Interferometers and minimum-uncertainty states

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Minimum-uncertainty states in variables which are quadratic in mode creation and annihilation operators can be used to increase the accuracy of interferometric measurements. For a general interferometer we show how to find the relevant uncertainty relation. We consider two specific examples, a Mach-Zehnder interferometer and a Mach-Zehnder interferometer in which the first beam splitter has been replaced by a four-wave mixer. For both devices su(2) squeezed minimum-uncertainty states can be used to achieve phase-measurement accuracies of $1/N$, where N is the total photon number at the input. We also describe a method of producing approximate versions of these states.

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I. INTRODUCTION

Squeezed states have been a topic of major interest in quantum optics for the past ten years [1]. Much work, both theoretical and experimental, has been done on their properties and on methods of producing them. One of the principal reasons for the interest in squeezed states is their utility in increasing the accuracy of interferometric measurements [2].

An interferometer typically has two input ports and two output ports. The quantity to be measured, usually a phase shift, is determined by measuring the difference in the numbers of photons emerging from the two output ports. The minimum phase shift which one can measure, the accuracy of the interferometer, is determined by the fluctuations in the input light. If a coherent state is sent into one of the input ports and the vacuum into the other, then the accuracy is $1/\sqrt{N}$, where N is the mean number of photons in the input state. If a standard squeezed state with squeezing parameter $r>0$ is sent into the second port instead of the vacuum, then the accuracy becomes $e^{-r}/\sqrt{N}$. It is possible to do better.

This was shown in a general and elegant analysis of interferometers by Yurke, McCall, and Klauder [3]. They showed that a Mach-Zehnder interferometer can be analyzed using the group SU(2) and that other interferometers, in which the beam splitters are replaced by four-wave mixers, can be analyzed using SU(1,1). These observations have been employed by subsequent authors to examine how different input states are transformed by beam splitters and four-wave mixers [4,5]. Their group-theoretical analysis allowed Yurke, McCall, and Klauder to find an input state which allows one to measure a phase shift of order $1/N$.

In a separate line of work the idea of squeezing was generalized to operators more complicated than the quadrature components of a field mode. In particular, operators quadratic in the mode creation and annihilation operators have been considered [6–8]. For example, one can find squeezed states for the operators which constitute the Schwinger representation of the angular-momentum operators [9]. Because these operators form a representation of the su(2) Lie algebra, the resulting states are called su(2) squeezed states.

The angular-momentum operators satisfy an uncertainty relation which is an inequality. States which obey this relation as an equality will be referred to as minimum-uncertainty states. A particular subset of su(2) squeezed states are also minimum-uncertainty states. These states were first derived by Aragone *et al.*, who called them "intelligent states" [10,11].

This paper will show that these states are useful for interferometry and can lead to phase-measurement accuracies of order $1/N$. We show this for both a standard Mach-Zehnder interferometer and a modified Mach-Zehnder interferometer in which the first beam splitter is replaced by a four-wave mixer. The modified interferometer can achieve the same accuracy as the standard one with a lower level of su(2) squeezing. We will also describe how to produce approximate versions of these states. We are thus able to extend the results of Yurke, McCall, and Klauder and also to show that an explicit connection exists between higher-order squeezing and improved interferometric measurements.

We also consider more general interferometers and show that each interferometer has a set of minimum-uncertainty states associated with it. The interferometers can consist of different combinations of beam splitters, four-wave mixers, and degenerate parametric amplifiers. The specific arrangement determines the operators which describe the phase-measurement accuracy of the device. These operators obey an uncertainty relation which, in turn, leads to a family of minimum-uncertainty states. These states should prove useful in analyzing the performance of the interferometer. In the case of the standard and modified Mach-Zehnder interferometers, this procedure yields the su(2) minimum-uncertainty states.

II. DESCRIPTION OF INTERFEROMETERS

In this section we review the description of a Mach-Zehnder interferometer developed in Ref. [3] and show

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how it is related to minimum-uncertainty states. We also show how these considerations can be generalized to interferometers containing elements other than beam splitters, such as four-wave mixers or degenerate parametric amplifiers.

A Mach-Zehnder interferometer consists of two beam splitters. Each beam splitter has two input ports and two output ports (see Fig. 1). It can be described by a unitary operator U, the scattering matrix, which relates the in annihilation operators to the out annihilation operators by

$$a_{1\text{out}}=U^{-1}a_{1\text{in}}U\ ,\quad a_{2\text{out}}=U^{-1}a_{2\text{in}}U\ . \tag{2.1}$$

Because the beam splitter is a linear device, we can represent the action of U by a unitary 2×2 matrix, i.e.,

$$\begin{bmatrix} a_{1\text{out}} \\ a_{2\text{out}} \end{bmatrix} = \begin{bmatrix} U_{11} U_{12} \\ U_{21} U_{22} \end{bmatrix} \begin{bmatrix} a_{1\text{in}} \\ a_{2\text{in}} \end{bmatrix} . \tag{2.2}$$

If we work with states instead of operators, then U is the operator which transforms the input state $|\text{in}\rangle$ into the output state $|\text{out}\rangle$,

$$|\text{out}\rangle = U|\text{in}\rangle\ . \tag{2.3}$$

At this point we shall adopt the convention that operators without an "in" or an "out" subscript are "in" operators. Because most of the operators which we shall be considering will be in operators, this convention will prove to be convenient.

It is also useful to look at the action of U on the operators [3,4]

$$\begin{aligned} J_1 &= (a_1^\dagger a_2 + a_1 a_2^\dagger)/2\ , \\ J_2 &= -i(a_1^\dagger a_2 - a_1 a_2^\dagger)/2\ , \\ J_3 &= (a_1^\dagger a_1 - a_2^\dagger a_2)/2\ . \end{aligned} \tag{2.4}$$

These operators are the Schwinger representation of the angular-momentum operators [9]. They obey the su(2) commutation relations

$$[J_k, J_m] = i\epsilon_{kmn}J_n\ , \tag{2.5}$$

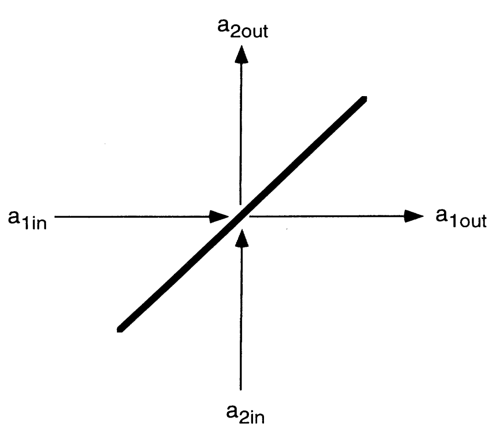

FIG. 1. Beam splitter. The annihilation operators $a_{1\text{in}}$ and $a_{2\text{in}}$ correspond to the input fields and $a_{1\text{out}}$ and $a_{2\text{out}}$ correspond to the output fields.

where k, m, and n run from 1 to 3 and ϵ_{kmn} is the completely antisymmetric tensor of rank 3. As was shown by Yurke, McCall, and Klauder, a Mach-Zehnder interferometer can be described in terms of these operators alone [3]. This is because what is measured at the output of the interferometer is J_3 and the beam-splitter transformations act like rotations which transform the angular-momentum operators among themselves. Therefore, the measurement of J_3 at the output corresponds to the measurement of a variable which is a linear combination of J_1, J_2, and J_3 at the input.

The operator U can be expressed as the exponential of i times a linear combination of the operators in Eq. (2.4) [3,4]. We shall be interested in two particular examples. The first is described by the operator $U_1 = \exp(-i\pi J_1/2)$, which corresponds to the 2×2 matrix [see Eq. (2.2)]

$$\begin{bmatrix} a_{1\text{out}} \\ a_{2\text{out}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} a_{1\text{in}} \\ a_{2\text{in}} \end{bmatrix} , \tag{2.6}$$

and the second is given by $U_2 = \exp(i\pi J_1/2)$, which corresponds to

$$\begin{bmatrix} a_{1\text{out}} \\ a_{2\text{out}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} a_{1\text{in}} \\ a_{2\text{in}} \end{bmatrix} . \tag{2.7}$$

We are now ready to form a Mach-Zehnder interferometer (see Fig. 2). The first beam splitter is described by U_1 and the second by U_2. The device producing a phase shift ϕ in one of the legs is described by the unitary operator $U(\phi) = \exp(-i\phi a_1^\dagger a_1)$. It is changes in the phase shift which we wish to measure.

If we consider the interferometer as a whole, the output state is related to the input state by

$$|\text{out}\rangle = U_2 U(\phi) U_1 |\text{in}\rangle\ . \tag{2.8}$$

In order to measure changes in the phase shift ϕ, we usually measure the difference between the photon numbers at the output ports, i.e.,

$$J_{3\text{out}} = (a_{1\text{out}}^\dagger a_{1\text{out}} - a_{2\text{out}}^\dagger a_{2\text{out}})/2\ .$$

Changes $\delta\phi$ in the phase angle are detected by the

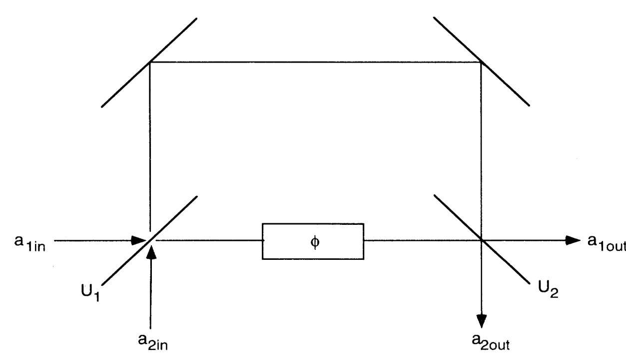

FIG. 2. A Mach-Zehnder interferometer. The first beam splitter is described by the unitary operator U_1 and the second by the unitary operator U_2. The device between the beam splitters produces a phase shift ϕ.

changes they create in the expectation value of $J_{3\text{out}}$. Since there are fluctuations in $J_{3\text{out}}$, a phase change is detectable only if it induces a change in $\langle J_{3\text{out}}\rangle$ which is larger than $\Delta J_{3\text{out}}$. Therefore, the minimum detectable phase change is given by

$$\delta\phi = \Delta J_{3\text{out}} \Big/ \left| \frac{d\langle J_{3\text{out}}\rangle}{d\phi} \right| . \tag{2.9}$$

We are interested in which input states will produce a small value of $\delta\phi$, so it is useful to express Eq. (2.9) in terms of the input state. Using Eqs. (2.6)–(2.8), one finds that

$$\begin{aligned}\langle J_{3\text{out}}\rangle &= \langle \text{out}|J_3|\text{out}\rangle \\ &= -\sin\phi\langle \text{in}|J_1|\text{in}\rangle + \cos\phi\langle \text{in}|J_3|\text{in}\rangle ,\end{aligned} \tag{2.10}$$

from which one concludes that

$$\frac{d\langle J_{3\text{out}}\rangle}{d\phi} = -\cos\phi\langle \text{in}|J_1|\text{in}\rangle - \sin\phi\langle \text{in}|J_3|\text{in}\rangle . \tag{2.11}$$

To consider small changes about $\phi=0$ we shall evaluate the above equation at $\phi=0$. We also find that

$$\Delta J_{3\text{out}} = \Delta J_{3\text{in}} = [\langle \text{in}|J_3^2|\text{in}\rangle - \langle \text{in}|J_3|\text{in}\rangle^2]^{1/2} ,$$

so that

$$\delta\phi = \Delta J_3 / |\langle J_1\rangle| , \tag{2.12}$$

where we again remark that quantities without "in" or "out" subscripts are "in" quantities.

The commutation relations obeyed by J_1, J_2, and J_3 imply that these operators satisfy the uncertainty relation

$$\Delta J_2 \Delta J_3 \geq |\langle J_1\rangle|/2 . \tag{2.13}$$

This, in conjunction with Eq. (2.12), implies that

$$\delta\phi \geq 1/(2\Delta J_2) . \tag{2.14}$$

This relation will be satisfied as an equality by states which satisfy Eq. (2.13) as an equality, i.e., minimum-uncertainty states in J_2 and J_3. Such a minimum-uncertainty state with a large value of ΔJ_2 would allow us to measure small changes in the phase. This strongly suggests we examine the minimum-uncertainty states in order to see what kind of accuracies they will produce.

Before doing so, however, let us see whether such a connection between minimum detectable phases and minimum-uncertainty states holds for other kinds of interferometers. The interferometers we consider consist of a device with two input ports, followed by something which produces a phase shift, followed by a second device with two input ports. So far, we have been assuming that the two-port devices (we label devices by the number of input ports) are beam splitters, but this need not be the case. Another possibility is a four-wave mixer, which is described by the 2×2 matrix [3,5]

$$\begin{bmatrix} a_{1\text{out}} \\ a_{2\text{out}}^{\dagger} \end{bmatrix} = \begin{bmatrix} \mu & \nu \\ \nu^* & \mu \end{bmatrix} \begin{bmatrix} a_{1\text{in}} \\ a_{2\text{in}}^{\dagger} \end{bmatrix} , \tag{2.15}$$

where μ is real, ν is complex, and $\mu^2 - |\nu|^2 = 1$. Another possible two-port device is a beam splitter followed by two degenerate parametric amplifiers, one at each output port of the beam splitter. If this device is followed by a standard beam splitter, one has an interferometer of the type considered by Raymer and Yang [12].

Let us consider the general interferometer depicted in Fig. 3. The action of the first four-port device is described by a unitary operator U_A and the action of the second by a unitary operator U_B. The output state is given in terms of the input state by

$$|\text{out}\rangle = U_B U(\phi) U_A |\text{in}\rangle . \tag{2.16}$$

Let us also define the state $|\Psi\rangle = U_A|\text{in}\rangle$, which is the state of the field after the first two-port device. Suppose we are measuring a variable W at the output in order to determine small changes $\delta\phi$ in the phase. As before, we can measure a phase change if the change it induces in the expectation value of W at the output is larger than the fluctuations in W. Therefore, the minimum detectable phase is

$$\delta\phi = \Delta W_{\text{out}} \Big/ \left| \frac{d\langle W_{\text{out}}\rangle}{d\phi} \right| . \tag{2.17}$$

Let us assume again the we are looking at small phase changes about $\phi=0$. This assumption is not necessary, but it is made for the sake of simplicity. We find that

$$\begin{aligned} \Delta W_{\text{out}}|_{\phi=0} &= [\langle\Psi|V^2|\Psi\rangle - \langle\Psi|V|\Psi\rangle^2]^{1/2} \equiv \Delta V|_{\Psi} , \\ \left.\frac{d\langle W_{\text{out}}\rangle}{d\phi}\right|_{\phi=0} &= \frac{d}{d\phi}\langle \text{in}|U_A^{-1}U(\phi)^{-1}U_B^{-1}WU_B U(\phi)U_A|\text{in}\rangle|_{\phi=0} \\ &= i\langle\Psi|[N_1, V]|\Psi\rangle , \end{aligned} \tag{2.18}$$

where $V = U_B^{-1}WU_B$ and $N_1 = a_1^{\dagger}a_1$. The minimum detectable phase change is then given by

$$\delta\phi = \Delta V|_{\Psi} / |\langle\Psi|[N_1, V]|\Psi\rangle| . \tag{2.19}$$

The numerator and denominator of this expression are

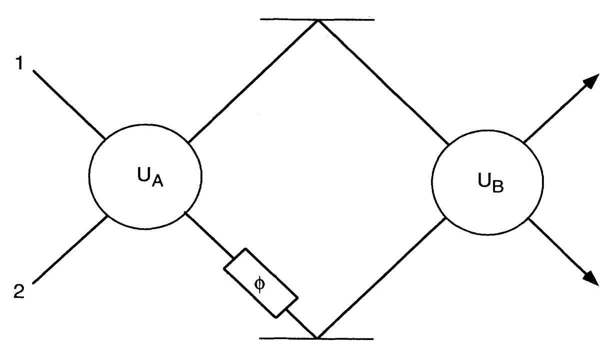

FIG. 3. A general interferometer. The first two-port device is described by the unitary operator U_A and the second by the unitary operator U_B.

again constituents of an uncertainty relation. In particular, we have that

$$\Delta V|_\Psi \Delta N_1|_\Psi \geq |\langle\Psi|[N_1,V]|\Psi\rangle|/2\ , \tag{2.20}$$

which gives us the condition for $\delta\phi$,

$$\delta\phi \geq 1/(2\Delta N_1|_\Psi)\ . \tag{2.21}$$

Equality will hold in this relation for states which satisfy Eq. (2.20) as an equality, i.e., minimum-uncertainty states in N_1 and V. This again suggests that these minimum-uncertainty states will be useful in exploring the accuracy of the interferometer. One would find a minimum-uncertainty state $|\Psi\rangle$, which yields a small minimum measurable phase angle, and then apply U_A^{-1} to find the corresponding input state. The ultimate utility of these states depends upon how the total photon number at the input is related to ΔN_1 for the state $|\Psi\rangle$. This is because the interferometer accuracy is expressed in terms of the input photon number. Therefore, the final step of the analysis is to express $\Delta N_1|_\Psi$ in terms of N and to substitute the result into Eq. (2.21).

We now return to the specific case of the Mach-Zehnder interferometer. For this we need the su(2) minimum-uncertainty states. We shall discuss these states and their properties.

III. su(2) MINIMUM-UNCERTAINTY STATES

Consider the uncertainty relation

$$\Delta J_1 \Delta J_2 \geq |\langle J_3\rangle|/2\ . \tag{3.1}$$

We would like to find the states which satisfy this relation as an equality. They satisfy the eigenvalue equation [10,13]

$$(J_1+i\lambda J_2)|\psi\rangle=\beta|\psi\rangle\ , \tag{3.2}$$

where λ is real. These states are related via a rotation to those which satisfy Eq. (2.13) as an equality. This will be discussed subsequently in more detail.

The parameters β and λ can be related to the properties of the state $|\psi\rangle$. The eigenvalue β is related to the expectation values of J_1 and J_2, i.e.,

$$\langle\psi|J_1|\psi\rangle=\mathrm{Re}(\beta)\ ,\quad \langle\psi|J_2|\psi\rangle=(1/\lambda)\mathrm{Im}(\beta)\ . \tag{3.3}$$

The parameter λ is a squeezing parameter. This can be seen from the following relations, which follow from Eq. (3.2) (see Appendix A):

$$(\Delta J_1)^2=(\lambda/2)\langle J_3\rangle\ ,\quad (\Delta J_2)^2=(1/2\lambda)\langle J_3\rangle\ . \tag{3.4}$$

Therefore, if $|\lambda|>1$, the state $|\psi\rangle$ is squeezed in J_2, and if $|\lambda|<1$, then the squeezing is in J_1.

Before proceeding further, we need to specify the space of states in which we want to solve Eq. (3.2). The representations of su(2) are labeled by the parameter j which assumes non-negative integer and half-integer values. Every state in the carrier space of the representation corresponding to j is an eigenstate of the Casimir operator $J^2=J_1^2+J_2^2+J_3^2$, with eigenvalue $j(j+1)$. This carrier space is spanned by the states $|j,m\rangle$, where m is the eigenvalue of J_3 and runs from $-j$ to j in integer steps. Therefore, the state $|j,m\rangle$ satisfies the eigenvalue equations

$$J^2|j,m\rangle=j(j+1)|j,m\rangle$$

and

$$J_3|j,m\rangle=m|j,m\rangle\ .$$

Let us connect these rather abstract, but familiar, considerations back to the Schwinger representation of su(2). If we define the total photon number $N=a_1^\dagger a_1+a_2^\dagger a_2$, then N commutes with J_1, J_2, and J_3, and J^2 is given by

$$J^2=(N/2)[(N/2)+1]\ , \tag{3.5}$$

so that j is just the eigenvalue of $N/2$. Therefore, each representation of su(2) corresponds to a fixed total photon number. The operator J_3 is one-half the difference between the number of photons in the first mode and that in the second. Therefore, the state $|j,m\rangle$ corresponds to a state with $j+m$ photons in mode 1 and $j-m$ photons in mode 2, i.e.,

$$|j,m\rangle=|j+m\rangle_1\otimes|j-m\rangle_2\ . \tag{3.6}$$

Here, $|j+m\rangle_1$ designates a mode-1 number state with $j+m$ photons and $|j-m\rangle_2$ designates a mode-2 number state with $j-m$ photons.

We want to solve Eq. (3.2) separately for each representation of su(2), i.e., for fixed j. This means that the solutions will be linear combinations of the state $|j,m\rangle$, with j fixed and m running from $-j$ to j. We shall be interested in a certain subset of the solutions corresponding to $\lambda>1$. Equations (3.4) imply that these states are squeezed in J_2. The states are given explicitly by

$$|\psi(j,m_0,\lambda)\rangle e=c_{jm_0}(\lambda)e^{-i\theta J_1} \times\sum_{m=-j}^{m_0}(-i\sqrt{\lambda^2-1})^{j+m}[1/(m_0-m)!] \times[(j-m)!/(j+m)!]^{1/2}|j,m\rangle\ . \tag{3.7}$$

In this equation, $c_{jm_0}(\lambda)$ is a normalization constant and the angle θ lies between $\pi/2$ and π. It is specified by the condition $\lambda\cos\theta=-1$. The eigenvalue corresponding to $|\psi(j,m_0,\lambda)\rangle$ is $\beta=-im_0\sqrt{\lambda^2-1}$. The derivation of this equation is given in Appendix A. Another, more compact form for the states is

$$|\psi(j,m_0,\lambda)\rangle=c'_{jm_0}(\lambda)e^{-i\theta J_1} \times\exp(iJ_-/\sqrt{\lambda^2-1})|j,m_0\rangle\ , \tag{3.8}$$

where θ is as before, J_- is the angular-momentum lowering operator, and $c'_{jm_0}(\lambda)$ is a different normalization constant. Therefore, we have a two-parameter set of minimum-uncertainty states: one parameter m_0 is discrete, and the other, $\lambda>1$, is continuous.

We want to calculate ΔJ_1 and ΔJ_2 for the states $|\psi(j,m_0,\lambda)\rangle$. From Eqs. (3.4) we see that if we find

$\langle\psi(j,m_0,\lambda)|J_3|\psi(j,m_0,\lambda)\rangle$, we can immediately find these uncertainties. In order to find this expectation value, we need to find $c_{jm_0}(\lambda)$. From Eq. (3.7) we see that the normalization constant is given by

$$|c_{jm_0}(\lambda)|^2=\left[\sum_{m=-j}^{m_0}(\lambda^2-1)^{j+m}[1/(m_0-m)!]^2 \times[(j-m)!/(j+m)!]\right]^{-1} . \tag{3.9}$$

The expectation of J_3 can also be found. This is done by noting that

$$\begin{aligned} e^{i\theta J_1}J_3e^{-i\theta J_1}&=(\sin\theta)J_2+(\cos\theta)J_3\\ &=(i/2)(\sin\theta)(J_1-J_+)+(\cos\theta)J_3 , \end{aligned} \tag{3.10}$$

where $J_\pm=J_1\pm iJ_2$ are the angular-momentum raising and lowering operators. Use of Eqs. (3.7) and (3.10) gives

$$\langle\psi(j,m_0,\lambda)|J_3|\psi(j,m_0,\lambda)\rangle=|c_{jm_0}(\lambda)|^2(1/\lambda)\sum_{m=-j}^{m_0}(\lambda^2-1)^{j+m}[1/(m_0-m)!]^2 \times[(j-m)!/(j+m)!][(\lambda^2-1)(m_0-m)-m] . \tag{3.11}$$

We shall be interested in the states $|\psi(j,m_0,\lambda)\rangle$ in two different parameter regimes. The first is the highly squeezed $\lambda\to\infty$ (or, more specifically, $\lambda\gg j$) regime. For very large λ we retain only the terms with the highest powers of λ in the equations for the normalization constant and the expectation value of J_3. This leads to

$$\begin{aligned} &\langle\psi(j,m_0,\lambda)|J_3|\psi(j,m_0,\lambda)\rangle\\ &\qquad\cong(1/\lambda)[(j+m_0)(j-m_0+1)-m_0] , \end{aligned} \tag{3.12}$$

which, in conjunction with Eqs. (3.4), implies that

$$(\Delta J_1)^2\cong[(j+m_0)(j-m_0+1)-m_0]/2 \tag{3.13}$$

for large λ.

The second regime is when j is large, λ is of order j, and $m_0\ll j$. We again need to develop approximate expressions for $|c_{jm_0}(\lambda)|^2$ and $\langle J_3\rangle$. Let us set $\lambda=xj$, where x is of order 1 or smaller (exactly how small it can be, we shall see shortly), which implies that $\lambda^2-1\cong x^2j^2$. An examination of the sums appearing in the expressions for $|c_{jm_0}(\lambda)|^2$ and $\langle J_3\rangle$ shows that the most important terms are those for which m is close to m_0. This is true because the factor $[1/(m_0-m)!]^2$ suppresses terms for which m_0-m is large. For m close to m_0 and $j\gg m_0$, we have

$$(j-m)!/(j+m)!\cong j^{2(m_0-m)}(j-m_0)!/(j+m_0)! . \tag{3.14}$$

Inserting this into the expression for $|c_{jm_0}(\lambda)|^2$, we find

$$\begin{aligned} |c_{jm_0}(\lambda)|^2&\cong\left[x^{2j}j^{2(m_0+j)}[(j-m_0)!/(j+m_0)!]\sum_{m=-j}^{m_0}x^{2m}[1/(m_0-m)!]^2\right]^{-1}\\ &\cong\{x^{2(j+m_0)}j^{2(m_0+j)}[(j-m_0)!/(j+m_0)!]I_0(2/x)\}^{-1} , \end{aligned} \tag{3.15}$$

where we have extended the range of the sum and introduced the modified Bessel function

$$I_0(z)=\sum_{k=0}^{\infty}(z/2)^{2k}/(k!)^2 . \tag{3.16}$$

The series

$$\sum_{m=-j}^{m_0}x^{2(m-m_0)}[1/(m_0-m)!]^2$$

starts to cut off when $m_0-m>1/x$, so that its replacement by $I_0(2/x)$ should be a good approximation as long as $(1/x)\ll j$. This gives us a restriction on how small x can be. A similar analysis of the expression for $\langle J_3\rangle$ gives

$$\begin{aligned} &\langle\psi(j,m_0,\lambda)|J_3|\psi(j,m_0,\lambda)\rangle\\ &\quad\cong|c_{jm_0}(\lambda)|^2(xj)^{2(m_0+j)}\\ &\qquad\times[(j-m_0)!/(j+m_0)!]jI_1(2/x)\\ &\quad\cong jI_1(2/x)/I_0(2/x) , \end{aligned} \tag{3.17}$$

where

$$I_1(z)=\sum_{k=0}^{\infty}(z/2)^{2k+1}/[k!(k+1)!] . \tag{3.18}$$

Making use of Eqs. (3.4), we find that in this regime ($j\gg1$, $j\gg m_0$, and x of order 1 or less but much larger than $1/j$),

$$(\Delta J_1)^2\cong(xj^2/2)I_1(2/x)/I_0(2/x) . \tag{3.19}$$

Finally, let us note that in Sec. II it was stated that minimum-uncertainty states in J_2 and J_3 would lead to very sensitive interferometric measurements, whereas in this section we have considered minimum-uncertainty states in J_1 and J_2. It is possible to convert one into the other by means of a rotation. Consider, in particular, the rotation

$$U_R = e^{-i\pi J_2/2} e^{-i\pi J_3/2} , \tag{3.20}$$

i.e., a rotation about the 3 axis by $\pi/2$ followed by one about the 2 axis by $\pi/2$. This rotation maps the operators J_1, J_2, and J_3 into each other as follows:

$$\begin{aligned} U_R J_1 U_R^{-1} &= J_2 , \\ U_R J_2 U_R^{-1} &= J_3 , \\ U_R J_3 U_R^{-1} &= J_1 . \end{aligned} \tag{3.21}$$

Therefore, the state $U_R|\psi(j,m_0,\lambda)\rangle$ satisfies the eigenvalue equation

$$[U_R(J_1+i\lambda J_2)U_R^{-1}]U_R|\psi(j,m_0,\lambda)\rangle = \beta U_R|\psi(j,m_0,\lambda)\rangle , \tag{3.22}$$

or

$$(J_2+i\lambda J_3)U_R|\psi(j,m_0,\lambda)\rangle = \beta U_R|\psi(j,m_0,\lambda)\rangle . \tag{3.23}$$

This means that $U_R|\psi(j,m_0,\lambda)\rangle$ is a minimum-uncertainty state in J_2 and J_3 which is squeezed in J_3 for $\lambda > 1$. We also note that the properties of J_2 in the state $U_R|\psi(j,m_0,\lambda)\rangle$ are identical to those of J_1 in $|\psi(j,m_0,\lambda)\rangle$. Therefore, for this state in the very-large-λ regime, we have

$$(\Delta J_2)^2 \cong [(j+m_0)(j-m_0+1)-m_0]/2 , \tag{3.24}$$

and in the regime $j \gg 1$, $j \gg m_0$, and λ of order j or smaller, we have

$$(\Delta J_2)^2 \cong (xj^2/2) I_1(2/x)/I_0(2/x) . \tag{3.25}$$

Finally, let us find the expectation values of J_1, J_2, and J_3 in the state $U_R|\psi(j,m_0,\lambda)\rangle$. Equations (3.3) and (3.21) give us that (for $\lambda \geq 1$)

$$\langle J_2\rangle = 0 , \quad \langle J_3\rangle = (-m_0/\lambda)\sqrt{\lambda^2-1} . \tag{3.26}$$

In the large-λ regime, we find from Eqs. (3.12) and (3.21) that

$$\langle J_1\rangle \cong [(j+m_0)(j-m_0+1)-m_0]/\lambda , \tag{3.27}$$

and in the regime $j \gg 1$, $j \gg m_0$, and λ of order j or smaller, we have

$$\langle J_1\rangle \cong j I_1(2/x)/I_0(2/x) . \tag{3.28}$$

Therefore, the vector $(\langle J_1\rangle, \langle J_2\rangle, \langle J_3\rangle)$ for $U_R|\psi(j,m_0,\lambda)\rangle$ lies in the 2-3 plane.

IV. APPLICATION TO THE MACH-ZEHNDER INTERFEROMETER

We are now in a position to examine the Mach-Zehnder interferometer and to determine its accuracy with an su(2) squeezed minimum-uncertainty state as an input. We shall also show how to produce a state which is close to $|\psi(j,m_0,\lambda)\rangle$ by using standard nonlinear optical devices and evaluate the accuracy of the interferometer with this input state.

Let the input state to the interferometer be $U_R|\psi(j,m_0,\lambda)\rangle$, where λ is large ($\lambda \gg j$). The fluctuations in J_2 will then be given by Eq. (3.24). Substituting this result into Eq. (2.14), we find that the minimum detectable phase shift is

$$\delta\phi = 1/\{\sqrt{2}[(j+m_0)(j-m_0+1)-m_0]^{1/2}\} , \tag{4.1}$$

which achieves a minimum value of $1/[2j(j+1)]^{1/2}$ for $m_0=0$. Because j is just half the total number of photons going into the interferometer, we see that the phase-measurement accuracy is of order $1/N$.

We shall first describe how to construct a state close to $|\psi(j,m_0,\lambda)\rangle$ and then show that it has the desired properties. We begin with the two-mode coherent state $|u,iu\rangle$, where u, the amplitude of mode 1, is real and the amplitude of mode 2, iu, is imaginary. We then send each mode through degenerate parametric amplifiers. Mode 1 is sent through an amplifier described by the squeeze operator

$$S_1 = \exp[-r(a_1^{\dagger 2}-a_1^2)/2] , \tag{4.2}$$

and mode 2 is sent through an amplifier described by

$$S_2 = \exp[r(a_2^{\dagger 2}-a_2^2)/2] . \tag{4.3}$$

S_1 and S_2 transform the operators a_1 and a_2, respectively, as

$$\begin{aligned} S_1^{-1}a_1S_1 &= (\cosh r)a_1-(\sinh r)a_1^\dagger , \\ S_2^{-1}a_2S_2 &= (\cosh r)a_2+(\sinh r)a_2^\dagger . \end{aligned} \tag{4.4}$$

The two squeezed modes are now sent into the two input ports of a four-wave mixer. This is described by the operator

$$S_{12} = \exp(2irK_1) , \tag{4.5}$$

where $K_1=(a_1^\dagger a_2^\dagger + a_1a_2)/2$. S_{12} transforms the operators a_1 and a_2 as

$$\begin{aligned} S_{12}^{-1}a_1S_{12} &= (\cosh r)a_1+i(\sinh r)a_2^\dagger , \\ S_{12}^{-1}a_2S_{12} &= (\cosh r)a_2+i(\sinh r)a_1^\dagger . \end{aligned} \tag{4.6}$$

The state $S_{12}S_1S_2|u,iu\rangle$ is similar to the state $|\psi(j,m_0,\lambda)\rangle$ without the $\exp(-i\theta J_1)$ factor. Its fluctuations in J_3 are small. We now note that if λ is large, then $\theta \cong \pi/2$. Therefore, we take for our approximation to $|\psi(j,m_0,\lambda)\rangle$ the state

$$|\Phi\rangle = e^{-i\pi J_1/2} S_{12}S_1S_2|u,iu\rangle . \tag{4.7}$$

The final transformation in Eq. (4.7), the operator $e^{-i\pi J_1/2}$, can be accomplished by a beam splitter.

Now let us examine the properties of the state $|\Phi\rangle$. For the expectation values of $\langle J_n\rangle, n=1,2,3$, and $\langle N\rangle$, we have

$$\begin{aligned}
&\langle J_1 \rangle = 0 \,, \\
&\langle J_2 \rangle = 0 \,, \\
&\langle J_3 \rangle = -u^2 + \tfrac{1}{2}\sinh^2(2r) \,, \\
&\langle N \rangle = 2u^2 + 2\sinh^2(2r) \,,
\end{aligned} \tag{4.8}$$

and the fluctuations in J_1 and J_2 are

$$\begin{aligned}
&(\Delta J_1)^2 = (u^2 e^{4r}/2) + \tfrac{1}{4}\sinh^2(2r) \,, \\
&(\Delta J_2)^2 = (u^2 e^{-4r}/2) + \tfrac{1}{4}\sinh^2(2r) \,.
\end{aligned} \tag{4.9}$$

Let us first note that if $u \gg e^{4r}$, then

$$\Delta J_1 \cong (u/\sqrt{2})e^{2r} \,, \quad \Delta J_2 \cong (u/\sqrt{2})e^{-2r} \,, \quad \langle J_3 \rangle \cong -u^2 \,, \tag{4.10}$$

so that $\Delta J_1 \Delta J_2 \cong |\langle J_3 \rangle|/2$, i.e., $|\Phi\rangle$ is approximately a minimum-uncertainty state. If the state $|\Phi'\rangle = U_R|\Phi\rangle$, were the input state to an interferometer, the minimum detectable phase, using the approximate expression in Eq. (4.10), is

$$\begin{aligned}
\delta\phi = \Delta J_3|_{\Phi'}/|\Phi'|J_1|\Phi'\rangle| &= \Delta J_2|_{\Phi}/|\langle\Phi|J_3|\Phi\rangle| \\
&\cong e^{-2r}/(u\sqrt{2}) \\
&\cong e^{-2r}/\sqrt{\langle N \rangle} \,,
\end{aligned} \tag{4.11}$$

which can be a considerable enhancement over the usual $1/\sqrt{\langle N \rangle}$ result. For r sufficiently large, i.e., when u is comparable to e^{4r}, the behavior of $|\Phi\rangle$ will deviate considerably from that of a minimum-uncertainty state. In fact, a more careful examination of the minimum detectable phase shows that with $|\Phi\rangle$ one achieves an accuracy of $(1/\langle N \rangle)^{3/4}$ rather than the $1/\langle N \rangle$ which can be achieved by a true minimum-uncertainty state. The details of this calculation are given in Appendix B.

V. FOUR-WAVE-MIXER INTERFEROMETER

Let us now consider an interferometer with an element which provides gain. In particular, let us replace the first beam splitter in a Mach-Zehnder interferometer by a four-wave mixer (Fig. 4). What we shall find is that $1/\langle N \rangle$ accuracy is achievable with lower amounts of su(2) squeezing than in a standard Mach-Zehnder interferometer.

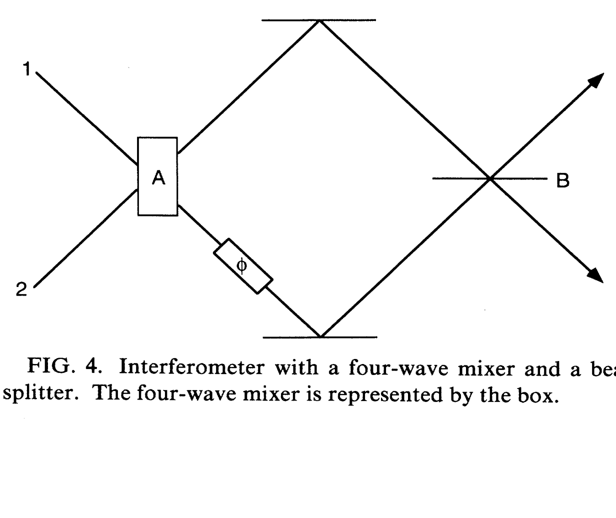

FIG. 4. Interferometer with a four-wave mixer and a beam splitter. The four-wave mixer is represented by the box.

It might be suspected from a quick examination of Sec. II that su(2) minimum-uncertainty states are the ones to use in this new device. The analysis there shows that the operators in the expression for the minimum detectable phase shift depend on the part of the interferometer which the light passes through after the phase shift has been introduced. As our new interferometer is the same as the Mach-Zehnder interferometer after this point, the same states which were of use there, the su(2) minimum-uncertainty states, should also be of use in analyzing our new device.

It is useful, however, to go briefly through the analysis in order to illustrate the use of the general formalism. Let us choose our beam splitter to be described by the operator $\exp(-i\pi J_2/2)$, i.e.,

$$U_B = e^{-i\pi J_2/2} \,, \tag{5.1}$$

and also assume the we are measuring J_3 at the output. This means that the operator V is given by

$$V = U_B^{-1} J_3 U_B = -J_1 \,, \tag{5.2}$$

and the minimum detectable phase change is

$$\delta\phi = \Delta J_1|_{\Psi}/|\langle\Psi|[N_1, J_1]|\Psi\rangle| \,, \tag{5.3}$$

where $|\Psi\rangle$ is the state of the two-mode field just after the four-wave mixer. Because $N = N_1 + N_2$ commutes with J_1, and N_1 can be expressed as $N_1 = J_3 + N/2$, we can express $\delta\phi$ as

$$\delta\phi = \Delta J_1|_{\Psi}/|\langle\Psi|[J_3, J_1]|\Psi\rangle| \,. \tag{5.4}$$

If $|\Psi\rangle$ is an su(2) minimum-uncertainty state in the variables J_1 and J_3, then Eq. (5.4) becomes

$$\delta\phi = 1/(2\Delta J_3|_{\Psi}) \,. \tag{5.5}$$

The input states for which Eq. (5.5) holds are found by applying U_A^{-1} to an su(2) minimum-uncertainty state in the variables J_1 and J_3. The operator U_A describes the action of the four-wave mixer. In order to be specific, we shall assume that U_A is given by $\exp(2ir'K_1)$, where r' is a real, positive number. This is the same as the operator S_{12} in Sec. IV if $r' = 2r$.

First, we need the su(2) minimum-uncertainty states in the variables J_1 and J_3. If we define the rotation

$$U_0 = e^{i\pi J_3/2} e^{i\pi J_2/2} \,. \tag{5.6}$$

then U_0 transforms the operators J_1, J_2, and J_3 as

$$\begin{aligned}
&U_0 J_1 U_0^{-1} = J_3 \,, \\
&U_0 J_2 U_0^{-1} = J_1 \,, \\
&U_0 J_3 U_0^{-1} = J_2 \,.
\end{aligned} \tag{5.7}$$

The state $U_0|\psi(j, m_0, \lambda)\rangle$ is a minimum- uncertainty state in J_1 and J_3 and is squeezed in J_1 for $\lambda > 1$.

We must now relate the number of photons in the state $|\Psi\rangle$ to the number in the input state, $|\text{in}\rangle = U_A^{-1}|\Psi\rangle$. It is at this point that the analysis of this interferometer differs from that of the standard Mach-Zehnder interferometer. In a Mach-Zehnder interferometer, the photon number is the same in $|\Psi\rangle$ as in $|\mathrm{in}\rangle$. This is no longer true if the initial beam splitter is replaced by a four-wave mixer. In particular, we find that

$$\langle \mathrm{in}|N||\mathrm{in}\rangle = 2\{\langle\Psi|K_3|\Psi\rangle\cosh r' - \langle\Psi|K_2|\Psi\rangle\sinh r'\} - 1 \ , \tag{5.8}$$

where

$$K_2 = -i(a_1^\dagger a_2^\dagger - a_1 a_2)/2$$

and

$$K_3 = (N+1)/2 \ .$$

The operators K_1, K_2, and K_3 form a representation of the su(1,1) Lie algebra.

Equation (5.8) leads rather quickly to the conclusion that the state $U_A^{-1}U_0|\psi(j,m_0,\lambda)\rangle$ is not the optimal input state. The operator K_2 acting on a state with $j=j_0$ produces a state which is a linear combination of states with $j=j_0+1$ and $j=j_0-1$. The state $U_0|\psi(j_0,m_0,\lambda)\rangle$ is a linear combination of states, all of which have the same value of j. Therefore, K_2 acting on this state leads to a state which is a linear combination of states with j values of j_0+1 and j_0-1. Because states with different values of j are orthogonal, this leads to the conclusion that $\langle\Psi|K_2|\Psi\rangle=0$ if

$$|\Psi\rangle = U_0|\psi(j_0,m_0,\lambda)\rangle \ .$$

In the large-λ limit, this leads to a minimum detectable phase shift of

$$\delta\phi \cong (\sqrt{2}\cosh r')/\langle \mathrm{in}|N|\mathrm{in}\rangle \ . \tag{5.9}$$

This is essentially the result in Eq. (4.1) multiplier by $\cosh r'$ and a poorer result than that of the Mach-Zehnder interferometer. On the other hand, we would expect that by adding an element that provides gain, we would not degrade the performance of the device. This suggests that we have not chosen our input state properly.

We can correct this problem by realizing that any state of the form

$$|\Phi\rangle = \sum_{j=j_1}^{j_2} d_j|\psi(j,m_0,\lambda)\rangle \ , \tag{5.10}$$

where the d_j are arbitrary, will be a solution to Eq. (3.2) with eigenvalue $\beta=-im_0\sqrt{\lambda^2-1}$ and is, therefore, a minimum-uncertainty state in J_1 and J_2. If we choose for our state $|\Psi\rangle$ the state $|\Psi\rangle=U_0|\Phi\rangle$, we shall still be dealing with su(2) minimum-uncertainty states, and Eq. (5.5) for $\delta\phi$ will still hold.

We now must find the number of photons in the state

$$|\mathrm{in}\rangle = \exp(-ir'K_1)|\Psi\rangle \ .$$

Making use of Eq. (5.8), we see that

$$\langle \mathrm{in}|N|\mathrm{in}\rangle = 2\left[(\cosh r')\sum_{j=j_1}^{j_2}|d_j|^2\langle\psi(j,m_0,\lambda)|U_0^{-1}K_3U_0|\psi(j,m_0,\lambda)\rangle - (\sinh r')\left[\sum_{j=j_1}^{j_2-1} d_{j+1}^* d_j\langle\psi(j+1,m_0,\lambda)|U_0^{-1}K_2U_0|\psi(j,m_0,\lambda)\rangle + \mathrm{c.c.}\right]\right] - 1 \ , \tag{5.11}$$

where c.c. denotes complex conjugate. The first matrix element in this equation is relatively simple to evaluate because K_3 commutes with U_0, and $|\psi(j,m_0,\lambda)\rangle$ is an eigenstate of K_3 with eigenvalue $j+\frac{1}{2}$. The evaluation of the second matrix element requires considerably more effort. Let us first note that

$$U_0^{-1}K_2U_0 = (Y_{21}-Y_{22})/2 \ , \tag{5.12}$$

where the operators Y_{21} and Y_{22} are given by

$$Y_{21} = i(a_1^{\dagger 2} - a_1^2)/2 \ , \quad Y_{22} = i(a_2^{\dagger 2} - a_2^2)/2 \ . \tag{5.13}$$

These operators appear in the definition of a form of higher-order squeezing known as amplitude-squared squeezing [7]. We also find that $[J_1,(Y_{21}-Y_{22})]=0$, so that the factor $e^{-i\theta J_1}$ which appears in the definition of $|\psi(j,m_0,\lambda)\rangle$ cancels out. Finally, we have that for $\lambda>1$,

$$\begin{aligned}\langle\psi(j+1,m_0,\lambda)|U_0^{-1}K_2U_0|\psi(j,m_0,\lambda)\rangle = c_{j+1,m_0}(\lambda)^* c_{jm_0}(\lambda) \sum_{m=-j-1}^{m_0}\sum_{m'=-j}^{m_0} &(-1)^{j+m'}(i\sqrt{\lambda^2-1})^{2j+m+m'+1}[1/(m_0-m)!][1/(m_0-m')!] \\ &\times[(j-m+1)!/(j+m+1)!]^{1/2}[(j-m')!/(j+m')!]^{1/2} \\ &\times\langle j+1,m|(Y_{21}-Y_{22})/2|j,m'\rangle\end{aligned} \tag{5.14}$$

and

$$\langle j+1,m|(Y_{21}-Y_{22})|j,m'\rangle$$
$$=(i/2)\{\delta_{m',m-1}[(j+m)(j+m+1)]^{1/2}$$
$$-\delta_{m',m+1}[(j-m+1)(j-m)]^{1/2}\}\ . \quad (5.15)$$

We shall be interested in these expressions in the regime where λ is of order j or smaller. For this device, this produces the best results; the results in the $\lambda\to\infty$ regime are similar to those in Eq. (5.9). As in Sec. III, we set $\lambda=x_j j$, where x_j is of order 1 or smaller, and make use of the approximations embodied in Eqs. (3.14) and (3.15). When $j\gg 1$, $j\gg m_0$, and x_j is of order 1 or less but much larger than $1/j$, we have

$$\langle\psi(j+1,m_0,\lambda)|U_0^{-1}K_2U_0|\psi(j,m_0,\lambda)\rangle$$
$$\cong-(ij/2)I_1(2/x_j)/I_0(2/x_j)\ . \quad (5.16)$$

Our result for the number of input photons is then

$$\langle \text{in}|N|\text{in}\rangle\cong(\cosh r')\sum_{j=j_1}^{j_2}2j|d_j|^2$$
$$+i(\sinh r')\sum_{j=j_1}^{j_2-1}j(d_{j+1}^*d_j-d_{j+1}d_j^*)$$
$$\times I_1(2/x_j)/I_0(2/x_j)\ . \quad (5.17)$$

Finally, let us assume that $\Delta j=j_2-j_1\ll j_1$, but $\Delta j\gg 1$. We also define $j_0=(j_1+j_2)/2$ and then set $d_j=(-i)^j/\sqrt{\Delta j}$ For $\langle \text{in}|N|\text{in}\rangle$, we now find

$$\langle \text{in}|N|\text{in}\rangle\cong 2j_0\{\cosh r'-[I_1(2/x)/I_0(2/x)]\sinh r'\}\ , \quad (5.18)$$

where $\lambda=x_0j_0$.

Now we must find the minimum detectable phase shift for this state. From Eqs. (5.5), (5.7), and (3.4), we have

$$\delta\phi=1/(2\Delta J_3|_\Psi)=1/(2\Delta J_1|_\Phi)=1/(2\lambda\langle\Phi|J_3|\Phi\rangle)^{1/2}\ . \quad (5.19)$$

With the same choice of d_j as in the previous paragraph and by making use of Eq. (3.17), we have

$$\delta\phi\cong(1/j_0)\{I_0(2/x_0)/[2x_0I_1(2/x_0)]\}^{1/2}\ . \quad (5.20)$$

We now wish to express this in terms of the number of input photons. As we saw in Sec. III, Eq. (5.20) will be valid for $1/x_0\ll j$. Let us assume that x_0 is chosen so that $j\gg 1/x_0\gg 1$ so that we can use the large-argument expressions for I_0 and I_1. In particular, for $y\gg 1$ [14],

$$I_0(y)\cong e^y[1+(1/8y)]/\sqrt{2\pi y}\ ,$$
$$I_1(y)\cong e^y[1-(3/8y)]/\sqrt{2\pi y}\ . \quad (5.21)$$

In this limit Eqs. (5.18) and (5.20) become

$$\langle \text{in}|N|\text{in}\rangle=2j_0[(x_0e^{r'}/8)+e^{-r'}]\ , \quad (5.22)$$

$$\delta\phi\cong 1/(j_0\sqrt{2x_0})\ , \quad (5.23)$$

respectively. If we now use Eq. (5.22) to eliminate j_0 in Eq. (5.23), we find

$$\delta\phi\cong[\sqrt{2}/\langle \text{in}|N|\text{in}\rangle][(\sqrt{x_0}e^{r'}/8)+1/(\sqrt{x_0}e^{r'})]\ . \quad (5.24)$$

The expression in brackets is a function of $\sqrt{x_0}e^{r'}$ only and reaches a minimum when $\sqrt{x_0}e^{r'}=2\sqrt{2}$. Therefore, for the minimum value of $\delta\phi$, we have

$$\delta\phi\cong 1/(\langle \text{in}|N|\text{in}\rangle)\ . \quad (5.25)$$

If we choose $x_0\ll 1$ and $e^{r'}\gg 1$, we are in the regime in which the approximations which we have made in deriving Eq. (5.25) are valid. Therefore, we can achieve a $1/N$ accuracy with this device as well.

Note that this interferometer achieves a $1/N$ accuracy at lower levels of su(2) squeezing than does the Mach-Zehnder interferometer. In particular, we found that the squeezing parameter λ must be of order j or larger to achieve $1/N$ accuracies in the Mach-Zehnder interferometer, while in the modified device this accuracy can be achieved when λ is much smaller than j.

It is of interest to see if the approximate su(2) squeezed states which we found in Sec. IV are useful here. The answer, with some modifications to the states and the interferometer, is yes. First, note that if we do not apply the final rotation in Eq. (4.7), the resulting state is approximately a minimum-uncertainty state J_1 and J_3, which is what we require for $|\Psi\rangle$ in our modified interferometer. On the other hand, it is J_3 rather than J_1 which is squeezed, which is the opposite of what is desired. This can be corrected by applying a rotation of $\pi/2$ about axis 2, which corresponds to adding another beam splitter to the device. We want the operator S_{12} in Eq. (4.7) to correspond to the action of the four-wave mixer in the interferometer. This means the new beam splitter which accomplishes the rotation about axis 2 must be inserted between the four-wave mixer and the original beam splitter. Therefore, our final device, shown in Fig. 5, consists of three elements, a four-wave mixer described by S_{12} followed by two beam splitters, each of which is described by $\exp(-i\pi J_2/2)$. The input state to the entire device is taken to be $|\text{in}\rangle=s=S_1S_2|u,iu\rangle$, and the resulting phase-measurement accuracy is (assuming that $u\gg e^{4r}$)

$$\delta\phi=e^{-3r}/(\langle \text{in}|N|\text{in}\rangle)^{1/2}\ . \quad (5.26)$$

For moderate values of r, this represents a considerable improvement over the standard $1/\sqrt{N}$ accuracy.

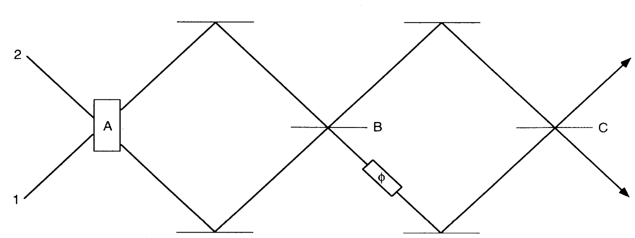

FIG. 5. An interferometer consisting of a four-wave mixer and two beam splitters.

VI. CONCLUSION

We have shown that su(2) minimum-uncertainty states allow one to achieve phase-measurement accuracies of order $1/N$ in both standard and nonstandard interferometers. We have also presented a scheme for generating approximate versions of these states.

The results of this paper and those of Refs. [3] and [4] strongly suggests that interferometers are most usefully analyzed in terms of variables which are quadratic in the mode creation and annihilation operators. Squeezed states—and minimum-uncertainty squeezed states, in particular—should be of considerable use in analyzing the behavior of these devices and in obtaining highly accurate phase measurements.

The utility of these states is not limited to interferometry. As shown by the recent work of Wineland *et al.*, su(2) squeezed atomic states can also be used to improve the accuracy of spectroscopic measurements [15]. The su(2) squeezed minimum-uncertainty states should prove useful here as well.

There are two issues on which further work would be useful. The first is the ultimate accuracy of the interferometric measurements. Yurke, McCall, and Klauder showed that it is at least $1/N$, but can one do better? This question arises especially in the case in which the interferometer includes elements with gain, such as four-wave mixers or degenerate parametric amplifiers. In the case of interferometers containing laser amplifiers, Gea-Banacloche has concluded that active and passive devices have the same accuracies [16]. Whether this is true in general is not known.

The second issue, closely related to the first, is the accuracy which can be obtained in a series of measurements. Measurements on quantum systems generally must be repeated because of the fluctuations inherent in the system, i.e., a series of measurements must be made. As has been shown recently by Braunstein, Lane, and Caves, the analysis of such measurements is far from simple [17].

Despite the fact that interferometers are rather old devices, there are still open questions remaining about their performance. We believe that quadratic squeezed states can play a role in answering some of these questions.

ACKNOWLEDGMENTS

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APPENDIX A

We derive some of the properties of the su(2) minimum-uncertainty states. As was stated in Sec. III, these states are solutions of the eigenvalue equation

$$(J_1+i\lambda J_2)|\psi\rangle=\beta|\psi\rangle\ , \tag{A1}$$

where λ is real. From this equation a number of properties follow immediately. Taking expectation values gives

$$\langle\psi|J_1|\psi\rangle=\mathrm{Re}(\beta)\ ,\quad \langle\psi|J_2|\psi\rangle=(1/\lambda)\mathrm{Im}(\beta)\ . \tag{A2}$$

Operating on both sides of Eq. (A1) with $J_1-i\lambda J_2$, taking expectation values, and making use of Eq. (A2) gives

$$(\Delta J_1)^2+\lambda^2(\Delta J_2)^2=\lambda\langle J_3\rangle\ . \tag{A3}$$

If we apply $J_1+i\lambda J_2$ to both sides of Eq. (A1), take expectation values, and then take the real part, we find

$$(\Delta J_1)^2=\lambda^2(\Delta J_2)^2\ . \tag{A4}$$

Equations (A3) and (A4) yield Eqs. (3.4).

In order to solve Eq. (A1) we first define a state $|\psi'\rangle$ which is related to $|\psi\rangle$ by a rotation about axis 1,

$$|\psi'\rangle=e^{i\theta J_1}|\psi\rangle\ , \tag{A5}$$

where θ is to be determined later. Inserting Eq. (A5) into Eq. (A1), we find

$$[J_1+i\lambda(J_2\cos\theta-J_3\sin\theta)]|\psi'\rangle=\beta|\psi'\rangle\ . \tag{A6}$$

Let us now choose $\cos\theta=-1/\lambda$ and θ to be in the range $\pi\geq\theta\geq\pi/2$. Note that this implies that $|\lambda|\geq 1$. In order to find solutions for $|\lambda|<1$, a different definition of $|\psi'\rangle$ is necessary. Here we shall consider only the case $|\lambda|\geq 1$. With the choice of θ given above, Eq. (A6) becomes

$$(J_- - i\sqrt{\lambda^2-1}J_3)|\psi'\rangle=\beta|\psi'\rangle\ . \tag{A7}$$

If we now expand $|\psi'\rangle$ as

$$|\psi'\rangle=\sum_{m=-j}^{j}c_m|j,m\rangle\ , \tag{A8}$$

Eq. (A7) reduces to the recurrence relation

$$\begin{aligned} c_{m+1}&=\{(\beta+im\sqrt{\lambda^2-1})/[(j+m+1)\\ &\qquad\times(j-m)]^{1/2}\}c_m\ ,\\ &\qquad\qquad m\neq j\ ,\\ c_j(\beta+ij\sqrt{\lambda^2-1})&=0\ ,\quad m=j\ . \end{aligned} \tag{A9}$$

The second equation tells us that either $c_j=0$ or $\beta=-ij\sqrt{\lambda^2-1}$. If $c_j=0$, this implies that one of the coefficients multiplying c_m in the above equation must vanish, i.e., $\beta=-im_0\sqrt{\lambda^2-1}$ for some m_0. Therefore, the coefficients c_m corresponding to the solution $\beta=-im_0\sqrt{\lambda^2-1}$ are given by

$$\begin{aligned} c_m&=\left[\prod_{k=-j}^{m-1}\{[i(k-m_0)\sqrt{\lambda^2-1}]/[(j+k+1)(j-k)]^{1/2}\}\right]c_{-j}\\ &=(-i\sqrt{\lambda^2-1})^{j+m}[(m_0+j)!/(m_0-m)!][(j-m)!/(j+m)!(2j)!]^{1/2}c_{-j} \end{aligned} \tag{A10}$$

for $m \le m_0$ and $c_m = 0$ for $m > m_0$. Finally, grouping the m-independent constants

$$(m_0+j)!c_{-j}/[(2j)!]^{1/2}$$

into the normalization constant $c_{jm_0}(\lambda)$, we obtain Eq. (3.7).

APPENDIX B

We show that the state $|\Phi'\rangle = U_R|\Phi\rangle$, where $|\Phi\rangle$ is defined in Eq. (4.7), leads to an accuracy of $1/\langle N\rangle^{3/4}$ in a Mach-Zehnder interferometer. This means we must evaluate

$$\delta\phi = \Delta J_3|_{\Phi'}/|\langle\Phi'|J_1|\Phi'\rangle| = \Delta J_2|_{\Phi}/|\langle\Phi|J_3|\Phi\rangle| \ . \tag{B1}$$

We shall assume the $\langle N\rangle \gg 1$ and $r \gg 1$ so the we can neglect e^{-r} compared to e^r. In this regime we have, from Eqs. (4.8) and (4.9),

$$\begin{aligned} \delta\phi &= [(u^2/32s)+s]^{1/2}/|u^2-2s| \ , \\ \langle N\rangle &= 2u^2+4s \ , \end{aligned} \tag{B2}$$

where $s = (e^r/2)^4$. What we now want to do is to minimize $\delta\phi$ with respect to u and s, while keeping $\langle N\rangle$ fixed.

In order to accomplish this, we solve the second of the above equations for u^2 and substitute the result into the first. For $\delta\phi$, this gives

$$\delta\phi = [(\langle N\rangle/64s) - \tfrac{1}{16} + s]^{1/2}/|(\langle N\rangle/2) - 4s| \ . \tag{B3}$$

Squaring this, differentiating with respect to s, and setting the result equal to zero gives

$$4s^3 + (\langle N\rangle s^2/2) + (3\langle N\rangle s/16) - (\langle N\rangle^2/128) = 0 \ . \tag{B4}$$

We are only interested in solutions of this equation which are greater than zero because of the definition of s. The solution satisfying this condition is approximately (to highest order in $\langle N\rangle$)

$$s = (\langle N\rangle)^{1/2}/8 \ . \tag{B5}$$

Substitution of this result into Eq. (B3) yields

$$\delta\phi = (1/\langle N\rangle)^{3/4} \ . \tag{B6}$$

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