary part in the coefficient in front of $E(t)$ while Eq. (4) does not have this. This means that in order to eliminate this imaginary coefficient at $E(t)$ from (1) we must take a reference frame for the field rotating with the cavity frequency ω.

Further, the coefficients of both $x(\tau)$ and at $y(\tau)$ in Eq. (4) are real, while even after transformation to the rotating reference frame the coefficient at $P(t)$ in (2) is complex. In establishing the equivalence, the phase shift between $E(t)$ and $P(t)$ must be eliminated together with the transformation to the rotating reference frame.

Finally, one has to shift the scale for $S(t)$ onto the value of Nd_0 in order to eliminate the constant term from (3) which is absent in (6), and to invert this scale in order to bring the signs in the new equation for the polarization to a form coinciding with Eq. (5).

C. Change of reference frame and scale

First we perform the transformation of the amplitudes of the optical field $E(t)$ and the polarization $P(t)$ and the changes in the population difference $S(t)$.

The explicit form of this intermediate transformation is

$$\mathcal{E}(t) = E(t)e^{i(\omega t+\varphi_0)}, \quad \mathcal{P}(t) = P(t)e^{i\omega t},$$
$$\mathcal{S}(t) = -[S(t) - Nd_0], \tag{8}$$

where the value of the compensating phase shift φ_0 includes both the phase of the imaginary unit i and the phase $\arg(g)$ of the complex dipole matrix element of the atomic transition between upper and lower levels

$$\varphi_0 = \frac{\pi}{2} - \arg(g). \tag{9}$$

The intermediate equations can then be written as follows:

$$\frac{d}{dt}\mathcal{E}(t) = -\kappa\mathcal{E}(t) + |g|\mathcal{P}(t), \tag{10}$$

$$\frac{d}{dt}\mathcal{P}(t) = -\gamma_\perp \mathcal{P}(t) + i(\omega - \varepsilon)\mathcal{P}(t) + |g| N d_0 \mathcal{E}(t) - |g| \mathcal{E}(t) \mathcal{S}(t), \tag{11}$$

$$\frac{d}{dt}\mathcal{S}(t) = -\gamma_\parallel \mathcal{S}(t) + 2|g|[\mathcal{E}^*(t)\mathcal{P}(t) + \mathcal{E}(t)\mathcal{P}^*(t)]. \tag{12}$$

D. Normalization

Second, Eqs. (10)–(12) need only proper normalizations of the time, and of the variables and parameters. Since the term in Eq. (5) that is linear in $y(\tau)$ has a coefficient with a real part equal to unity, the normalization of the time is evident:

$$\tau = \gamma_\perp t = \frac{t}{\gamma_\perp^{-1}}. \tag{13}$$

Hence, time now is measured in the units of the polarization relaxation time $\gamma_\perp^{-1}$.

By means of the following relations for the coefficients:

$$\sigma \equiv \frac{\kappa}{\gamma_\perp}, \tag{14}$$

$$b \equiv \frac{\gamma_\parallel}{\gamma_\perp}, \tag{15}$$

$$a \equiv 1 - ie = 1 - i\frac{\omega - \varepsilon}{\gamma_\perp}; \quad e \equiv \frac{\omega - \varepsilon}{\gamma_\perp}; \tag{16}$$

$$r \equiv \frac{d_0}{d_{1,0}^{\text{thr}}}, \tag{17}$$

where

$$d_{1,0}^{\text{thr}} = \frac{\kappa \gamma_\perp}{|g|^2 N} \tag{18}$$

is the well known value for the first threshold (onset of steady-state lasing) of the resonantly tuned laser, and by means of the following normalization of the dynamic variables

$$X(\tau) = \frac{2|g|}{\gamma_\perp}\mathcal{E}(t), \quad Y(\tau) = 2\frac{|g|^2}{\kappa\gamma_\perp}\mathcal{P}(t), \quad Z(\tau) = \frac{|g|^2}{\kappa\gamma_\perp}\mathcal{S}(t), \tag{19}$$

we finally get the complex *laser*-Lorenz equations:

$$\frac{d}{d\tau}X(\tau) = -\sigma X(\tau) + \sigma Y(\tau), \tag{20}$$

$$\frac{d}{d\tau}Y(\tau) = -aY(\tau) + rX(\tau) - X(\tau)Z(\tau), \tag{21}$$

$$\frac{d}{d\tau}Z(\tau) = -bZ(\tau) + \tfrac{1}{2}[X^*(\tau)Y(\tau) + X(\tau)Y^*(\tau)], \tag{22}$$

with the *only* difference from the original complex Lorenz model (4)–(6) being that in the laser case (20)–(22) the normalized pumping parameter r is a real parameter

$$\text{Im}(r) = 0, \tag{23}$$

while in the general case of Eqs. (4)–(6) this parameter (7) possesses a nonzero imaginary part which has been discussed already in [20,21]. For the first laser threshold we use d_1^{thr} (respectively, r_1^{thr}) where "thr" denotes the threshold and the subscript "1" denotes the first laser threshold; we use $d_{1,0}^{\text{thr}}$ (respectively, $r_{1,0}^{\text{thr}}$) for the first laser threshold with resonant tuning of the cavity, where the subscript "0" denotes the resonant detuning, $\omega = \varepsilon$.

E. Distinctive features of the *laser*-Lorenz model and a hierarchy of standard laser equations

We can conclude the following from this derivation of a particular form of the complex Lorenz equations (20)–(22) from the semiclassical equations for the single-mode homogeneously broadened laser (1)–(3).

(i) The complex Lorenz equations are *practically* applicable to the laser only in two particular cases; one is considered by Ning and Haken [31], which we discussed in Sec. II A [*Formally*, one can assert that there is a continuum of possibilities for such an identification defined by function $F(r_2, e) = 0$ which allows us to eliminate from consideration the parameter r_2]; in our article, we consider the particular case when the parameter r is a real number [20,21], i.e., if $\text{Im}(r) = r_2 = 0$.

(ii) In the laser case, the variables $X(\tau)$, $Y(\tau)$, and $Z(\tau)$ describe, respectively, the normalized amplitudes of the field and polarization and the population difference.

(iii) In the laser case, the reference frame both rotates with the cavity frequency ω and there is an inverted scale for the population difference with its origin shifted to the value of the unsaturated population difference.

(iv) In the laser case, the time is measured in units of the polarization relaxation time $\gamma_\perp^{-1}$.

(v) In the laser case, the phase shift between the optical field and polarization which is due to the Coulomb gauge and complexity of the dipole matrix element of the transition between upper and lower lasing levels, has been formally eliminated by the compensating phase shift in the definition of the field variable $X(\tau)$.

The hierarchy of the standard equations for the single-mode homogeneously broadened laser is presented in Table I.

It is clear from our derivation of the laser-related form of the complex Lorenz model from Eqs. (1)–(3) that one can successfully use either the description by means of laser physics terms and parameters (the left column in the Table I) or the description in the "hydrodynamical" terms of the Lorenz models, complex and real (the right column in the Table I). Downward arrows in both columns of Table I show the direction from a general form to a more particular one. Thus, Eqs. (1)–(3) in five dimensions for the *detuned* homogeneously broadened laser are completely isomorphic to the complex Lorenz model with $r_2 = 0$.

If one sets the detuning equal to zero, the parameter e in the complex Lorenz model vanishes as well. Therefore, the standard equations for the *resonantly tuned* laser remain five dimensional and are completely isomorphic to the complex Lorenz model with $e = r_2 = 0$.

The following warning is required here: one *cannot* yet identify these equations with the three-dimensional *real* Lorenz model because these equations contain, for in-

TABLE I. Hierarchy of standard laser equations.

Detuned homogeneously broadened single-mode laser, eqs. (1)–(3)	$\Longleftrightarrow$	*Complex* Lorenz model with $r_2 = 0$
$\Downarrow$		$\Downarrow$
The same but *tuned* laser: $\omega - \epsilon = 0$	$\Longleftrightarrow$	*Complex* Lorenz model with $e = r_2 = 0$
$\Downarrow$		$\Downarrow$
Tuned laser with fixed phase ($\arg(g) = \pm\pi/2$) and special initial conditions for $E(t)$ and $P(t)$	$\Longleftrightarrow$	*Real* Lorenz model, i. e. complex Lorenz model with $e = r_2 = 0$ and special initial conditions for $X(\tau)$ and $Y(\tau)$

stance, transient and noise driven, time-dependent processes which cannot, in principle, be described in three dimensions. However, for most deterministic transient processes, there is a rapid relaxation to a three-dimensional subspace of the five dimensions with the adoption of a particular fixed value of the absolute phase which remains asymptotically fixed.

In contrast, when Langevin noise sources are added to these equations, there is noise-induced diffusion of the absolute phase which requires the full five equations for a proper description [33]. Hence, capturing the effects of laser dynamics in the presence of spontaneous emission requires, necessarily, the complex rather than the real equations.

For deterministic evolution, it is easy to see that the five-dimensional equations for the *resonantly tuned* laser can be *effectively* replaced by the three-dimensional equations in the case when one takes the special choice of the relative phase such that $P(0)/E(0)$ is real at an arbitrary value of $S(0)$. The resulting equations will be *effectively* isomorphic to three-dimensional *real* Lorenz model.

We can now distinguish three levels both for the standard laser equations and for the Lorenz models in the laser case. At each level, one can use both the laser and the "hydrodynamical" presentations which are completely isomorphic as we have shown in this section.

We have collected in one place both the previous results [18–23] and our results about connections and overlaps between the standard laser equations and the Lorenz models. This helps to clarify the parameter regions and initial conditions which define the validity of applications to each particular case.

III. EXACT EXPRESSION FOR THE SECOND THRESHOLD

The laser-related complex Lorenz model itself is not so convenient for the study of stability because of the choice of a reference frame which rotates with the cavity frequency ω rather than the reference frame which rotates with the actual frequency of the stationary optical field [20,19,22]. This leads to the transformation of the steady solution, describing steady-state (constant intensity) laser operation, into a periodic solution which seems to us to have been properly termed "undue complexity" [19]. Even in the original work by Fowler, Gibbon, and McGuinness [26], wherein the complex Lorenz model was analyzed for the first time, the authors changed the reference frame by transforming to the reference frame which is commonly used in laser physics [13] for the description of the detuned laser. From the point of view of stability theory it is always easier to treat a fixed point rather than a periodic solution. This is the reason why previous workers [18–21] used the reference frame adopted in laser physics but not the complex Lorenz model itself even when they considered Eqs. (1)–(3), which are completely isomorphic to Eqs. (20)–(22).

A. Equations for the normalized variables in the reference frame rotating with the frequency of the stationary optical field

It can be easily shown [13,15–25], that in the reference frame rotating with the frequency

$$\Omega = \frac{\kappa\varepsilon + \gamma_\perp \omega}{\kappa + \gamma_\perp}, \tag{24}$$

the basic equations (1)–(3) have a nontrivial (lasing) steady solution. The shift of Ω from the cavity frequency ω is called "frequency pulling"; see, for example, in [13], pp. 121 and 187. So, taking the dynamic variables in this reference frame

$$\tilde{\mathcal{E}}(t) = E(t)e^{i(\Omega t + \varphi_0)}, \quad \tilde{\mathcal{P}}(t) = P(t)e^{i\Omega t},$$
$$\tilde{\mathcal{S}}(t) = -[S(t) - Nd_0], \tag{25}$$

where the compensating phase shift φ_0 was defined earlier by relation (9), and introducing a parameter δ containing the absolute detuning $(\omega - \varepsilon)$:

$$\delta = \frac{\omega - \varepsilon}{\kappa + \gamma_\perp}, \tag{26}$$

we get the system of equations

$$\frac{d}{dt}\tilde{\mathcal{E}}(t) = -\kappa(1 + i\delta)\tilde{\mathcal{E}}(t) + |g|\tilde{\mathcal{P}}(t), \tag{27}$$

$$\frac{d}{dt}\tilde{\mathcal{P}}(t) = -\gamma_\perp(1 - i\delta)\tilde{\mathcal{P}}(t) + |g|Nd_0\tilde{\mathcal{E}}(t) - |g|\tilde{\mathcal{E}}(t)\tilde{\mathcal{S}}(t), \tag{28}$$

$$\frac{d}{dt}\tilde{\mathcal{S}}(t) = -\gamma_\parallel\tilde{\mathcal{S}}(t) + 2|g|[\tilde{\mathcal{E}}^*(t)\tilde{\mathcal{P}}(t) + \tilde{\mathcal{E}}(t)\tilde{\mathcal{P}}^*(t)]. \tag{29}$$

The parameter δ scales the detuning of the frequency Ω of the steady-state lasing solution from the cavity frequency ω and the transition resonance frequency ε for which the values are, respectively, $\kappa\delta$ and $\gamma_\perp\delta$.

1. Discussion of the parameter δ

Before proceeding, it is helpful to discuss in detail the dependence of δ on the other parameters of the system.

Since the relaxation rate κ which characterizes the loss of the cavity

$$\kappa > \gamma_\perp + \gamma_\parallel, \tag{30}$$

should be compared with $\gamma_\perp$ and $\gamma_\parallel$ when analyzing the second threshold, it is better to rewrite the quantity δ as a product of two separate factors using the common normalization of all parameters to the parameter $\gamma_\perp$ which is, therefore, a fixed constant:

$$\delta\left(\frac{\omega - \varepsilon}{\gamma_\perp};\frac{\kappa}{\gamma_\perp}\right) = \frac{\omega - \varepsilon}{\kappa + \gamma_\perp} = \frac{\omega - \varepsilon}{\gamma_\perp}\left(\frac{\kappa}{\gamma_\perp} + 1\right)^{-1}. \tag{31}$$

The first factor in this product depends solely on the cavity detuning as measured in the natural units of the transition linewidth (since $\gamma_\perp$ is now a fixed normalizing parameter) while the second factor characterizes the relative loss of the cavity. This means that if one wants to vary the quantity δ changing only the detuning at fixed relative cavity loss, one has to vary only the first factor. Varying the relative cavity loss at fixed detuning requires variation of the second factor only.

Thus, the parameter δ contains simultaneously both the dependence on detuning and on the relative loss of the laser cavity in contradistinction to the common misconception that δ is just a normalized detuning. If the value of κ is also to be varied, $(\kappa + \gamma_\perp)$ cannot be used as a constant normalizing factor.

It is easy to illustrate this statement. For instance, one has δ equal to zero in two *physically different* cases: either when detuning goes to zero $[(\omega-\varepsilon)/\gamma_\perp \to 0]$ or when the cavity is very bad $(\kappa/\gamma_\perp \to \infty)$. Hence, simply setting the value of δ equal to zero does *not* specify completely the physical situation—further specification of the value of either $(\omega-\varepsilon)/\gamma_\perp$ or $\kappa/\gamma_\perp$ is required for understanding the physical meaning of a zero value of the parameter δ.

Another but also ambiguous situation occurs for large values of δ. It is clear that the detuning $(\omega-\varepsilon)/\gamma_\perp$ is large in this case. However, to what extent is the detuning large?

The answer is *not* unique and *strongly* depends on whether or not the limit of large δ is taken simultaneously with the bad-cavity limit $\kappa/\gamma_\perp \to \infty$. We can say in advance that the limit of large δ in the extremely bad-cavity limit can be reached only for masers while either limit $(\kappa/\gamma_\perp \gg 1$ or $|\omega-\varepsilon|/\gamma_\perp \gg 1)$ is difficult to achieve for any laser (see also the discussion in the Sec. VI C). In the experimental observations of second threshold in homogeneously broadened lasers, it was not possible to operate much beyond $\kappa/\gamma_\perp \le 3$ and $|\omega-\varepsilon|/\gamma_\perp \le 10$.

Since the standard laser equations are completely isomorphic to the laser-related complex Lorenz model, we can reformulate the remarks of this subsection in terms of the parameters σ, e, and b. Recalling the definitions (14), (15), and (16), we rewrite δ as a function of e and σ:

$$\delta(e;\sigma) = \frac{e}{\sigma + 1}. \tag{32}$$

In this form the previous remarks are even more clear—the parameter e describes *solely* the cavity detuning while the parameter σ describes the relative loss of the cavity.

Conclusion: if one performs some limits in δ one should *necessarily* specify the behavior of both σ and e.

2. Explicitly symmetrical form of equations

Our aim in this subsection is to make a normalization of the new dynamical variables to simplify the form of both the steady solution and the coefficients of the equations in variations near this steady solution.

In this way we also get an explicitly symmetrical form of the equations, i.e., a form which is covariant with respect to the rotations of the steady solutions.

To this end it is convenient to define an "angular" variable

$$\rho = \arctan\delta = \arctan\frac{\omega - \varepsilon}{\kappa + \gamma_\perp} = \arctan\frac{e}{\sigma + 1}. \tag{33}$$

The normalized time is now measured in the units of $(\gamma_\perp\sqrt{1+\delta^2})^{-1}$:

$$\tilde{\tau} = \left(\gamma_\perp\sqrt{1+\delta^2}\right)t = t\Big/\left(\gamma_\perp\sqrt{1+\delta^2}\right)^{-1}. \tag{34}$$

By means of relations for the coefficients

$$\sigma = \frac{\kappa}{\gamma_\perp}, \quad \tilde{b} = \frac{b}{\sqrt{1+\delta^2}} = \frac{\gamma_\parallel}{\gamma_\perp\sqrt{1+\delta^2}},$$
$$\tilde{r} = \frac{r}{1+\delta^2} = \frac{d_0}{(1+\delta^2)d_{1,0}^{\text{thr}}} = \frac{d_0}{d_1^{\text{thr}}}, \tag{35}$$

where

$$d_1^{\rm thr} = (1+\delta^2) d_{1,0}^{\rm thr} = \frac{\kappa\gamma_\perp(1+\delta^2)}{|g|^2 N} \tag{36}$$

is now the value for the first laser threshold of the *detuned* laser, and by means of the following normalization of the dynamic variables:

$$\tilde{X}(\tilde{\tau}) = \frac{2|g|}{\gamma_\perp\sqrt{1+\delta^2}}\tilde{\mathcal{E}}(t), \quad \tilde{Y}(\tilde{\tau}) = \frac{2|g|^2}{\kappa\gamma_\perp(1+\delta^2)}\tilde{\mathcal{P}}(t),$$
$$\tilde{Z}(\tilde{\tau}) = \frac{|g|^2}{\kappa\gamma_\perp(1+\delta^2)}\tilde{\mathcal{S}}(t), \tag{37}$$

we obtain another representation for the complex *laser-Lorenz* equations:

$$\frac{d}{d\tilde{\tau}}\tilde{X}(\tilde{\tau}) = -\sigma e^{i\rho}\tilde{X}(\tilde{\tau}) + \sigma\tilde{Y}(\tilde{\tau}), \tag{38}$$

$$\frac{d}{d\tilde{\tau}}\tilde{Y}(\tau) = -e^{-i\rho}\tilde{Y}(\tilde{\tau}) + \tilde{r}\tilde{X}(\tilde{\tau}) - \tilde{X}(\tilde{\tau})\tilde{Z}(\tilde{\tau}), \tag{39}$$

$$\frac{d}{d\tilde{\tau}}\tilde{Z}(\tilde{\tau}) = -\tilde{b}\tilde{Z}(\tilde{\tau}) + \tfrac{1}{2}[\tilde{X}^*(\tilde{\tau})\tilde{Y}(\tilde{\tau}) + \tilde{X}(\tilde{\tau})\tilde{Y}^*(\tilde{\tau})]. \tag{40}$$

The striking feature of Eqs. (38)–(40) is that now they have steady-state solutions which can be written down in very simple form:

$$\tilde{X}_0 = A\exp(\phi), \quad \tilde{Y}_0 = A\exp(\phi+\rho), \quad \tilde{Z}_0 = \tilde{r}-1, \tag{41}$$

where the magnitude A of the normalized stationary field and polarization is equal to

$$A = \sqrt{\tilde{b}\sqrt{1+\delta^2}(\tilde{r}-1)} = \sqrt{b\left(\frac{r}{1+\delta^2}-1\right)} \tag{42}$$

and the parameter ϕ is arbitrary:

$$0 \le \phi \le 2\pi, \tag{43}$$

while the "angular" parameter ρ characterizing both the degree of the detuning and the relative loss of the cavity has been introduced earlier by means of relation (33).

3. Symmetrical equations in variations with respect to the stationary solutions

It is more convenient to use a set of the real variables $\{x_1, x_2, x_3, x_4, x_5\}$:

$$\tilde{X}(\tilde{\tau}) = x_1(\tilde{\tau}) + i x_2(\tilde{\tau}), \tag{44}$$

$$\tilde{Y}(\tilde{\tau}) = x_3(\tilde{\tau}) + i x_4(\tilde{\tau}), \tag{45}$$

$$\tilde{Z}(\tilde{\tau}) = x_5(\tilde{\tau}). \tag{46}$$

In terms of these variables, Eqs. (38)–(40) acquire the following explicitly symmetrical form with a simple trigonometrical parametrization of the coefficients:

$$\frac{d}{d\tilde{\tau}}x_1 = -\sigma\cos(\rho)x_1 + \sigma\sin(\rho)x_2 + \sigma x_3, \tag{47}$$

$$\frac{d}{d\tilde{\tau}}x_2 = -\sigma\sin(\rho)x_1 - \sigma\cos(\rho)x_2 + \sigma x_4, \tag{48}$$

$$\frac{d}{d\tilde{\tau}}x_3 = \tilde{r}x_1 - \cos(\rho)x_3 - \sin(\rho)x_4 - x_1x_5, \tag{49}$$

$$\frac{d}{d\tilde{\tau}}x_4 = \tilde{r}x_2 + \sin(\rho)x_3 - \cos(\rho)x_4 - x_2x_5, \tag{50}$$

$$\frac{d}{d\tilde{\tau}}x_5 = -\tilde{b}x_5 + x_1x_3 + x_2x_4. \tag{51}$$

The stationary solutions turn out to be written in a very simple form as well [compare with relations (3.45) and (3.46) in Sec. 3.4 of [26]]:

$$\{x_1^{\rm st}, x_2^{\rm st}, x_3^{\rm st}, x_4^{\rm st}, x_5^{\rm st}\} = A\left\{\cos(\phi), \sin(\phi), \cos(\phi+\rho), \sin(\phi+\rho), \frac{\tilde{r}-1}{A}\right\}. \tag{52}$$

Now we introduce the variations near the solutions (52):

$$q_n = x_n - x_n^{\rm st}, \qquad n = 1,\ldots,5. \tag{53}$$

The equations for the variations are written as

$$\frac{d}{d\tilde{\tau}}q_1 = -\sigma\cos(\rho)q_1 + \sigma\sin(\rho)q_2 + \sigma q_3, \tag{54}$$

$$\frac{d}{d\tilde{\tau}}q_2 = -\sigma\sin(\rho)q_1 - \sigma\cos(\rho)q_2 + \sigma q_4, \tag{55}$$

$$\frac{d}{d\tilde{\tau}}q_3 = q_1 - \cos(\rho)q_3 - \sin(\rho)q_4 - A\cos(\phi)q_5 - q_1q_5, \tag{56}$$

$$\frac{d}{d\tilde{\tau}}q_4 = q_2 + \sin(\rho)q_3 - \cos(\rho)q_4 - A\sin(\phi)q_5 - q_2q_5, \tag{57}$$

$$\frac{d}{d\tilde{\tau}}q_5 = A\cos(\rho+\phi)q_1 + A\sin(\rho+\phi)q_2 + A\cos(\phi)q_3 + A\sin(\phi)q_4 - \tilde{b}q_5 + q_1q_3 + q_2q_4. \tag{58}$$

The matrix of the linear part of Eqs. (54)–(58) is therefore written as follows:

$$L_{\text{las}} = \begin{pmatrix} -\sigma\cos\rho & \sigma\sin\rho & \sigma & 0 & 0 \\ -\sigma\sin\rho & -\sigma\cos\rho & 0 & \sigma & 0 \\ 1 & 0 & -\cos\rho & -\sin\rho & -A\cos\phi \\ 0 & 1 & \sin\rho & -\cos\rho & -A\sin\phi \\ A\cos(\rho+\phi) & A\sin(\rho+\phi) & A\cos\phi & A\sin\phi & -\tilde{b} \end{pmatrix}. \quad (59)$$

Straightforward calculation immediately shows that this matrix is a degenerate matrix:

$$\det L_{\text{las}} \equiv 0, \quad (60)$$

and, hence, at least one characteristic root λ, i.e., one of the solutions of the characteristic equation

$$I^{(5)} \equiv \det\left(L_{\text{las}} - \tilde{\lambda}\hat{\mathbf{1}}\right) = 0 \quad (61)$$

is equal to zero. It turns out to be that there is only one zero root of Eq. (61), and we get the characteristic equation for this matrix in the form of a partially factorized polynomial of the fifth order in λ:

$$I^{(5)} = \tilde{\lambda} I^{(4)} = \tilde{\lambda}(\tilde{c_0}\tilde{\lambda}^4 + \tilde{c_1}\tilde{\lambda}^3 + \tilde{c_2}\tilde{\lambda}^2 + \tilde{c_3}\tilde{\lambda} + \tilde{c_4}) = 0. \quad (62)$$

We note the exception of the first threshold where there is an additional root exactly equal to zero. At the second threshold a pair of characteristic roots with zero real parts appears in addition to this zero root. From the point of view of stability theory a pair of purely imaginary roots is equivalent to one zero root [34]. This characteristic equation was obtained in [19], and in the next subsection we show that Eq. (62) and the characteristic equation derived in [19] are the same.

We write down the coefficients c_i, $i = 1, \ldots, 4$, for this equation both in terms of the coefficients of the system (38)–(40):

$$\tilde{c_0} = 1, \quad (63)$$

$$\tilde{c_1} = \tilde{b} + 2(1+\sigma)\cos\rho, \quad (64)$$

$$\tilde{c_2} = (1-\sigma)^2 + 4\sigma\cos^2\rho + 2\tilde{b}(1+\sigma)\cos\rho + A^2, \quad (65)$$

$$\tilde{c_3} = \tilde{b}\left\{(1-\sigma)^2 + 4\sigma\cos^2\rho\right\} + A^2(1+3\sigma)\cos\rho, \quad (66)$$

$$c_4 = 2A^2\sigma(1+\sigma)\cos^2\rho, \quad (67)$$

and, in the next subsection, in terms of the parameters of the complex Lorenz system (4)–(6).

B. The Hurwitz minor T_3, the Liénard-Chipart stability criterion, and the exact explicit expression for the second threshold

1. *Characteristic equation for the problem*

Equations (38)–(40) and the equations for their variations (54)–(58) near the stationary solutions (52) are very convenient to the study of the Hopf-like bifurcation at the second threshold because of their symmetrical form and simple trigonometrical parametrization of the coefficients (which are the components of the stationary solution). This study will be reported separately.

However, in this subsection it is more convenient to use the original notation for the complex Lorenz model [26] because of the need to compare the results we will derive here with the previous results in [19,20,22]. We compare our results mainly with results obtained in the work [19] which is up to now, probably, the work containing the most analytical results about the threshold properties of the standard laser equations under consideration in the present article. Studies of the various aspects of stability for the standard model of the single-mode homogeneously broadened laser have been done in [19] in the reference frame rotating with the cavity frequency ω; the time has been measured in the units of κ^{-1}, and the normalization of the variables was different. This might have made the comparison of our results with [19] difficult, however, very fortunately, the results from [19] have been reproduced in Sec. V of the review [22] in more frequently used notation, which makes a direct comparison with our results easier.

In terms of the original complex Lorenz model we can rewrite the characteristic equation using a redefinition of the characteristic number

$$\lambda = \tilde{\lambda}\cos\rho. \quad (68)$$

Thus, apart from the common factor of $\cos^4\rho = (1+\delta^2)^{-2}$, Eq. (62) takes the form

$$I^{(4)} = c_0\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 = 0, \quad (69)$$

where the parameters of the original complex Lorenz model are now used in the following expressions for coefficients:

$$c_0 = 1, \quad (70)$$

$$c_1 = 2(1+\sigma) + b, \quad (71)$$

$$c_2 = b(r-1) + (1+\sigma)(1+\sigma+2b) + \left\{(1-\sigma)^2 - b\right\}\delta^2(e;\sigma), \quad (72)$$

$$c_3 = b(1+3\sigma)(r-1) + b(1+\sigma)^2 + b\left\{(1-\sigma)^2 - (1+3\sigma)\right\}\delta^2(e;\sigma), \quad (73)$$

$$c_4 = 2b\sigma(1+\sigma)(r-1) - 2b\sigma(1+\sigma)\delta^2(e;\sigma). \quad (74)$$

We see that coefficients of the characteristic equation contain only the squared normalized detuning e^2 through the squared parameter function $\delta^2(e;\sigma)$.

The quantity $\delta^2(e;\sigma)$ entered into the characteristic coefficients (70)–(74) not only by means of the squared (normalized) field amplitude $A^2 = b(r-1-\delta^2)/(1+\delta^2)$ in the equations for variations (54)–(58) but also separately as a part of the coefficient $a = 1-ie$ in the original complex Lorenz equations. Therefore, if we perform expansions of the second threshold in the powers of δ^2, in order to be consistent we must collect the total factor at each power of δ^2 (i.e., including that part in the expression for A^2) in order to get the true coefficients in such expansions.

The neglect of this fact in previous work [19,20] led to quantitatively inconsistent results. This is why we have collected the total factors at $\delta^2(e;\sigma)$ in the characteristic coefficients (72)–(74).

The characteristic equations (62)–(69) are the basic ones for exact derivation of the explicit expression for the second threshold. Let us now make sure that we have the same characteristic equation as that derived previously by others.

Recalling the expressions obtained and used in [19] and [22], p. 61, we found there the characteristic equation

$$\lambda \sum_{n=0}^{4} a_n \lambda^n = 0 \tag{75}$$

with the following coefficients a_n:

$$a_4 = 1, \tag{76}$$

$$a_3 = 2 + 2\tilde{\kappa} + \tilde{\gamma}, \tag{77}$$

$$a_2 = 2\tilde{\gamma}(1+\tilde{\kappa}) + (1+\tilde{\kappa})^2 + (2C - 1 - \tilde{\Delta}^2)\tilde{\gamma} + (1-\tilde{\kappa})^2\tilde{\Delta}^2, \tag{78}$$

$$a_1 = \tilde{\gamma}\{(1+\tilde{\kappa})^2 + (1+3\tilde{\kappa})(2C-1-\tilde{\Delta}^2) + (1-\tilde{\kappa})^2\tilde{\Delta}^2\}, \tag{79}$$

$$a_0 = 2\tilde{\kappa}\tilde{\gamma}(1+\tilde{\kappa})(2C-1-\tilde{\Delta}^2). \tag{80}$$

Making the correspondence between the physical quantities

$$\begin{aligned} \tilde{\kappa} &\Leftrightarrow \sigma, \\ \tilde{\gamma} &\Leftrightarrow b, \\ 2C &\Leftrightarrow r, \\ \tilde{\Delta}^2 &\Leftrightarrow \delta^2, \end{aligned} \tag{81}$$

and between the characteristic coefficients

$$a_n = c_{4-n}, \quad n = 1, \ldots, 4, \tag{82}$$

we have verified that the characteristic equations (69)–(74) which we are using in the present work, and the characteristic equations (75)–(80) used by previous workers [19,20,22], are the same equations. This means, of course, that we studied the same Hurwitz minors as were studied previously.

2. *Liénard-Chipart stability criterion*

In this subsection we will derive the formal solution for the second threshold. This formal solution has two different branches. We show in the next subsection that one of the branches is not physical because it gives the value of the second threshold which lies *below* the first laser threshold r_1^{thr} for the detuned single-mode homogeneously broadened laser

$$r_1^{\text{thr}} = 1 + \delta^2(e;\sigma) = 1 + \frac{e^2}{(\sigma+1)^2}. \tag{83}$$

By virtue of the characteristic equations (70)–(74) the Hurwitz minors of our problem are defined by the following matrix [35]:

$$\begin{pmatrix} c_1 & 1 & 0 & 0 \\ c_3 & c_2 & c_1 & 1 \\ 0 & c_4 & c_3 & c_2 \\ 0 & 0 & 0 & c_4 \end{pmatrix}$$

and written as

$$T_1 = c_1 > 0, \tag{84}$$

$$T_2 = c_1 c_2 - c_3, \tag{85}$$

$$T_3 = c_3 T_2 - c_1^2 c_4. \tag{86}$$

Since the nontrivial stationary solution (52) exists only above the first threshold (83):

$$r - r_1^{\text{thr}} = r - 1 - \delta^2(e;\sigma) > 0, \tag{87}$$

all the coefficients c_i, $i = 1, \ldots, 4$ turn out to be positive

$$c_i > 0, \quad i = 1,2,3,4. \tag{88}$$

The latter inequality means that the *exponential instability* of the nontrivial stationary solutions (52) under consideration is determined solely by the sign of the third Hurwitz minor T_3 [35]. This particular case of the general Routh-Hurwitz criterion is called in the theory of matrices the Liénard-Chipart stability criterion [35]. In general, when the real part is equal to zero for at least one of the characteristic roots while the other characteristic roots have negative real parts, the rigorous investigation of the stability may no longer be possible in the framework of linearization, and special theory should be applied—stability theory for critical cases; see for review [36]. However, if, as in this case, the zero eigenvalue corresponds to a symmetry (the indeterminacy of the absolute phase) which is not broken by the subsequent bifurcation, the analysis can continue with a study only of the minor determinant governing the roots with initially negative real parts [37].

Calculating the minor T_3, we find that one might write this minor as a quadratic equation with respect to the normalized unsaturated population difference r. This gives us an equation for determination of those values of r at which the minor T_3 changes sign:

$$T_3(r) = kr^2 + pr + q = 0. \tag{89}$$

This quadratic equation (which is a biquadratic equation with respect to the steady-state intensity of the field A) was formally solved in the original work by Fowler, Gibbon, and McGuinness [26] and later, also formally, in the work by Ning and Haken [29]. No comparative analysis was done for two possible solutions of this equation and for the value $r_1^{\rm thr}$ of the first laser threshold, nor was an analytical study of the influence of detuning on these quantities provided. This situation has left open the problem of the analytical study of the dependence of the second threshold on the parameters of the model and its relation to other threshold values (i.e., to the value of the second threshold for the resonantly tuned laser, and to the value of the first threshold) as well as the problem of analysis of asymptotic expressions for the second threshold over the whole physical region of parameters.

Up to now, the main attempts to carry out parts of these analyses were, in our opinion, made in [19–22]. A discussion and comparison of our results and theirs can be found in Secs. VI A and VI B.

Thus, one of the roots of this quadratic equation (or maybe both roots in some physical region of parameters)

$$r^{\pm} = \frac{-p \pm \sqrt{p^2 - 4kq}}{2k} \tag{90}$$

is the sought-for value for the second threshold for the single-mode homogeneously broadened laser. The upper index of these roots $r^{\pm}$ corresponds to the sign in front of the square root.

The coefficients p and q in the Eq. (89) are very cumbersome and one can find their explicit general forms in Appendix A.

The most easily calculated coefficient k

$$k = b^2(1 + 3\sigma)(b + 1 - \sigma) \tag{91}$$

should be negative because the Hurwitz minor $T_3(r)$ as a quadratic function of r defined by (89) should be negative for large values of r when the stationary solution (52) is unstable according to known results [16–23]. This implies that even for the detuned laser we retain the usual bad-cavity condition [4,5]:

$$\sigma \ > \ b + 1 \tag{92}$$

as a necessary condition for obtaining a second threshold.

To define the true second threshold from the formal expression (90) one has to use the following requirements.

(i) The second threshold should be a real number.

(ii) The second threshold should not be less than the first threshold $r_1^{\rm thr} = 1 + \delta^2(e;\sigma)$.

We have proven that, under the bad-cavity condition (92), both roots (90) are real numbers because the discriminant of the quadratic equation (89) is always positive:

$$p^2 - 4kq > 0. \tag{93}$$

Hence, both roots (90) can be considered as a possible second threshold and they do not cross each other at any physical values of the parameters under the bad-cavity condition. Since that proof is very tedious, we have also put it in Appendix A.

3. Elimination of the root r^+

Now we check whether or not the root r^+ is more than the lasing threshold. We show that this root r^+ is *always less* than the first laser threshold $r_1^{\rm thr}$.

Proposition 3.1. At a zero value of the detuning and at an arbitrary finite value of the relative cavity loss, i.e., when $e = \delta(e;\sigma) = 0$, the root r^+ is a *negative* number.

Proof 3.1. Taking the limit of zero e^2 in the expression (90) with the plus sign in front of the square root, one gets

$$\lim_{\delta^2 \to 0} r^+ \ = \ \frac{-2 - b - 6\sigma - b\sigma - 6\sigma^2 - 2b\sigma^2 - 2\sigma^3}{b + 3b\sigma}, \tag{94}$$

which is obviously less than zero for all positive values of b and σ.

The physical consequence of *Proposition 3.1* is that for the *resonantly tuned* laser the root r^+ has no physical meaning and the only second threshold is given by the root r^- in expression (90).

However, one must be attentive to the behavior of the root r^+ with growth of the parameter $\delta(e;\sigma)$. For instance, at $\sigma = 3$ and $b = 1$ the root r^+ grows rapidly becoming positive at $\delta^2(e;\sigma) > 7$ (Fig. 1). Even in this particular case, the competition between two increasing functions of $\delta^2(e;\sigma)$, the root r^+ and the first threshold $r_1^{\rm thr} = 1 + \delta^2(e;\sigma)$, is evident. If r^+ becomes more than the lasing threshold $r_1^{\rm thr}$ a second physically meaningful branch of second thresholds will appear.

Fortunately, this is not the case and we will show this in *Proposition 3.2.* The difference $r^+ - r_1^{\rm thr}$ has no zeros for real values of $\delta(e;\sigma)$, and r^+ is, therefore, always less than $r_1^{\rm thr}$ (according to the previous proposition) for physical values of $\delta(e;\sigma)$, σ and b.

Proof 3.2. Straightforward solution of the equation

$$r^+ \ - \ r_1^{\rm thr} = 0 \tag{95}$$

with respect to $\delta^2(e;\sigma)$, gives after very long transformations:

$$\delta^2(e;\sigma) = \begin{cases} -\dfrac{(\sigma + b + 1)^2}{(\sigma - 1)^2}, \\ -\dfrac{(\sigma + 1)^2}{(\sigma - 1)^2}. \end{cases} \tag{96}$$

Thus, Eq. (95) has either one (or both) of the roots (96) or no roots at all depending on the values of the param-

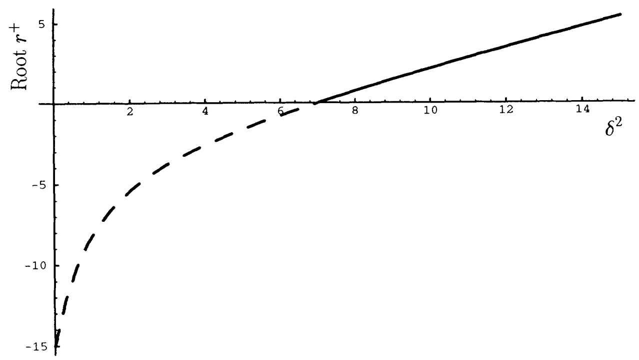

FIG. 1. A numerical example why one cannot neglect the root r^+ of the equation (89) without careful consideration. The root r^+ is shown vs squared detuning δ^2 at $\sigma = 3.0$ and $b = 1.0$. The values of r^+ become positive (solid line) when $\delta^2 > 7.0$. For these values of parameters, this root should be obligatory compared with the value of the first threshold $r_1^{\rm thr}$

eters σ and b. In any case, relation (96) is an obvious contradiction because there is a positive quantity on the left-hand side and there are explicitly negative quantities on the right-hand side of this relation.

The solutions (96) are *symbolic* solutions of equations $r^+ - r_1^{\rm thr} = 0$ and $r^- - r_1^{\rm thr} = 0$. However, if one sets, for instance, $\sigma = 3$ and $b = 3/4$, *symbolic* solution (96) is not a *numerical* solution of the equation $r^- - r_1^{\rm thr} = 0$. This simply means that this equation has no solutions for these values of σ and b.

Thus, the root r^+ can never cross the value of the first laser threshold $r_1^{\rm thr}$, remaining *below* it.

4. Verification of the position of the root r^-

Now we verify whether or not the root r^- is less than the lasing threshold. If so, for some domain of physical parameters, this would mean that the root r^- cannot be a second threshold inside that domain of the parameters, being therein less than the first threshold. But we demonstrate that the root r^- is *always greater* than the first threshold.

Proposition 3.3. At zero detuning, at arbitrary finite value of the relative cavity loss and under the bad-cavity condition, the difference $r^- - r_1^{\rm thr}$ is a positive number:

$$\lim_{\delta^2\to 0}\left\{ r^- - r_1^{\rm thr}\right\} = \frac{\sigma(3+b+\sigma)}{\sigma-b-1} - 1 \geq 8. \qquad (97)$$

Proof 3.3. According to the contents of Sec. IV, the absolute minimum for $r^-(= r_2^{\rm thr})$ occurs at zero detuning and is greater than or equal to 9 while the minimum of $r_1^{\rm thr}$ also occurs at zero detuning and is equal identically to 1.

Proposition 3.4. The difference $r^- - r_1^{\rm thr}$ has no zeros at real values of $\delta(e;\sigma)$, and r^- is always greater than $r_1^{\rm thr}$ for physical values of $\delta(e;\sigma)$, σ and b.

Proof 3.4. Solving the equation

$$r^- - r_1^{\rm thr} = 0 \qquad (98)$$

with respect to $\delta^2(e;\sigma)$, we find once again *symbolic* relations (96). Let us recall once again that a *symbolic* solution can exist in a domain of the parameters where the corresponding *numerical (or exact)* solution does not exist. If so, there are no roots at all which do not violate our consideration.

Further proof just repeats the *Proof 3.2.* Thus, the root r^- remains always above the first threshold $r_1^{\rm thr}$.

The physical conclusion from *Propositions 3.1–3.4* is that among two roots (90) of the basic equation (89) derived from the Liénard-Chipart stability criterion [35] only the root r^- has the physical meaning as the second threshold for the single-mode homogeneously broadened laser.

5. Exact analytical expression for the second threshold at arbitrary physical values of $(\omega-\varepsilon)/\gamma_\perp = e$, $\kappa/\gamma_\perp = \sigma$, and $\gamma_\parallel/\gamma_\perp = b$

Thus, we get for the value of the second threshold the following exact explicit expression:

$$r_2^{\rm thr}(\delta^2(e;\sigma),\sigma,b) = \frac{-p-\sqrt{p^2-4kq}}{2k} = \frac{N(\delta^2(e;\sigma),\sigma,b)}{D(\sigma,b)}. \qquad (99)$$

To make the expressions more compact we are using here the parameter $\delta(e;\sigma)$, which contains the dependence on the normalized detuning $e = (\omega-\varepsilon)/\gamma_\perp$. The numerator of this exact expression has the following explicit form:

$$\begin{aligned} N(\delta^2,\sigma,b) = {} & 2+3b+b^2+2\delta^2-b\delta^2-b^2\delta^2+4\sigma+9b\sigma+2b^2\sigma+4\delta^2\sigma \\ & -7b\delta^2\sigma-8b^2\delta^2\sigma+17b\sigma^2+5b^2\sigma^2-8\delta^2\sigma^2+9b\delta^2\sigma^2+b^2\delta^2\sigma^2 \\ & -4\sigma^3+3b\sigma^3-4\delta^2\sigma^3-b\delta^2\sigma^3-2\sigma^4+6\delta^2\sigma^4 \\ & +(-1+\sigma)\,(2+b+2\sigma)(1+2b+b^2+2\delta^2-2b^2\delta^2+\delta^4-2b\delta^4 \\ & \qquad +b^2\delta^4+4\sigma+6b\sigma+2b^2\sigma+8\delta^2\sigma+4b\delta^2\sigma-16b^2\delta^2\sigma \\ & \qquad +4\delta^4\sigma-2b\delta^4\sigma-2b^2\delta^4\sigma+6\sigma^2+6b\sigma^2+b^2\sigma^2+4\delta^2\sigma^2 \\ & \qquad +32b\delta^2\sigma^2-14b^2\delta^2\sigma^2-2\delta^4\sigma^2+10b\delta^4\sigma^2+b^2\delta^4\sigma^2+4\sigma^3+2b\sigma^3 \\ & \qquad -8\delta^2\sigma^3+28b\delta^2\sigma^3-12\delta^4\sigma^3-6b\delta^4\sigma^3+\sigma^4-6\delta^2\sigma^4+9\delta^4\sigma^4)^{1/2}, \end{aligned} \qquad (100)$$

while the denominator is just

$$D(\sigma,b) = -2k = 2b^2(1+3\sigma)(\sigma-b-1). \tag{101}$$

Such cumbersome expressions as expression (100) are, very unfortunately, typical for the Hurwitz minor method in stability theory. The main difficulty is to understand the behavior of $r_2^{\rm thr}$ as of a function of the main variable, the normalized detuning e^2, over the range of the set of the physical values of the other parameters, the normalized relaxation rates σ and b.

C. Analytical expansions of the second threshold in powers of detuning $e = (\omega - \varepsilon)/\gamma_\perp$ for arbitrary $\sigma = \kappa/\gamma_\perp$ and $b = \gamma_\parallel/\gamma_\perp$

In this subsection we can consider the normalized relaxation σ as a fixed finite parameter but not as a variable going to infinity. Thus, the limits of zero detuning e and zero parameter $\delta(e;\sigma)$ will coincide.

Expanding expression (99) into a series with respect to powers of the squared normalized detuning e^2, we get

$$\begin{aligned} r_2^{\rm thr}(e^2 \ll 1,\sigma,b) &= \tilde R_0 + \tilde R_2 e^2 + \tilde R_4 e^4 + \tilde R_6 e^6 + O(e^8)\\ &= -\frac{\sigma(3+b+\sigma)}{1+b-\sigma} + \frac{\sigma\left(-5-2b-b^2+5\sigma^2\right)e^2}{(1+\sigma)^2(-1-b+\sigma)(1+b+\sigma)}\\ &\quad+\frac{4(-1+\sigma)\sigma(1-b+\sigma)(-b+2\sigma)(2+b+2\sigma)e^4}{(1+\sigma)^5(1+b+\sigma)^3}\\ &\quad+4(-1+\sigma)\sigma(1-b+\sigma)(-b+2\sigma)(2+b+2\sigma)\\ &\quad\times\frac{\left(-1+b^2-3\sigma-2b\sigma+7b^2\sigma+\sigma^2-14b\sigma^2+3\sigma^3\right)e^6}{(1+\sigma)^8(1+b+\sigma)^5} + O(e^8). \end{aligned} \tag{102}$$

In addition to this expansion, in order to make comparisons easier with earlier results, we give here the same expansion but in powers of the σ-dependent parameter $\delta^2(e;\sigma)$:

$$\begin{aligned} r_2^{\rm thr}(\delta^2 \ll 1,\sigma,b) &= R_0 + R_2\delta^2 + R_4\delta^4 + R_6\delta^6 + O(\delta^8)\\ &= -\frac{\sigma(3+b+\sigma)}{1+b-\sigma} - \frac{\sigma\left(-5-2b-b^2+5\sigma^2\right)}{(1+b-\sigma)(1+b+\sigma)}\delta^2\\ &\quad+\frac{4(-1+\sigma)\sigma(1-b+\sigma)(-b+2\sigma)(2+b+2\sigma)}{(1+\sigma)(1+b+\sigma)^3}\delta^4\\ &\quad+4(-1+\sigma)\sigma(1-b+\sigma)(-b+2\sigma)(2+b+2\sigma)\\ &\quad\times\frac{\left(-1+b^2-3\sigma-2b\sigma+7b^2\sigma+\sigma^2-14b\sigma^2+3\sigma^3\right)}{(1+\sigma)^2(1+b+\sigma)^5}\delta^6 + O(\delta^8). \end{aligned} \tag{103}$$

Both expression (102) and expression (103) provide us, of course, with the true value of the second threshold for the resonantly tuned laser [18–23] (or, in other words, for the bifurcation point at which the nontrivial steady-state solution of the *real* Lorenz model loses stability [28]):

$$\begin{aligned} r_{2,0}^{\rm thr} &\equiv r_2^{\rm thr}(e^2=0)\\ &= r_2^{\rm thr}(\delta^2=0) = \tilde R_0 = R_0 = \frac{\sigma(3+b+\sigma)}{\sigma-b-1}. \end{aligned} \tag{104}$$

IV. ABSOLUTE MINIMUM OF THE SECOND THRESHOLD FOR THE *RESONANTLY TUNED* LASER

What is the minimum value of the second threshold for the *resonantly tuned* laser?

Differentiating the expression (104) with respect to the variables σ and b, we get

$$\frac{\partial r_{2,0}^{\rm thr}}{\partial b} = \frac{2\sigma(1+\sigma)}{(1+b-\sigma)^2}, \tag{105}$$

$$\frac{\partial r_{2,0}^{\rm thr}}{\partial \sigma} = \frac{-3-4b-b^2-2\sigma-2b\sigma+\sigma^2}{(1+b-\sigma)^2}. \tag{106}$$

The derivative (105) is always positive at positive σ.

Therefore, $r_{2,0}^{\rm thr}$ as a function of b is a monotonically growing function which has its obvious minimum at the lower boundary, i.e., at $b = 0$.

The derivative (106) has a unique zero at the following *positive* values of σ:

$$\sigma_0(b) = 1 + b + \sqrt{4 + 6b + 2b^2}. \quad (107)$$

At $b = 0$ the value of $\sigma_0(b)$ is equal to 3.

Setting the function $\sigma_0(b)$ and zero value of b into the expression for $r_{2,0}^{\text{thr}}$, one can obtain the *minimum* value of the second threshold for the *resonantly tuned* laser:

$$\min\left(r_{2,0}^{\text{thr}}\right) = 9. \quad (108)$$

While the expansion for $r_{2,0}^{\text{thr}}$ at $\sigma = \sigma_0(b)$ in powers of b in the vicinity of $b = 0$ has the following form:

$$r_{2,0}^{\text{thr}}(\sigma = \sigma_0(b)) = 9 + 6b - \frac{b^2}{8} + \frac{3b^3}{32} + O(b^4), \quad (109)$$

for large b this function is asymptotically a straight line:

$$r_{2,0}^{\text{thr}}(\sigma = \sigma_0(b)) \approx 9 + \left\{ \frac{(1+\sqrt{2})(2+\sqrt{2})}{\sqrt{2}} \right\} b$$

$$\approx 9 + 5.828436. \quad (110)$$

Thus, the value of the minimum for the second threshold lies not far (less than one order of the magnitude) above the first laser threshold for the resonantly tuned laser which is equal in these units exactly to 1.

This value of $\sigma_0(0)$ at which the minimum occurs is not approximately equal to 3.5 [22] but equal exactly to 3. The specific dependence of σ_0 on b which realizes this minimum for nonzero values of b is given by relation (107).

The minima of the second threshold $r_{2,0}^{\text{thr}}(\sigma = \sigma_0(b))$ have been collected in Table II together with the specific values of $\sigma_0(b)$ which realize these minima. We note that the transition to instability at the value of $b = \gamma_\parallel/\gamma_\perp = 0.25$ was predicted for the ammonia laser [38,39] and observed [40–43].

For atomic gas lasers and solid-state lasers with strong homogeneous broadening, b may be much smaller than one, while the physical upper limit of the two-level model for b is 2, though multilevel models have a physical upper limit for b of 1 [22]. We include the values of $b = 8/3$ and $b = 10$ for comparison with other studies of the Lorenz model for nonlaser systems [28].

One can see from Table II that the value of $r_{2,0}^{\text{thr}}(\sigma = \sigma_0(b))$ grows with b but not too rapidly for the most physically accessible region for lasers between $b = 0.0$ and $b = 0.25$.

The values of $r_{2,0}^{\text{thr}}$ are not so inaccessible for the physically accessible range of values for b, being very close to the absolute minimum which is equal to 9.

The discussion in this short section has shown that it may be experimentally easier to observe the transition to instability of the standard laser equations or of the Lorenz model in the case when the relation $\gamma_\parallel/\gamma_\perp = b$ is very small and when the value of the ratio $\kappa/\gamma_\perp = \sigma$ is matched to the value of σ_0 according to its dependence (107) on $\gamma_\parallel/\gamma_\perp$.

TABLE II. The minimal values of the second threshold $r_{2,0}^{\text{thr}}$ for various values of b which are achieved at the specified value of $\sigma = \sigma_0(b)$.

b	$\sigma_0(b)$	$r_{2,0}^{\text{thr}}$
0.0	3.0	9.0
0.001	3.0025	9.006
0.01	3.02499	9.05999
0.1	3.24942	9.59884
0.25	3.62171	10.4934
0.5	4.23861	11.9772
1.0	5.4641	14.9282
2.0	7.89898	20.798
$8/3 \approx 2.666\ldots$	9.51664	24.7
10.0	27.2481	67.4962

V. INITIAL PULSATION FREQUENCY: GENERAL AND ASYMPTOTIC ANALYTICAL EXPRESSIONS

One can derive from the characteristic equation (69) the expression for the initial pulsation frequency by taking in that characteristic equation the characteristic number λ as a purely imaginary quantity:

$$\lambda = i\,\Lambda_0. \quad (111)$$

Then Eq. (69) splits into a system of two equations: the equation for the determination of the value of the second threshold (89)

$$T_3(r;\sigma,b) = c_1c_2c_3 - c_3^2 - c_1^2c_4 = 0,$$

which has been already solved in Sec. III B, and the equation for the value of Λ_0^2

$$\Lambda_0^2 = \left(\frac{c_3}{c_1}\right)\Bigg|_{r=r_2^{\text{thr}}}, \quad (112)$$

where the normalized intensity of noncoherent pumping r is equal to the value of the second threshold. In [19] and [22] the initial pulsation frequency is denoted by Ω. We are using here the designation Λ_0 for the initial pulsation frequency because we have already used the symbol Ω for other purposes.

It is very important to note that due to the positiveness of the quantities c_3 and c_1 defined by relations (73) and (71), respectively, the solution of Eq. (112) exists at *all* physical values of parameters under the bad-cavity condition above the lasing threshold $r_1^{\text{thr}} = 1 + \delta^2(e;\sigma)$.

The latter result means that the bifurcation at the second threshold occurs *always* at the presence of two imaginary characteristic roots. We have set the explicit analytical expression for the initial pulsation frequency Λ_0 in Appendix B.

The expansion in powers of $\delta^2(e;\sigma)$ for the square of the initial pulsation frequency at the second threshold has the form

$$\Lambda_0^2(\delta^2(e;\sigma) \ll 1) = \frac{-2b\sigma\,(1+\sigma)}{1+b-\sigma} - \frac{4b\,(-1+\sigma)\,\sigma\,(-b+2\sigma)}{(1+b-\sigma)\,(1+b+\sigma)}\delta^2 + \frac{4b\,(-1+\sigma)\,\sigma\,(1-b+\sigma)\,(-b+2\sigma)\,(1+3\sigma)}{(1+\sigma)\,(1+b+\sigma)^3}\delta^4$$
$$+4b\,(-1+\sigma)\,\sigma\,(1-b+\sigma)\,(-b+2\sigma)\,(1+3\sigma)$$
$$\times\frac{\left(-1+b^2-3\sigma-2b\sigma+7b^2\sigma+\sigma^2-14b\sigma^2+3\sigma^3\right)}{(1+\sigma)^2(1+b+\sigma)^5}\delta^6+O(\delta^8). \tag{113}$$

We immediately see that in the limit of large σ taken *after* the limit (113) of a small value of parameter $\delta(e;\sigma)$ the following expression is obtained:

$$\Lambda_0^2(\delta^2 \ll 1; \sigma \gg b+1) \approx 2b\sigma(1+4\delta^2+12\delta^4+36\delta^6). \tag{114}$$

Let us also consider now other asymptotic cases.

In the *single* limit of large δ^2 taken in the general expression (B1) we get the following form:

$$\Lambda_0^2(\delta^2(e;\sigma) \gg 1) = -\frac{(-1+\sigma)^2\,(1-b+3\sigma)}{1+b-\sigma}\delta^2 - \frac{(1+\sigma)\left(-1+b^2-3\sigma-4b\sigma+5b^2\sigma+\sigma^2-12b\sigma^2+3\sigma^3\right)}{(-1+b-3\sigma)\,(1+b-\sigma)}$$
$$+\frac{4b\sigma(1+\sigma)^2\,(1-b+\sigma)\,(-b+2\sigma)\,(1+3\sigma)}{(-1+\sigma)^2(1-b+3\sigma)^3}\frac{1}{\delta^2}+O\left(\frac{1}{\delta^4}\right). \tag{115}$$

At the same time, in the *single* limit of large σ we have

$$\Lambda_0^2(\sigma \gg b+1) = (-1+3\delta^2)\sigma^2 + \frac{2\,(1+\delta^2)\,(1-3\delta^2+3b\delta^2)\,\sigma}{-1+3\delta^2}$$
$$+(1+\delta^2)\,(-1+3\delta^2)^{-3}(1-9\delta^2-10b\delta^2-6b^2\delta^2$$
$$+27\delta^4+12b\delta^4-48b^2\delta^4-27\delta^6+54b\delta^6+54b^2\delta^6)+O\left(\frac{1}{\sigma}\right). \tag{116}$$

These two expressions coincide in the double limit of large σ and large δ^2, providing us with the following limit expression:

$$\Lambda_0^2(\delta^2 \gg 1; \sigma \gg b+1) \approx 3\sigma^2\delta^2. \tag{117}$$

If we consider the expression for Λ_0^2 at large σ and moderate $\delta^2 = 1/3$, we get

$$\Lambda_0^2(\delta^2 = 1/3; \sigma \gg b+1) \approx 3 + \frac{4\sqrt{2}\sqrt{\sigma}}{\sqrt{3}} + \frac{2\sqrt{2}\sigma^{\frac{3}{2}}}{\sqrt{3}}. \tag{118}$$

VI. DISCUSSION OF THE PUMP VALUE REQUIRED FOR THE SECOND THRESHOLD

A. Brief overview of previous analytical results

It is now possible to compare our results with previous analytical results for this dynamical model [19,20,22]. In order to make the comparison of all results easier, we give this brief overview of the earlier results. Some of these results were formulated in another time scale—more exactly with time normalized to κ^{-1} [19], while in this article we use the common normalization of time to $\gamma_\perp^{-1}$, some other results contain misprints [22]. At last, there are both a minor inconsistency in the derivation of these results which has led to different values of the coefficients in expansions for the second threshold and initial pulsation frequency, and incomplete specification of the physical meaning of the asymptotic case of large $\delta^2(e;\sigma)$ at large values of σ (see the preliminary discussion in Sec. III A 1).

In particular, these results concern the analytical expressions for the second threshold and for the initial pulsation frequency of the Hopf-like bifurcation of the steady solution at the second threshold. They have been accomplished previously only under certain approximations.

Generally, the earlier results were calculated for the fixed value of $\gamma_\parallel/\gamma_\perp = b = 1$ while in this article we have restored in all expressions the dependence on b and therefore we can easily proceed to the particular case of $b = 1$.

Further, in previous work the value of $\kappa/\gamma_\perp = \sigma$ has been either fixed ($\sigma = 3$ in [20]) or taken in the limit of $\sigma \gg 2$ (for $b = 1$) at various limits [19–22] for $\delta^2(e;\sigma)$.

The following expression has been derived in [20] for the expansion of the second threshold in powers of the parameter $\delta^2(e;\sigma)$ at $b = 1$ and $\sigma = 3$:

$$r_2^{\text{thr}}(\delta^2 \ll 1; \sigma = 3; b = 1)$$
$$\approx 3 + 16.8\delta^2 + (324 + 194.4\delta^2 + 262.44\delta^4)^{1/2}$$
$$\approx 21 + 22.2\delta^2 + 7.29\delta^4. \tag{119}$$

Other approximate results have been given in [19,22] under the condition $\sigma \gg 2$, which is a particular case of the asymptotic expressions in the bad-cavity limit $\sigma \gg b+1$ at $b = 1$.

Three situations have been considered in the particular case of the bad-cavity limit: small detuning $\delta^2(e;\sigma) \ll 1$, moderate detuning $\delta^2(e;\sigma) = 1/3$, and large detuning $\delta(e;\sigma) \gg 1$.

Unfortunately, when reproducing the expressions from the original work [19] in the review [22] the rescaling factor of σ^2 for the initial pulsation frequency was neglected; we have restored it below.

According to [19], for small values of the parameter $\delta(e;\sigma)$ the second threshold value is

$$r_2^{\rm thr}(\delta^2 \ll 1;\sigma \gg 2) \approx 1+\delta^2+\frac{\sigma(1+\delta^2)^2}{1-3\delta^2} \approx 1+\delta^2+\sigma(1+5\delta^2+7\delta^4+3\delta^6), \quad (120)$$

and the initial pulsation frequency is given by

$$\Lambda_0^2(\delta^2 \ll 1;\sigma \gg 2) \approx \frac{2\sigma(1+\delta^2)}{1-3\delta^2} \approx 2\sigma(1+4\delta^2+3\delta^4). \quad (121)$$

For a moderate value of $\delta^2(e;\sigma) = 1/3$, the corresponding expressions are

$$r_2^{\rm thr}(\delta^2=\tfrac{1}{3};\sigma \gg 2) \approx \tfrac{4}{3}\sqrt{\tfrac{2}{3}}\sigma^{3/2}, \quad (122)$$

$$\Lambda_0^2(\delta^2=\tfrac{1}{3};\sigma \gg 2) \approx 2\sqrt{\tfrac{2}{3}}\sigma^{3/2}. \quad (123)$$

For large values of $\delta^2(e;\sigma)$ one finds in [19] the following expressions:

$$r_2^{\rm thr}(\delta^2 \gg 1;\sigma \gg 2) \approx 1+\delta^2+\frac{2\sigma^2(3\delta^2-1)}{3} \approx 1+\delta^2+2\sigma^2\delta^2, \quad (124)$$

$$\Lambda_0^2(\delta^2 \gg 1;\sigma \gg 2) \approx \sigma^2(3\delta^2-1). \quad (125)$$

B. Various asymptotic expressions

In this subsection, on the basis of our general results, we derive and consider miscellaneous asymptotic expressions both for the second threshold and for the initial pulsation frequency. We compare our general expressions with earlier results, discuss differences, and specify more clearly the physical meaning of the asymptotic expressions.

1. Comparison with previous results and some remarks

Let us now compare our results with earlier results [19,20,22], which we have collected in Sec. VI A. For instance, at $\sigma = 3$ and $b = 1$ used in earlier studies one can get from the general expansion (103) the following particular expansion:

$$r_2^{\rm thr}(\delta^2 \ll 1;\sigma=3,b=1) \approx 21+\tfrac{111}{5}\delta^2+\tfrac{162}{25}\delta^4-\tfrac{243}{125}\delta^4 \approx 21+22.5\delta^2+6.48\delta^4-1.944\delta^6. \quad (126)$$

All coefficients in powers of $\delta^2(e;\sigma)$ in this expansion are different from the coefficients in the earlier expansion (119). However, more important is the information which we get from the additional [in comparison with (119)] term of the sixth power of δ. It gives a negative contribution to the value of the second threshold at the values of $\sigma = 3$ and $b = 1$. Thus, making just one step beyond previous studies, we found that the perturbative approach reveals, even in the particular case, a possibility of a decreasing of the second threshold with increasing of detuning. A general treatment of this problem is given in the next section.

In order to understand the origin of the quantitative difference from previous results (i.e., in coefficients of these particular expansions) we have repeated all calculations keeping the quantity $(r-1-\delta^2)$ as completely δ^2 independent, and have reproduced the result (119). Thus, the difference appears to have originated in an inconsistent treatment of the quantity $(r-1-\delta^2)$ in [19] as a δ^2-independent quantity.

This inconsistency is quite minor; however, it generally leads to values of the coefficients which are regularly different from the true values. Fortunately, the qualitative meaning of the previous results remains valid.

In the double limit of small $\delta^2(e;\sigma)$ and large σ the second threshold depends linearly on the σ variable and contains only positive corrections in the powers of the parameter $\delta(e;\sigma)$ (up to the sixth power) to its value in the case of exactly resonant tuning:

$$r_2^{\rm thr}(\delta^2 \ll 1;\sigma \gg b+1) \approx \sigma(1+5\delta^2+16\delta^4+48\delta^6)+O(\delta^8). \quad (127)$$

Comparing our expression with the analogous expression (120), one can conclude that again the coefficients of powers of $\delta(e;\sigma)$ in (120) are different from the true values and, in addition, there is strange appearance of unity in the expansion (120).

The qualitative difference between our results and the earlier results in this limit of small δ^2 and large σ is quite minor: the different coefficients of the powers of δ^2 do not disturb sufficiently the value of $r_2^{\rm thr}$, and unity can be safely neglected in comparison with large σ. However, the earlier expansion (120) is not suitable for numerical treatment.

At these values of δ^2 and σ, the previous particular asymptotic expression (121) for the initial pulsation frequency Λ_0 contains a coefficient at δ^4 which is different from our expression (114) when it is evaluated in the particular case $b = 1$.

The authors of [19] have also considered the case of $\delta^2 = 1/3$. In this case, we get from (103) at arbitrary values of b:

$$r_2^{\rm thr}(\delta^2=1/3;\sigma\gg b+1)\approx\sigma^{3/2}\left(\frac{4\sqrt{6}}{9\sqrt{b}}+\frac{4\,(-2+b)}{9b\sqrt{\sigma}}+\frac{(4\sqrt{6}+18\sqrt{6}b+16\sqrt{6}b^2)}{27b^{\frac{3}{2}}\sigma}+\left(\frac{74}{27}-\frac{28}{27b}+\frac{4b}{3}\right)\frac{1}{\sigma^{3/2}}\right). \quad (128)$$

The first term in this our expansion gives in the particular case of $b=1$ the same result as expression (121). However, a new feature is visible from our expansion (128) that is b dependent: with increasing b the value of the second threshold is slowly decreasing.

Our expansion (118) for Λ_0^2 in this region of parameters has the same leading term as the previous result (122).

The limit expression for the second threshold in the case of large detuning has (after a number of possible transformations) a quite simple form,

$$r_2^{\rm thr}(\delta^2\gg 1;\sigma,b)$$
$$=\delta^2\frac{-2+b+b^2+2\sigma-b\sigma+2\sigma^2-2\sigma^3}{b+b^2-b\sigma}. \quad (129)$$

Taking in this expression the additional limit of $\sigma\gg b+1$, we get a very simple dependence of the second threshold on large δ^2 and the normalized relaxation rates σ and b:

$$r_2^{\rm thr}(\delta^2\gg 1;\sigma\gg b+1)=\frac{2\sigma(\sigma-1)+b-2}{b}\delta^2\approx\frac{2\sigma^2}{b}\delta^2. \quad (130)$$

Recalling the analogous result (124) from [19] and setting $b=1$ in our general expression (130), we find that the inconsistent treatment of $(r-1-\delta^2)$ as an δ^2-independent quantity leads to a qualitatively different result: now the superfluous addition of δ^2 in the limit of large δ^2 can give an appreciable absolute correction while the relative correction may be small because $2\sigma^2\gg 1$.

Summarizing our comparison with previous results from [19] and [20], we have not only added a few general limit expressions to the earlier results but have also improved the earlier results. However, our study might not have been possible without those previous studies which made our work much easier.

Once again we found that $r_2^{\rm thr}$ decreases with increasing of b in the presence of detuning, which is opposite to the increase of $r_{2,0}^{\rm thr}$ when b is increasing.

2. *On the order of performance of double limits*

Another interesting question is whether or not these double limits in σ and δ^2 depend on the order of performing them or not? In both cases (129) and (130), which we have already considered, we first varied the detuning δ^2 and only after this took the large values of σ. Let us see what happens if we will first take the limit of the large σ, and after this will vary the value of δ^2.

In the limit of the large σ we get the very simple expression

$$r_2^{\rm thr}(\delta^2;\sigma\gg b+1)=\sigma^2\frac{2}{3b}(3\delta^2-1). \quad (131)$$

This relation gives the same value of the second threshold in the limit of the large δ^2 as expression (130). However, in the opposite limit of small detuning δ^2 we cannot now get expression (127) and, moreover, we get a negative unphysical value.

We found that the result of the double limit of small δ^2 and large normalized relaxation rate of the field σ strongly depends on the order of performing this double limit because of the implicit dependence of δ^2 on σ given in (26) and (31) and discussed in Sec. III A 1.

Thus, in order to avoid an ambiguity in the interpretation of the double limit of large δ^2 and large σ we have to specify that the double limit of large σ and large δ^2 means not only the inequalities $\sigma\gg b+1$ and $\delta^2\gg 1$ but rather also

$$e^2\gg(\sigma+1)^2\gg(b+1)^2. \quad (132)$$

C. Laser and maser asymptotes

The latter inequalities mean in terms of the atomic and field relaxation rates and frequencies the following conditions:

$$(\omega-\varepsilon)^2\gg(\kappa+\gamma_\perp)^2\gg(\gamma_\parallel+\gamma_\perp)^2. \quad (133)$$

It is clear that this condition can make physical sense in the framework of the single-mode approximation only for masers, as pointed out by Oraevskiy [44]. He proposed [45] that it is more probable to observe experimentally the transition to chaos as described in the Lorenz model when dealing with masers rather than with lasers. However, for the moment there is no experimental verification of this very interesting proposal.

If the analogous double limit is performed not in the variables σ and δ^2 but in the variables σ and e^2, one avoids the unnecessary additional requirement (133). It turns out to be that in the limit of large σ taken without the additional requirement (133) the second threshold does *not* possess a dependence on the normalized detuning e^2 at all. The corresponding expression has the form

$$r_2^{\rm thr}(e^2;\sigma\gg b+1)=(1+b)^2\sigma+2(2+b)(1+b)^2+\frac{(4+6b+2b^2+5e^2)(1+b)^3}{1+b}\frac{1}{\sigma}+O(\frac{1}{\sigma^2}). \quad (134)$$

The leading term in this expansion, which contains the squared normalized detuning $(\omega-\varepsilon)^2/\gamma_\perp^2=e^2$, vanishes in the limit of large $\kappa/\gamma_\perp=\sigma$, and this asymptotic expression does not depend on the detuning e^2 as should be expected (see the discussion in Sec. III A 1).

For lasers which have been studied thus far the practical limit for achieving bad cavities and sufficiently high pump values to reach the second threshold, limit the parameters to $\sigma\leq 5$, $|e|\leq 5$, and thus $\delta^2\leq 1$. This means that neither of the two limits (either that of large cavity loss or that of large detuning) in the double limit for the second threshold has relevance to any presently realizable laser that we know.

Thus, the conclusion is important: the simultaneous double limit of large δ^2 and σ (or the double limit of large e^2 and σ) makes physical sense only for masers, and all our previous results which are derived in these double limits are applicable only to masers but not to lasers.

For example, for the cryogenic hydrogen maser, the following numerical values have been reported in the recent work [46] by Mandel *et al.*: $\gamma_\perp \approx \gamma_\parallel \approx 1$ and $\kappa \approx 10^5$ (in $\sec^{-1}$). It is evident after our analysis that for such a maser the expressions obtained in the double limit of extremely bad cavity and large detuning have physical sense.

Although the NMR laser cannot be precisely described by the Lorenz model because of the presence of additional nonlinear polarization damping [47], the values of the relaxation constants ($\sigma \approx 4.875$ and $b \approx 2 \times 10^{-4}$) give rise to hopes that for a NMR laser it might be possible to realize a regime of operation similar to that described by simple expressions obtained in the double limits discussed before.

VII. GENERAL PROOF THAT INCREASING THE DETUNING INCREASES THE SECOND THRESHOLD FOR THE HOMOGENEOUSLY BROADENED SINGLE-MODE LASER

A. Motivation for the general proof (failure of perturbative approach)

Both the exact analytical expressions (99)–(101) for the second threshold and the expansion (103) carry a lot of the information about the actual position of the second threshold as a multiparameter function of the square of the normalized detuning $(\omega - \varepsilon)/\gamma_\perp = e$.

The natural question is: does the second threshold only increase with detuning or can the second threshold also sometimes decrease with increasing detuning?

The common conception based on many approximate studies (see reviews [22] and [23]) answers that increasing the detuning can only increase the value of the pump required to observe the second threshold. We will prove this exactly; however, for the moment, we will try to get a feeling for the situation by considering the approximate expansions.

One might think for the moment that the values of the parameter $\delta(e;\sigma)$ are not more than, for example, 1/2, in the most physical values of the parameters ω, ε, $\gamma_\perp$, and κ of the initial laser system (1)–(3).

If the first few coefficients of the expansion (103) of the second threshold in powers of the normalized detuning are positive in this region of the physical parameters, then we can assert that inside the most physical region of the values of parameters the second threshold only increases with increasing detuning.

Thus, let us check the definiteness in sign of the coefficients of the expansion (103). To this end, we introduce a new subsidiary variable [48,49]

$$\xi = \sigma - b - 1 > 0, \tag{135}$$

which is always positive due to the bad-cavity condition (92).

Substituting the variable (135) into (103), we see that first two terms in (103) are positive at all physical values of σ and b [or, in terms of the initial system (1)–(3), at all physical values of κ, $\gamma_\perp$, and $\gamma_\parallel$ obeying the bad-cavity condition (92)]

$$R_2 = \frac{(1+b+\xi)\,(8b+4b^2+10\xi+10b\xi+5\xi^2)}{\xi\,(2+2b+\xi)} > 0, \tag{136}$$

$$R_4 = \frac{4\,(2+\xi)\,(b+\xi)\,(1+b+\xi)\,(2+b+2\xi)\,(4+3b+2\xi)}{(2+b+\xi)\,(2+2b+\xi)^3} > 0, \tag{137}$$

while the third term in this expansion is not definite in sign:

$$\begin{aligned} R_6 &= 4\,(2+\xi)\,(b+\xi)\,(1+b+\xi)\,(2+b+2\xi)\,(4+3b+2\xi) \\ &\quad\times \frac{(-8b-12b^2-4b^3+8\xi-10b\xi-12b^2\xi+10\xi^2-5b\xi^2+3\xi^3)}{(2+b+\xi)^2(2+2b+\xi)^5}. \end{aligned} \tag{138}$$

However, one may consider the perturbation of the value of the second threshold by the latter term as negligible for *small detuning.* Actually, recalling the original definition (26) of δ^2:

$$\delta^2 = \frac{(\omega-\varepsilon)^2}{(\kappa+\gamma_\perp)^2}$$

and values of the parameters κ and $\gamma_\perp$ for the class-C lasers [50–53] one might conclude that the first three terms describe the behavior of the second threshold in the most physical region $\delta^2 < 1$ quite precisely.

In this region one can ignore safely all higher-order terms in the expansion (103) and assert on the basis of the approximate expansion (103) that the conclusion of [22,23] that the second threshold increases when the detuning increases is valid *practically* always. However, one cannot ignore the possibility of a failure of the above considerations for *higher* values of detuning.

Moreover, the presence of the coefficient R_6 (which is not definite in sign and can dominate in the expansion at large detunings) is a mathematical basis for certain doubts whether this common conception is valid in general, and, in particular, for large values of $\delta^2(e;\sigma)$.

Thus, this evident failure of the perturbative approach strongly motivates the general nonperturbative consideration of the influence of the detuning on the second threshold.

B. The proof

So, we prove in this subsection that the values of the second threshold $r_2^{\rm thr}$ at *nonzero* values of detuning [i.e., when $e^2 = (\omega-\varepsilon)^2/\gamma_\perp^2 \neq 0$ or, which is the same for exact treatment, $\delta^2 = (\omega-\varepsilon)^2/(\kappa+\gamma_\perp)^2 \neq 0$] are *always* (i.e., at arbitrary values of κ and $\gamma_\perp$ under the bad-cavity condition) greater than the value of the second threshold $r_{2,0}^{\rm thr}$ for the resonantly tuned laser.

For simplicity of notation in this subsection we denote

$$x = \delta^2(e;\sigma). \tag{139}$$

Thus, all operations with respect to x will be the operations with respect to the parameter $\delta^2(e;\sigma)$ which is proportional to the square of the absolute detuning.

The difference between the value of the second threshold $r_2^{\rm thr}$ for the *detuned* laser and the value of the second threshold for the resonantly tuned laser acquires the following form:

$$\Delta r(x) \equiv r_2^{\rm thr} - r_{2,0}^{\rm thr} = M^{-1}\left(-\alpha + \beta x + \gamma\sqrt{a(x-x_0)^2 + \frac{\alpha^2}{\gamma^2}}\right), \tag{140}$$

where all quantities α, β, γ, and a are strictly positive under the bad-cavity condition, and the quantity x_0 is not definite in sign.

In relation (140), the denominator M is equal to

$$M = 2b\,(\sigma - b - 1)\,(1 + 3\sigma)\,.$$

It is obviously a positive quantity under the bad-cavity condition and does not depend on the detuning $x = \delta^2(e;\sigma)$.

To show explicitly the positiveness of the above-mentioned quantities we have to introduce once again the positive variable ξ instead of the parameter σ:

$$\xi = \sigma - b - 1 \;>\; 0.$$

Considering α, β, γ, and a as functions of two *positive* parameters ξ and b we can write down the following explicitly positive expressions (after symbolic simplifications):

$$\alpha = (b+\xi)\,(2+b+\xi)\,(2+2b+\xi)\,(4+3b+2\xi) \;>\; 0, \tag{141}$$

$$\beta = 16b^2 + 20b^3 + 6b^4 + 40b\xi + 66b^2\xi + 23b^3\xi + 16\xi^2 + 66b\xi^2 + 34b^2\xi^2 + 20\xi^3 + 23b\xi^3 + 6\xi^4 \;>\; 0, \tag{142}$$

$$\gamma = (b+\xi)\,(4+3b+2\xi) \;>\; 0, \tag{143}$$

$$a = (b+\xi)^2(4+2b+3\xi)^2 \;>\; 0. \tag{144}$$

We have succeeded to prove that the difference $\Delta r(x)$ is always positive using expressions (141)–(144) and the special representation of quantities through the elliptic integrals. However, this direct proof is extremely complicated and it is not worth presenting it here.

Instead, we have also found another proof which is much simpler and based on the properties of the derivative of the threshold difference $\Delta r(x)$ with respect to x.

Differentiating $\Delta r(x)$ with respect to x, we get

$$\frac{\partial(\Delta r(x))}{\partial x} = M^{-1}\,\frac{\sqrt{\beta^2 a(x-x_0)^2 + \left(\dfrac{\alpha\beta}{\gamma}\right)^2} + \gamma a(x-x_0)}{\sqrt{a(x-x_0)^2 + \dfrac{\alpha^2}{\gamma^2}}}. \tag{145}$$

From (145) and from (142)–(144), we immediately extract a very important conclusion: *the derivative $\partial(\Delta r(x))/\partial x$ is asymptotically, at $x \to \infty$, a positive quantity:*

$$\lim_{x\to\infty}\left\{\frac{\partial(\Delta r(x))}{\partial x}\right\} = M^{-1}(\beta + \gamma\sqrt{a}) > 0, \tag{146}$$

which means that asymptotically, at $x \to \infty$, the second threshold $r_2^{\rm thr}$ grows monotonically with increasing detuning $x = \delta^2$ independent of the normalized relaxation rates $\sigma = \kappa/\gamma_\perp$ and $b = \gamma_\parallel/\gamma_\perp$. The expression (146) gives also the corresponding general asymptotic form of the threshold ratio at large detuning, i. e., of the ratio of the second threshold to the first threshold.

But what happens in between, at moderate values of detuning?

To understand this, we use the following two propositions.

Proposition 7.1. At zero detuning, $(\omega - \varepsilon) = 0$ (or, in other terms, $x = \delta^2 = e^2 = 0$), the derivative (145) is strictly positive at all physical values of the parameters under the bad-cavity condition.

Proof 7.1. Substituting the expression for x_0

$$x_0 = \frac{X_1}{X_2}, \tag{147}$$

where

$$X_1 = -[(2+b+\xi)(8b+12b^2+4b^3-8\xi+10b\xi+12b^2\xi \\ -10\xi^2+5b\xi^2-3\xi^3)], \tag{148}$$

$$X_2 = (b+\xi)^2(4+2b+3\xi)^2, \tag{149}$$

into expression (145) and setting the detuning equal to zero, we get after symbolic simplifications the following explicitly positive expression:

$$\frac{\partial(\Delta r(x=0))}{\partial x} = M^{-1}\frac{2b\,(1+b+\xi)\,(4+3b+3\xi)\,(8b+4b^2+10\xi+10b\xi+5\xi^2)}{2+2b+\xi} > 0. \tag{150}$$

Thus, at this stage we have shown that the derivative of the second threshold is strictly positive at zero detuning. This means that at least at the vicinity of zero detuning the value of the second threshold is growing with increasing detuning.

Let us now consider the region of nonzero detuning, when $x > 0$.

Proposition 7.2. The derivative (145) either has no zeros at $x > 0$ or has only one zero at $x > 0$.

Proof 7.2. Since the denominator of the derivative (145) is always positive, the derivative (145) can have a zero when

$$F_1(x) = F_2(x), \tag{151}$$

where

$$F_1(x) = \sqrt{\beta^2 a(x-x_0)^2 + \left(\frac{\alpha\beta}{\gamma}\right)^2}, \tag{152}$$

$$F_2(x) = -\gamma a(x-x_0). \tag{153}$$

The function $F_2(x)$ is a straight line which goes from the fourth quadrant into the second quadrant. The function $F_1(x)$ is a curve which lies inside the upper corner created by two straight lines: $f_-(x) = -\beta\sqrt{a}(x-x_0)$ and $f_+(x) = \beta\sqrt{a}(x-x_0)$. At zero value of x the value of $F_1(x)$ is equal to $(\alpha\beta/\gamma)$ and $F_1(x)$ approaches the straight lines $f_-(x)$ and $f_+(x)$ from above when x goes from x_0 to $\pm\infty$.

Therefore, the functions $F_1(x)$ and $F_2(x)$ can intersect each other *only* one time. The necessary condition for this is obviously written as follows:

$$\beta\sqrt{a} \;<\; \gamma a. \tag{154}$$

If the condition (154) does not hold then the functions $F_1(x)$ and $F_2(x)$ have no intersections at all. Thus, here we have shown that the derivative of the second threshold either has only one zero or has no zeros at all.

Conclusion from the propositions. Since the derivative (145) is strictly positive at zero detuning (according to *Proposition 7.1*) and is strictly positive asymptotically, at $x \to \infty$ [according to (146)], it is positive everywhere for $x > 0$ excluding, probably, only one point where it might be equal to zero (according to *Proposition 7.2*). It cannot be negative at $x > 0$ because in this case the derivative (145) would have at least *two* zeros which would be in evident contradiction with *Proposition 7.2.*

Since the derivative of the second threshold is positive everywhere including at zero detuning and excluding maybe only one nonzero value of detuning, the second threshold at nonzero detuning is always greater than the second threshold for zero detuning at *all* physical values of the parameters of the problem under the bad-cavity condition.

We have strictly proven here that increasing the detuning has only a stabilizing effect on the homogeneously broadened single-mode laser. This is now a rigorous mathematical statement.

VIII. COMPACT VISUALIZATION OF THE SECOND THRESHOLD

The second threshold $r_2^{\rm thr}$ can be defined on different but isomorphic sets of variables. If one uses the original relaxation rates κ, $\gamma_\parallel$, and $\gamma_\perp$ of the single-mode homogeneously broadened laser, the threshold can be understood as a function of these three variables and of the square of the detuning $(\omega-\varepsilon)^2$. The parameter domain in which the threshold is defined at a fixed value of the detuning is the infinite area determined by the bad-cavity condition:

$$\gamma_\parallel + \gamma_\perp \;<\; \kappa$$

for positive values of these parameters.

It is clear that the infiniteness of the domain of existence of the second threshold does not allow viewing of the 3d surface of the second threshold as a whole.

The situation is not better if one uses the "hydrodynamical" parameters $\sigma = \kappa/\gamma_\perp$ and $b = \gamma_\parallel/\gamma_\perp$ at a fixed detuning because the domain of existence of the second threshold is not compact yet but is defined by another version of the bad-cavity condition:

$$b + 1 < \sigma$$

for positive values of σ and b. Hence, one cannot look at the whole surface of the second threshold because the domain of its existence is still unbounded.

In order to compactify the domain of existence of the second threshold, let us note that the parameter κ is larger than the parameters $\gamma_\parallel$ and $\gamma_\perp$ (the parameter σ is an upper bound on the sum $b+1$ if the second threshold is to exist).

Therefore, one can easily compactify the domain of the existence of the second threshold by choosing as a normalization this largest parameter κ (or the parameter σ).

In view of this, let us introduce new parameters:

$$\Gamma_\perp = \frac{\gamma_\perp}{\kappa} = \frac{1}{\sigma}, \tag{155}$$

$$\Gamma_\parallel = \frac{\gamma_\parallel}{\kappa} = \frac{b}{\sigma}. \tag{156}$$

In terms of these parameters the domain of the existence of the second threshold is the interior of the triangle defined by the reformulated bad-cavity condition:

$$\Gamma_\perp + \Gamma_\parallel < 1, \tag{157}$$

and by the conditions of the positiveness of these parameters:

$$\Gamma_\perp > 0, \qquad \Gamma_\parallel > 0. \tag{158}$$

We note a further physical limit for the physics of the two-level optical systems given by $b \leq 2$ which limits $\Gamma_\parallel \leq 2\Gamma_\perp$.

The exact expression for the second threshold written in terms of $\Gamma_\perp$, $\Gamma_\parallel$ and δ^2 can be easily obtained by the direct symbolic substitution on a computer of the relations

$$\sigma = \frac{1}{\Gamma_\perp}, \tag{159}$$

$$b = \frac{\Gamma_\parallel}{\Gamma_\perp}, \tag{160}$$

into the exact expression derived in Sec. III B 5. However, it is not worth repeating here the form of the exact expression for the second threshold even if it is written in terms of other variables. The resulting expression is too cumbersome and does not carry new information.

The only but very important advantage of the choice of the parameters $\Gamma_\perp$ and $\Gamma_\parallel$ is that they compactify the domain in which the second threshold is defined.

We have plotted the 3d surface of the difference $\Delta r(\delta^2)$ between the second threshold r_2^{thr} for the *detuned* laser and the second threshold $r_{2,0}^{\text{thr}}$ for the *resonantly tuned* laser as a function of two variables $\Gamma_\perp$ and $\Gamma_\parallel$ for the following values of detuning: $|\omega - \varepsilon|/(\kappa + \gamma_\perp) = |\delta| = |e|/(\sigma + 1) =$ 0.1, 0.5, 1.0, 2.0, 3.0.

The surface is always above the $\Gamma_\perp\Gamma_\parallel$ plane according to the main statement about the influence of the detuning on the second threshold which we proved in the previous section.

In all figures the abscissa corresponds to $\Gamma_\perp$ and the ordinate corresponds to $\Gamma_\parallel$. The surface is defined only below the straight line where the sum of the abscissa and of the ordinate is equal to one.

At small values of detuning $|\delta| = 0.01$ one does not see (Figs. 2 and 3) any essential qualitative changes of the difference $\Delta r(\delta^2)$. The lowest values of the difference lie in the region which is close to small values of $\Gamma_\perp = \gamma_\perp/\kappa$ and to large (i.e., close to one) values of $\Gamma_\parallel = \gamma_\parallel/\kappa$.

According to the contour plot on Fig. 3, the maximal values of the second threshold are close to the straight lines $\Gamma_\perp = 0$ and $\Gamma_\perp + \Gamma_\parallel = 1$ at large (close to one) values of $\Gamma_\parallel$.

The qualitative changes become evident with increasing detuning. At the large values of detuning $|\delta| = 1.0$, and $|\delta| = 3.0$ (Figs. 4 and 5 and Figs. 6 and 7) the minimal values of the difference $\Delta r(\delta^2)$ move along the abscissa to the value of $\Gamma_\perp = 1.0$, being at the same time near the zero value of $\Gamma_\parallel = \gamma_\parallel/\kappa$.

Thus, this approximate information based on 3D plots shows that at high detunings the minimal values of the second threshold are located at moderate relative values

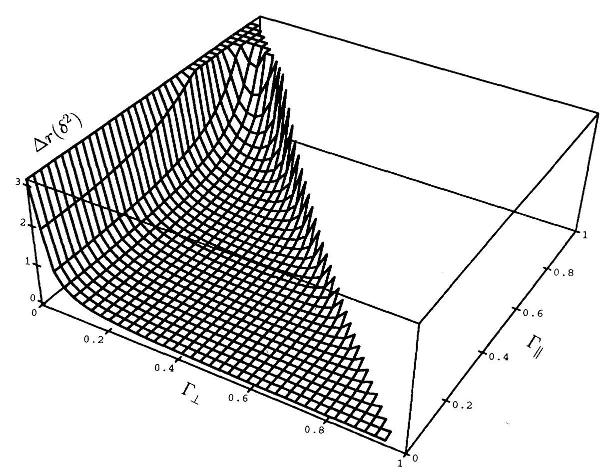

FIG. 2. The value of squared detuning δ^2 is equal to 0.01. The difference $\Delta r(\delta^2)$ between the second threshold r_2^{thr} for the *detuned* laser and the second threshold $r_{2,0}^{\text{thr}}$ for the *resonantly tuned* laser is shown as a function of two variables $\Gamma_\perp$ (the abscissa) and $\Gamma_\parallel$ (the ordinate). The surface of $\Delta r(\delta^2) = r_2^{\text{thr}} - r_{2,0}^{\text{thr}}$ is determined above the triangle defined by positive values of $\Gamma_\perp$ and $\Gamma_\parallel$ taken under the bad-cavity condition $\Gamma_\perp + \Gamma_\parallel < 1$.

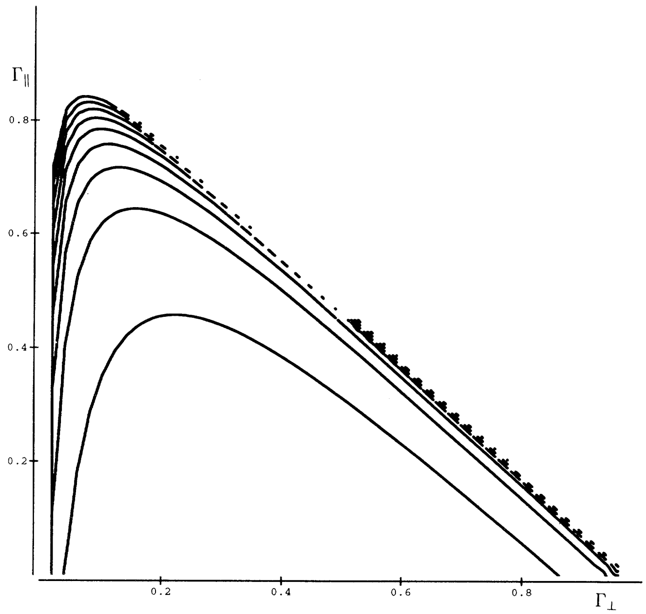

FIG. 3. The value of squared detuning δ^2 is equal to 0.01. The contour plot for the second threshold r_2^{thr} for the *detuned* laser is shown as a function of two variables $\Gamma_{\perp}$ (the abscissa) and $\Gamma_{\|}$ (the ordinate). The contour plot is determined above the triangle defined by positive values of $\Gamma_{\perp}$ and $\Gamma_{\|}$ taken under the bad-cavity condition $\Gamma_{\perp} + \Gamma_{\|} < 1$.

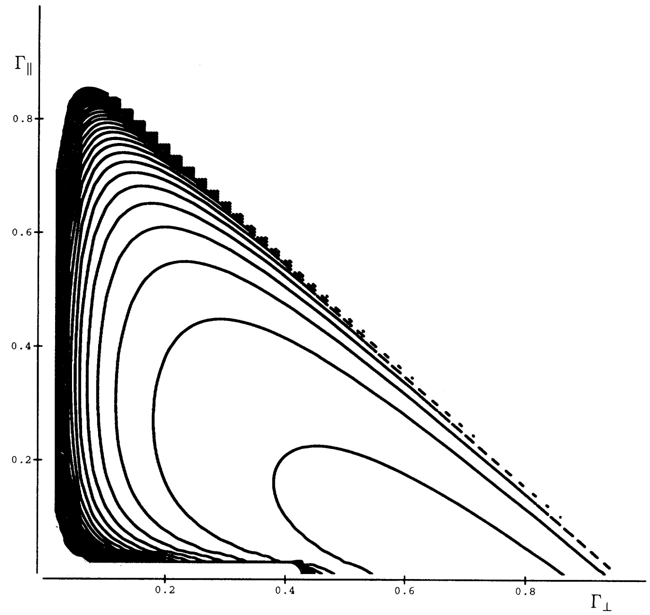

FIG. 5. The same as Fig. 3 but for $\delta^2 = 1.0$.

of κ, high values of $\gamma_{\perp}$, and near-zero values $\gamma_{\|}$.

The exact treatment of the minima of the second threshold will be done in the next section. However, Figs. 6 and 7 are more typical for masers rather than for lasers according to our discussion in Sec. VI C.

We also see that the second threshold is quite smooth and does not possess any local extrema. It is growing extremely fast near the boundaries of the domain of existence and it is relatively small at the middle of this domain.

IX. THE EFFECT OF DETUNING ON THE RATIO OF THE SECOND THRESHOLD TO THE FIRST THRESHOLD

While there is no reason, in general, to expect that higher pumping rates are easier to reach out of resonance than in resonance, it is at least conceptually interesting to see how these two thresholds scale in order to complete our understanding of the dependence of the two thresholds on the detuning parameter.

A. Minima of the second threshold at various detunings

In Sec. IV, we have shown that the absolute minimum of the second threshold r_2^{thr} occurs at zero detuning, $\delta^2 =$

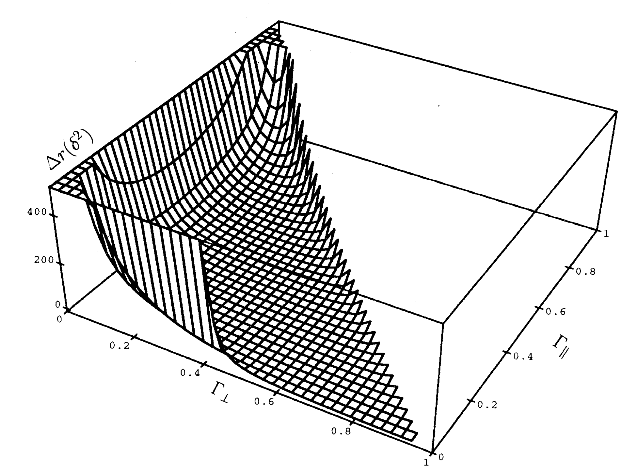

FIG. 4. The same as Fig. 2 but for $\delta^2 = 1.0$.

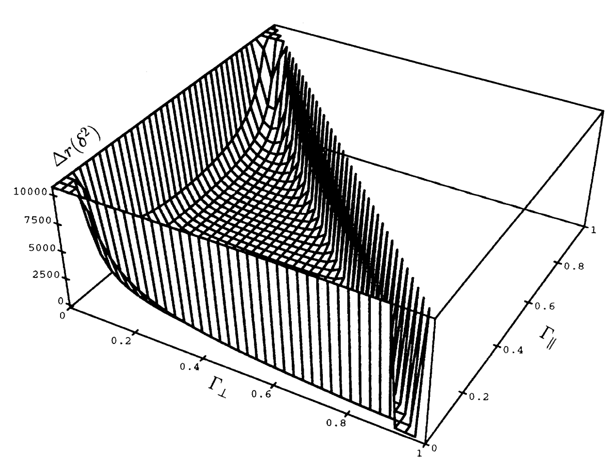

FIG. 6. The same as Fig. 2 but for $\delta^2 = 9.0$.

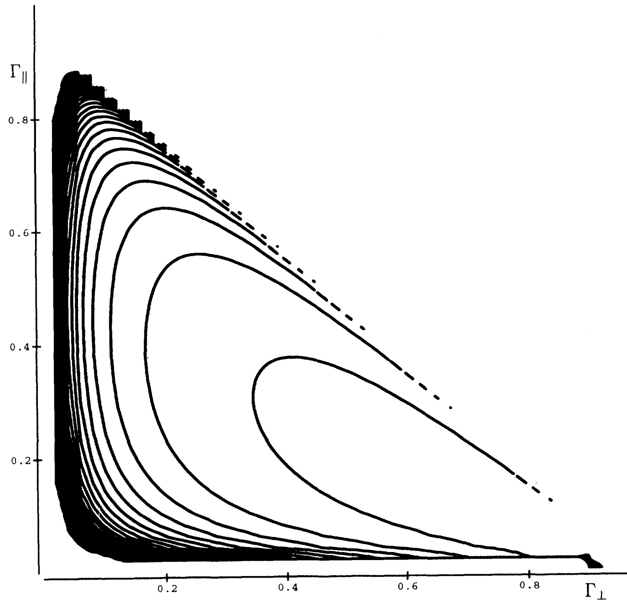

FIG. 7. The same as Fig. 3 but for $\delta^2 = 9.0$.

0, at $b = 0$ (or $\gamma_{\|} = \Gamma_{\|} = 0$) and at $\sigma = 3$, and it is equal exactly to 9. Since the value of the first threshold, $r_1^{\rm thr}$, is equal identically to 1 when the detuning is zero, the ratio $r_2^{\rm thr}/r_1^{\rm thr}$ is equal to 9 at this set of parameter values.

It is natural to expect that at least at small values of detuning δ the minimum of the second threshold $r^{\rm thr}$ is still along the straight line $\Gamma_{\|} = 0$. Moreover, we will show that a stronger statement holds: *all minima of the second threshold $r_2^{\rm thr}$ lie on the straight line $\Gamma_{\|} = \gamma_{\|} = 0$ independent of the detuning.*

To show this, we need both to know the shape (the profile) of the cross section between the second threshold $r_2^{\rm thr}$ and the plane defined by the equality $\gamma_{\|} = \Gamma_{\|} = 0$, and the sign of the derivative of the second threshold $r_2^{\rm thr}$ with respect to the variable $\Gamma_{\|}$ at the border of the domain of existence of the second threshold $r_2^{\rm thr}(\Gamma_\perp, \Gamma_{\|})$ defined by $\Gamma_{\|} = 0$.

It is clear that if the derivative of the second threshold $r_2^{\rm thr}(\Gamma_\perp, \Gamma_{\|})$ with respect to $\Gamma_{\|}$ taken at the minimum of the profile of the second threshold at $\Gamma_{\|} = 0$ is positive, then this minimum belonging to the profile of $r_2^{\rm thr}$ at $\Gamma_{\|} = 0$ is the true one. This follows, provided, of course, that there are no local minima far away from the border $\Gamma_{\|} = 0$. Our verification of the interior of the second threshold at chosen values of δ^2 has shown no such local remote extrema, i.e., has shown no zero absolute values of the gradient inside the domain of existence of the second threshold $r_2^{\rm thr}(\Gamma_\perp, \Gamma_{\|})$.

The profile of the second threshold $r_2^{\rm thr}$ in the plane $\Gamma_{\|} = 0$ or, which is the same, in the plane $\gamma_{\|} = 0$, is given by the following expression:

$$S_{\rm pr}(\Gamma_\perp, \delta^2) = \lim_{\Gamma_{\|}\to 0} r_2^{\rm thr} = \frac{N_{\rm pr}}{D_{\rm pr}}, \qquad (161)$$

where

$$N_{\rm pr} = -1 - 5\Gamma_\perp - 7\Gamma_\perp^2 - 3\Gamma_\perp^3 - 2\delta^2 + 2\Gamma_\perp\delta^2 - 2\Gamma_\perp^2\delta^2 + 2\Gamma_\perp^3\delta^2 - \delta^4 + 7\Gamma_\perp\delta^4 - 11\Gamma_\perp^2\delta^4 + 5\Gamma_\perp^3\delta^4, \qquad (162)$$

and

$$D_{\rm pr} = (-1 + \Gamma_\perp)\,\Gamma_\perp(1 + 2\Gamma_\perp + \Gamma_\perp^2 - 3\delta^2 + 2\Gamma_\perp\delta^2 + \Gamma_\perp^2\delta^2). \qquad (163)$$

At the same time, for the derivative of the second threshold $r_2^{\rm thr}$ with respect to $\Gamma_{\|}$ at the border $\Gamma_{\|} = 0$ one can write

$$\left\{\frac{\partial r_2^{\rm thr}}{\partial \Gamma_{\|}}\right\}_{\Gamma_{\|}=0} = \mathcal{G}(\Gamma_\perp, \delta)\mathcal{P}_4(\Gamma_\perp, \delta), \qquad (164)$$

where $\mathcal{G}(\Gamma_\perp, \delta^2)$ is a function which is positive-definite in sign and analytical under the bad-cavity condition, but $\mathcal{P}_4(\Gamma_\perp, \delta^2)$ is the following polynomial of the fourth order in $\Gamma_\perp$:

$$\mathcal{P}_4(\Gamma_\perp, \delta^2) = 1 + 4\Gamma_\perp + 6\Gamma_\perp^2 + 4\Gamma_\perp^3 + \Gamma_\perp^4 - 10\delta^2 - 8\Gamma_\perp\delta^2 + 12\Gamma_\perp^2\delta^2 + 8\Gamma_\perp^3\delta^2 - 2\Gamma_\perp^4\delta^2 + 5\delta^4 - 12\Gamma_\perp\delta^4 + 6\Gamma_\perp^2\delta^4 + 4\Gamma_\perp^3\delta^4 - 3\Gamma_\perp^4\delta^4. \qquad (165)$$

Thus, the sign and zeros of the whole derivative (164) depend only on the polynomial $\mathcal{P}_4(\Gamma_\perp, \delta^2)$ of the fourth order in $\Gamma_\perp$.

Zeros of the polynomial $\mathcal{P}_4(\Gamma_\perp, \delta^2)$ define the minima of the profile $S_{\rm pr}(\Gamma_\perp, \delta^2)$ which will be the minima of the whole second threshold $r_{\rm thr}^2$ if the derivative (164) is positive at this point.

Depending on the value of detuning δ, the derivative (164) can be both negative and positive as a function of $\Gamma_\perp$ on the physical interval of the values of $\Gamma_\perp$ between $-1 + 2\delta/\sqrt{1+\delta^2}$ and 1.0. We denote the critical value of $\Gamma_\perp$ where the derivative (164) becomes positive with growth of $\Gamma_\perp$ as $\{\Gamma_\perp\}_{\rm cr}$.

The criterion whether the minimum $\{\Gamma_\perp\}_{\rm min}$ of the second threshold is the true one is therefore the following simple inequality:

$$\{\Gamma_\perp\}_{\rm min} > \{\Gamma_\perp\}_{\rm cr}. \qquad (166)$$

As shown in Table III, at small values of the detuning δ^2 the point $\{\Gamma_\perp\}_{\rm cr}$ does not exist at all, and the derivative (164) is strictly positive according to the sign of the polynomial $\mathcal{P}_4(\Gamma_\perp, \delta^2)$. It is obvious that at these values of detuning the minima of the profile $S_{\rm pr}(\Gamma_\perp, \delta^2)$ are the true minima of the whole second threshold $r_2^{\rm thr}$; it is not necessary to use the inequality (166).

The situation changes at higher values of detuning because there appears a zero of the derivative (164) lying on the physical interval $\left(-1 + 2\delta/\sqrt{1+\delta^2}; 1.0\right)$. However, using the values of $\{\Gamma_\perp\}_{\rm cr}$, we see that the inequality (166) holds (albeit to show this, one needs very high precision, for instance, at $\delta^2 = 100.0^2$ the value of $\{\Gamma_\perp\}_{\rm min}$ differs from the value of $\{\Gamma_\perp\}_{\rm cr}$ only in the ninth digit, see the last two numbers in the last line of Table III).

TABLE III. The minimal values of the second threshold $r_2^{\rm thr}$ and of the ratio $r_2^{\rm thr}/r_1^{\rm thr}$ for various detunings.

δ^2	$\{r_2^{\rm thr}\}_{\rm min}$	$\{r_2^{\rm thr}/r_1^{\rm thr}\}_{\rm min}$	$\{\Gamma_\perp\}_{\rm min}$	$\{\Gamma_\perp\}_{\rm cr}$
$0.0001 = 0.01^2$	9.0015000549	9.00059999499	0.3333889	
$0.01 = 0.1^2$	9.1505437176	9.05994427491	0.338905641	
$0.25 = 0.5^2$	12.997288299	10.3978306392	0.470310748	0.2753816
$1.0 = 1.0^2$	25.756497690	12.8782488454	0.701373622	0.66817863
$4.0 = 2.0^2$	76.898272096	15.3796544192	0.893177820	0.8899717
$9.0 = 3.0^2$	161.94945584	16.1949455844	0.94836852	0.94766522
$16.0 = 4.0^2$	280.97038731	16.5276698418	0.970033754	0.96980225
$25.0 = 5.0^2$	433.98068092	16.6915646508	0.980534257	0.9804375
$36.0 = 6.0^2$	620.98644226	16.7834173579	0.986371027	0.98632394
$49.0 = 7.0^2$	841.98997559	16.8397995113	0.989936964	0.98991138
$64.0 = 8.0^2$	1096.9925893	16.8760451007	0.992270466	0.99225541
$81.0 = 9.0^2$	1385.9938937	16.9023645520	0.9938790803	0.99386966
$100.0 = 10.0^2$	1708.9950434	16.9207430044	0.9950341231	0.99502793
$225.0 = 15.0^2$	3833.9977863	16.9645919753	0.9977845457	0.997783316
$400.0 = 20.0^2$	6808.9987527	16.9800467648	0.99875214447	0.998751754
$2500.0 = 50.0^2$	42509.010012	16.9968012098	0.9998000549	0.999800045
$10000.0 = 100.0^2$	170009.01028	16.9991998953	0.99995000343	0.9999500028
100000.0	1700009.64199	16.9999264206	0.99999500003	0.999994997

Thus, at arbitrary detuning, the second threshold $r_2^{\rm thr}$ has minima lying in the plane defined by the relation $\gamma_\parallel = \Gamma_\parallel = 0$.

B. Saturation of the minima of the threshold ratio

Since the first threshold $r_1^{\rm thr}$ does not depend on the parameters $\Gamma_\perp$ and $\Gamma_\parallel$ at all, the ratio $r_2^{\rm thr}/r_1^{\rm thr}$ has the same minima as the second threshold $r_2^{\rm thr}$ itself but other values of them.

We know that the value of the second threshold is bounded from below by 9.0. In view of this, it is very interesting to note that the minima of the ratio $r_2^{\rm thr}/r_1^{\rm thr}$ have an evident saturation at the value of 17.0 with δ^2 going to infinity (Fig. 8):

$$\lim_{\delta^2\to\infty}\left\{\frac{S_{\rm pr}(\Gamma_\perp,\delta^2)}{(1+\delta^2)}\right\}_{\rm min} = 17.0. \qquad (167)$$

The verification of this, based on our analytical exact results, was done for higher values of the detuning δ than are shown in Table III and Fig. 8, and we have restricted the presentation of the results on the saturation effect to the most probable physical values.

The following warning is required here. Taking a straightforward formal limit of the expression

$$\lim_{\delta^2\to\infty}\left\{\frac{S_{\rm pr}(\Gamma_\perp,\delta^2)}{(1+\delta^2)}\right\}_{\rm min} = 1.0, \qquad (168)$$

where $S_{\rm pr}(\Gamma_\perp,\delta^2)$ is defined according to (161), MATHEMATICA provides us with the limit value equal to 1.0 but not to 17.0 as it should.

It is not so difficult to understand the reason by just looking at Figs. 9–11. We see that in (168) we got *the maximum of the unphysical branch for* $S_{\rm pr}(\Gamma_\perp,\delta^2)$ while the minimum of the physical branch of $S_{\rm pr}(\Gamma_\perp,\delta^2)$ has just collapsed into a single point in the limit of infinite δ^2 and becomes unattainable for the symbolic treatment by the computer program.

To show the saturation effect analytically, one should solve the equation

$$-1+6\Gamma_\perp^2+8\Gamma_\perp^3+3\Gamma_\perp^4+2\delta^2-20\Gamma_\perp\delta^2+20\Gamma_\perp^3\delta^2$$
$$-2\Gamma_\perp^4\delta^2+3\delta^4-4\Gamma_\perp\delta^4-6\Gamma_\perp^2\delta^4+12\Gamma_\perp^3\delta^4$$
$$-5\Gamma_\perp^4\delta^4=0, \qquad (169)$$

and then substitute all four roots obtained for $\Gamma_\perp$ into the expression for $S_{\rm pr}(\Gamma_\perp,\delta^2)$.

Dividing the results of substitutions by $(1+\delta^2)$, one should try to get the asymptotic behavior of the resulting expressions in the limit of $\delta\to\infty$. The result of one of

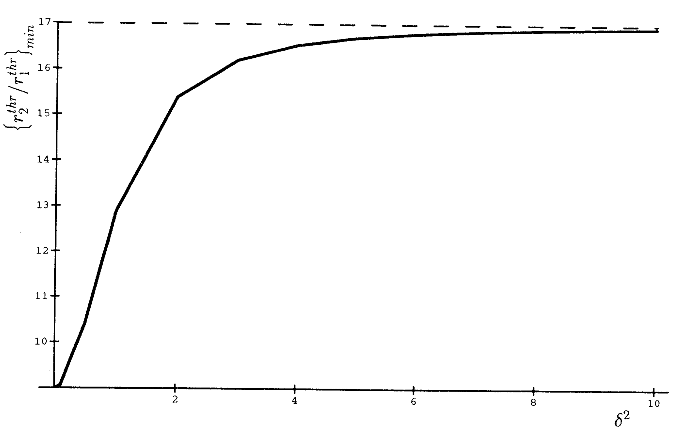

FIG. 8. The effect of the saturation of the minima of the threshold ratio to the value of 17.0. The minima of the threshold ratio $\{r_2^{\rm thr}/r_1^{\rm thr}\}_{\rm min}$ are shown vs detuning δ.

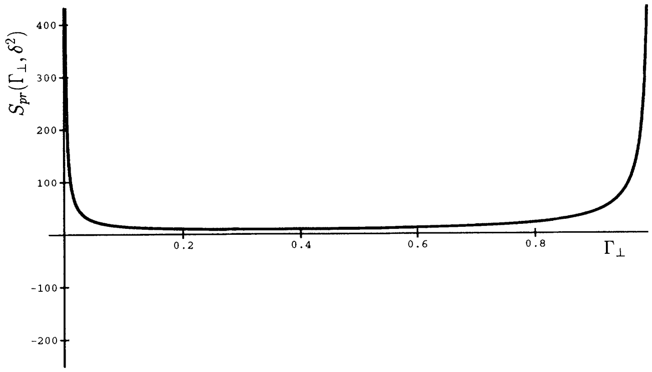

FIG. 9. The profile $S_{\rm pr}(\Gamma_\perp,\delta^2)$ of the second threshold $r^{\rm thr}$ in the plane $\Gamma_\parallel=\gamma_\parallel=0$ vs the normalized polarization relaxation rate $\Gamma_\perp$. The cross section $S_{\rm pr}(\Gamma_\perp,\delta^2)$ of the second threshold $r^{\rm thr}$ contains the minima of the second threshold. The value of the squared detuning δ^2 is equal to 0.01.

these limits (which is the only physical case) should be, according to the results obtained, equal to 17.0.

Unfortunately, we failed to show this completely analytically, because of limitations of the computers at our disposal. We hope that this will be shown symbolically later, on a more effective combination of computer and software.

In summary, we showed in this section that *independent of detuning* the second threshold of the single-mode unidirectional homogeneously broadened laser has its minima when the ratio of relaxation rates $\gamma_\parallel/\kappa$ is equal to zero or, in other words, when the population relaxation rate $\gamma_\parallel$ is really negligible in comparison with the cavity width κ under the bad-cavity condition.

We also showed that the minimum of the ratio of two thresholds, of the first (or lasing) threshold and of the second threshold (threshold of exponential instability), is limited (as a function of detuning) both from below and from above on the interval between 9.0 (at zero detuning) and 17.0 (at large detunings) (Fig. 8).

Such minima may be, in principle, attained in a proper experimental setup by means of reducing the ratio $\Gamma_\parallel=\gamma_\parallel/\kappa$ to zero and adjusting (at given detuning) the values of the ratio $\Gamma_\perp=\gamma_\perp/\kappa$ to the value given by the equation

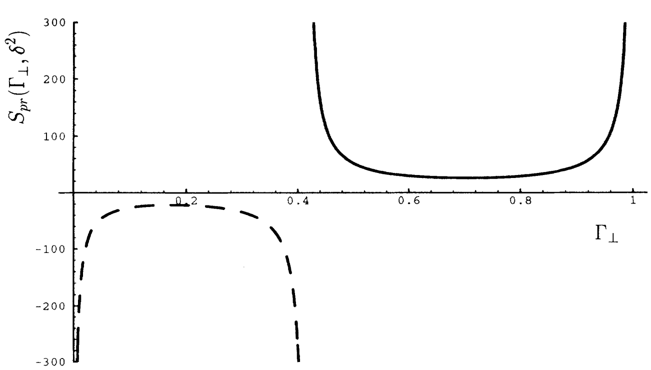

FIG. 10. The same as Fig. 9 but for $\delta^2=1.0$. The physical branch of $S_{\rm pr}(\Gamma_\perp,\delta^2)$ is plotted as a solid line while the unphysical branch is a dashed line.

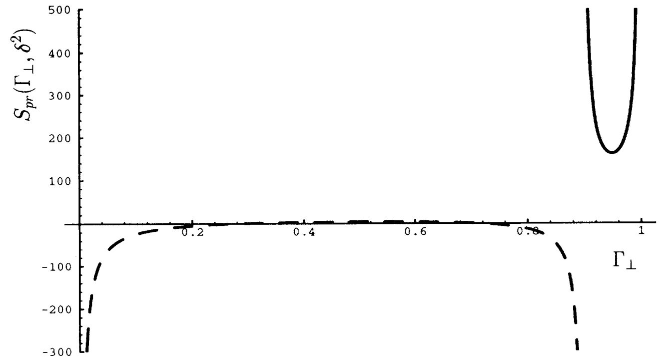

FIG. 11. The same as Fig. 10 but for $\delta^2=9.0$.

$$\mathcal{P}_4(\Gamma_\perp,\delta^2)=0. \tag{170}$$

Another interesting aspect of our analysis is that at high detunings the domain of the existence of the second threshold at $\gamma_\parallel=\Gamma_\parallel=0$ becomes narrower and narrower, converting into an almost zero-width region near $\Gamma_\perp=1.0$ (Figs. 9–11). While that domain has the natural right limit at $\{\Gamma_\perp\}_r=1$ due to the bad-cavity condition, the left limit is given by the expression

$$\{\Gamma_\perp\}_l=-1+\frac{2\delta}{\sqrt{1+\delta^2}}. \tag{171}$$

It is evident that the left limit $\{\Gamma_\perp\}_l$ becomes equal to the right one $\{\Gamma_\perp\}_r$ in the limit of infinite detuning $\delta^2\to\infty$. Thus, the domain $(\{\Gamma_\perp\}_l;\{\Gamma_\perp\}_r)$, which contains the true absolute minima $\{\Gamma_\perp\}_{\rm min}$ of the second threshold, becomes arbitrarily small at large detunings (Figs. 9–11).

X. SUMMARY OF THE RESULTS

We have treated fully analytically, at arbitrary physical values of parameters, the threshold functions for the *detuned* single-mode homogeneously broadened unidirectional laser, one of the most tutorial but still very physical models in laser physics.

As main results, we have obtained the following.

(i) It has been shown, by simple example, based on general analytical expressions, that the neglect of another possible branch for the second threshold, based on the opinion that this branch gives negative values for the threshold, was not consistent because that branch also can provide one with positive second threshold values. Nevertheless, the direct comparison of that branch with the first threshold, for all physical values of the parameters, shows that this branch is unphysical because it gives values which are less than the values of the first threshold.

(ii) We have proven that the physical branch of values for the second laser threshold is always greater than the values for the first threshold, i.e., there are no other physical restrictions on the parameter domain of the threshold functions except for the well-known bad-cavity condition.

(iii) It has been proven analytically, over the whole physical region of parameters, that the second threshold only increases with increasing of detuning. This is now a rigorous mathematical statement for the dynamical model under consideration.

(iv) It has been proven that the second threshold has its minima at various values of detuning when the ratio of the population relaxation rate to the cavity relaxation rate goes to zero. The second threshold has no local minima (at nonzero values of that ratio).

(v) It has been shown that the ratio of the second threshold to the first (lasing) threshold has minima which are bounded not only from below (by 9.0) but also from above, by 17.0. This is a completely unexpected result—to our knowledge, there was no hint before that those ratios when minimized would be bounded from above.

(vi) We have shown that the threshold ratio is a monotonically increasing function of detuning and, therefore, there is the effect of the saturation of the minima of the threshold ratio to the value of 17.0.

(vii) Based on the general expressions, we have obtained a number of limiting and asymptotic expressions for the second threshold and the initial pulsation frequency. This has allowed us not only to improve earlier approximate results which were obtained under more extreme approximations than ours (and which were also partially inconsistent) but also to separate in our discussions the laser asymptotes from the maser asymptotes. The latter allows us to distinguish parameter regions where the second threshold may be accessible for lasers and for masers.

(viii) It has been shown that the commonly used dimensionless normalization of the absolute value of detuning can lead to ambiguous situations mixing the effect of detuning with the extent of the "badness" of the cavity. In such situations, in previous works, a few inconsistent results have been obtained; we have shown how to improve them.

(ix) We have rigorously shown that the order in which double parameter limits are taken ("bad-cavity" limit and large detuning limit) is crucial for obtaining the correct asymptotic results for threshold functions.

(x) We have managed to compactify the domain of the existence of the second threshold by proper normalization, and to satisfy the common desire to look at the surface of the second threshold, by plotting it over a finite domain of the normalized relaxation rates. Quite a number of subsidiary plots give a complete representation of the topology and properties of the surface of the second threshold.

(xi) Trying to make clear the relations between several representations for the dynamical systems for a single-mode homogeneously broadened laser (which are still scattered in laser literature), we have given a complete and self-contained derivation of the laser versions of the complex and real Lorenz models from the semiclassical equations for such a laser. A clear hierarchy for those standard laser models, based on our derivation, has been presented.

ACKNOWLEDGMENTS

The authors are grateful to B. B. Govorkov, Jr. for the discussions of the contents of Sec. II. The tireless help of M. Bordin (ICTP, Trieste), A. Shiekh (ICTP, Trieste), and Susan Coghlan (CNLS, Los Alamos) in running MATHEMATICA is acknowledged with pleasure. The computer centers of the ICTP, Trieste, and of the CNLS, Los Alamos, are thanked for granting the opportunity to run large symbolic calculations on their computers. One of the authors (A.A.B.) expresses his sincere thanks to Professor A. Salam for the hospitality in the International Centre for Theoretical Physics (ICTP), Trieste, as well as to Professor G. Denardo for his permanent support of quantum-optics studies in the ICTP.

APPENDIX A: COEFFICIENTS OF THE BIQUADRATIC EQUATION AND THE SIGN OF ITS DISCRIMINANT

The coefficients of Eq. (89) have the following explicit forms:

$$k = b^2(1+3\sigma)(b+1-\sigma), \tag{A1}$$

$$\begin{aligned} p = b(&2+3b+b^2+2\delta^2-b\delta^2-b^2\delta^2+4\sigma+9b\sigma+2b^2\sigma+4\delta^2\sigma-7b\delta^2\sigma-8b^2\delta^2\sigma+17b\sigma^2+5b^2\sigma^2-8\delta^2\sigma^2\\ &+9b\delta^2\sigma^2+b^2\delta^2\sigma^2-4\sigma^3+3b\sigma^3-4\delta^2\sigma^3-b\delta^2\sigma^3-2\sigma^4+6\delta^2\sigma^4), \end{aligned} \tag{A2}$$

$$\begin{aligned} q = b\sigma(&6+5b+b^2-4\delta^2-6b\delta^2-2b^2\delta^2-10\delta^4+5b\delta^4+5b^2\delta^4+20\sigma+10b\sigma+b^2\sigma-28b\delta^2\sigma-8b^2\delta^2\sigma\\ &+12\delta^4\sigma-6b\delta^4\sigma-b^2\delta^4\sigma+24\sigma^2+13b\sigma^2+2b^2\sigma^2-2b\delta^2\sigma^2+2b^2\delta^2\sigma^2\\ &+8\delta^4\sigma^2+b\delta^4\sigma^2+12\sigma^3+4b\sigma^3+4b\delta^2\sigma^3-12\delta^4\sigma^3+2\sigma^4+4\delta^2\sigma^4+2\delta^4\sigma^4). \end{aligned} \tag{A3}$$

Since we use the bad-cavity condition (92), we consider all expressions in terms of parameters σ, b, and e, i.e., we replace the parameter $\delta(e;\sigma)$ by its expression (32) through σ and e.

After straightforward calculation the discriminant $\mathcal{D}$ of Eq. (89) can be presented in the form of a biquadratic function of the normalized detuning e:

$$\mathcal{D} = p^2 - 4kq = Ae^4 + Be^2 + C, \tag{A4}$$

where the coefficients A, B, and C have the following, apart from the common factor of $1/(\sigma+1)^4$, form:

$$A = 1 - 2b + b^2 + 4\sigma - 2b\sigma - 2b^2\sigma - 2\sigma^2 + 10b\sigma^2 + b^2\sigma^2 - 12\sigma^3 - 6b\sigma^3 + 9\sigma^4, \tag{A5}$$

$$B = 2 - 2b^2 + 12\sigma + 4b\sigma - 20b^2\sigma + 22\sigma^2 + 40b\sigma^2 - 48b^2\sigma^2 + 8\sigma^3 + 96b\sigma^3 - 44b^2\sigma^3 - 18\sigma^4 + 88b\sigma^4 - 14b^2\sigma^4 - 20\sigma^5 + 28b\sigma^5 - 6\sigma^6, \tag{A6}$$

$$C = 1 + 2b + b^2 + 8\sigma + 14b\sigma + 6b^2\sigma + 28\sigma^2 + 42b\sigma^2 + 15b^2\sigma^2 + 56\sigma^3 + 70b\sigma^3 + 20b^2\sigma^3 + 70\sigma^4 + 70b\sigma^4 + 15b^2\sigma^4 + 56\sigma^5 + 42b\sigma^5 + 6b^2\sigma^5 + 28\sigma^6 + 14b\sigma^6 + b^2\sigma^6 + 8\sigma^7 + 2b\sigma^7 + \sigma^8. \tag{A7}$$

To use explicitly the bad-cavity condition (92) we replace everywhere the parameter σ by the parameter ξ according to (135). The latter parameter [48,49] is always positive under the bad-cavity condition:

$$\xi = \sigma - b - 1 \ > \ 0. \tag{A8}$$

The coefficient A turns out to be explicitly positive in terms of ξ and b:

$$\frac{A}{(\sigma+1)^4} = 16b^2 + 16b^3 + 4b^4 + 32b\xi + 56b^2\xi + 20b^3\xi + 16\xi^2 + 64b\xi^2 + 37b^2\xi^2 + 24\xi^3 + 30b\xi^3 + 9\xi^4. \tag{A9}$$

After this one can rewrite the expression for the discriminant $\mathcal{D}$ as a sum of two terms:

$$\frac{\mathcal{D}}{A} = (e^2 + K_1)^2 + K_2 \ > \ 0, \tag{A10}$$

where the quantity K_2 is now *explicitly positive* in virtue of (A9):

$$\frac{K_2}{4A^2} = 64b\xi\,(2+\xi)\,(1+b+\xi)\,(2+b+\xi)^6\,(2+b+2\xi)\,(4+3b+3\xi)\,. \tag{A11}$$

Thus, we have shown that the discriminant of Eq. (89) is always positive under the bad-cavity condition.

APPENDIX B: INITIAL PULSATION FREQUENCY

The exact explicit analytical expression for the initial pulsation frequency Λ_0 at the second threshold $r_2^{\rm thr}$ has the following form:

$$\Lambda_0^2 = \frac{\mathcal{L}_1}{\mathcal{L}_2}, \tag{B1}$$

where

$$\begin{aligned}\mathcal{L}_1 = &-1 - b - \sigma - 4b\sigma + \sigma^2 - 3b\sigma^2 + \sigma^3 + \delta^2(-1 + b - \sigma - 2b\sigma + 5\sigma^2 + b\sigma^2 - 3\sigma^3)\\ &+(1-\sigma)(1 + 2b + b^2 + 2\delta^2 - 2b^2\delta^2 + \delta^4 - 2b\delta^4 + b^2\delta^4 + 4\sigma + 6b\sigma + 2b^2\sigma + 8\delta^2\sigma + 4b\delta^2\sigma\\ &-16b^2\delta^2\sigma + 4\delta^4\sigma - 2b\delta^4\sigma - 2b^2\delta^4\sigma + 6\sigma^2 + 6b\sigma^2 + b^2\sigma^2 + 4\delta^2\sigma^2 + 32b\delta^2\sigma^2 - 14b^2\delta^2\sigma^2 - 2\delta^4\sigma^2\\ &+10b\delta^4\sigma^2 + b^2\delta^4\sigma^2 + 4\sigma^3 + 2b\sigma^3 - 8\delta^2\sigma^3 + 28b\delta^2\sigma^3 - 12\delta^4\sigma^3 - 6b\delta^4\sigma^3 + \sigma^4 - 6\delta^2\sigma^4 + 9\delta^4\sigma^4)^{1/2}\end{aligned} \tag{B2}$$

and

$$\mathcal{L}_2 = 2\,(1 + b - \sigma)\,. \tag{B3}$$

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