Supersqueezed states

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We derive the supersqueeze operator for the supersymmetric harmonic oscillator, using Baker-Campbell-Hausdorff relations for the supergroup OSP(2/2). Combining this with the previously obtained superdisplacement operator, we derive the supersqueezed states. These are the supersymmetric generalization of the squeezed states of the harmonic oscillator.

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I. INTRODUCTION

Over the past 30 years much work has been done on coherent states [1–9] especially in the field of quantum optics. Beyond the harmonic-oscillator system, coherent states have also been developed for quantum (Schrödinger) systems with general potentials and for general Lie symmetries. These states are called (general) minimum-uncertainty coherent states [9] and (general) displacement-operator coherent states [3, 6–8].

There is also a different generalization of the coherent states of the harmonic-oscillator system. This is the concept of "squeezed" states [10, 11]. Most notably, squeezed states have been used in the context of quantum optics and in the context of gravitational wave detection. In quantum optics they describe the cases of "antibunched" and "bunched" light [12]. In gravitational wave detection they are used to describe "quantum nondemolition" or "action-back-evading" measurements [13].

Recently, we generalized [14] the concept of coherent states to supersymmetric systems, using the displacement-operator method for supergroups. We call the resulting states supercoherent states. For other approaches to this problem, see Refs. [15–20]. Our calculations require the use of a general technique for constructing Baker-Campbell-Hausdorff (BCH) relations [21–25] for supergroups, which had recently been developed [26–29]. Throughout our work, we have employed Rogers's definition [30] of supermanifolds and supergroups.

In Ref. [14], we discussed three systems: (i) the super Heisenberg-Weyl algebra, which defines the supersymmetric harmonic oscillator; (ii) an electron in a constant magnetic field, which is a supersymmetric quantum-mechanical system with a Heisenberg-Weyl algebra plus another bosonic degree of freedom; and (iii) the electron-monopole system, which has an OSP(1/2) supersymmetry. An obvious follow-up question was whether our supercoherent states for the harmonic oscillator could be generalized to supersqueezed states. It is the purpose of this paper to demonstrate the positive answer to this question.

In Sec. II we review the coherent and squeezed states of the harmonic oscillator from the minimum-uncertainty point of view. In Sec. III we do the same from the displacement-operator point of view, pointing out the equivalence of the two formulations [10]. In Sec. IV we review our harmonic-oscillator supercoherent states.

Our main objective is to obtain a supersqueeze operator and supersqueezed states incorporating bosonic and fermionic sectors of the OSP(2/2) supergroup. We start our derivation in Sec. V by obtaining the differential equations whose solutions enable us to write the supersqueeze operator as a product of the exponentials of single algebra elements. This is done using BCH relations for the supergroup OSP(2/2). We give the solution of the equations in Sec. VI. Section VII focuses on the case where only odd operators are involved in the squeeze. This yields fermionic squeezed states. The general supersqueezed states are obtained in Sec. VIII. We conclude with a discussion in Sec. IX.

II. MINIMUM-UNCERTAINTY COHERENT AND SQUEEZED STATES

The harmonic-oscillator Hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 \tag{1}$$

is quadratic in the operators x and p, which classically vary as $\sin(\omega t)$ and $\cos(\omega t)$. The commutation relation of the associated quantum operators ($\hbar = 1$)

$$[x, p] = i \tag{2}$$

defines an uncertainty relation

$$(\Delta x)^2(\Delta p)^2 \geq 1/4. \tag{3}$$

The minimum-uncertainty coherent states for the harmonic-oscillator potential can be defined as those *states that minimize the uncertainty relation (3), subject to the added constraint that the ground state is a member of the set.*

The states that minimize the uncertainty relation (3) are

$$\psi(x) = [2\pi\sigma^2]^{-1/2} \exp\left[-\left(\frac{x-x_0}{2\sigma}\right)^2 + ip_0x\right], \quad (4)$$

$$\sigma = s\sigma_0 = s/[2m\omega]^{1/2}. \quad (5)$$

When $s=1$, these Gaussian functions have the width of the ground state, so they are the coherent states. The states are labeled by two parameters $x_0 = \langle x\rangle$ and $p_0 = \langle p\rangle$.

The squeezed states of the harmonic oscillator can also be found from the minimum-uncertainty point of view [10]. In Eqs. (4) and (5) simply let $s \neq 1$. *The squeezed states of the harmonic oscillator are minimum-uncertainty Gaussian functions whose widths are not necessarily that of the ground state.* These states form a continuous three-parameter set. Their uncertainty product evolves with time as

$$[\Delta x(t)]^2[\Delta p(t)]^2 = \frac{1}{4}\left[1 + \frac{1}{4}\left(s^2 - \frac{1}{s^2}\right)^2 \sin^2(2\omega t)\right]. \quad (6)$$

III. DISPLACEMENT-OPERATOR COHERENT AND SQUEEZED STATES

Consider the displacement-operator approach using the oscillator algebra defined by $a, a^\dagger, a^\dagger a$, and I. The displacement-operator is the unitary exponentiation of the elements of the factor algebra, spanned by a and $a^\dagger$:

$$D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a] = \exp\left[-\frac{1}{2}|\alpha|^2\right] \exp[\alpha a^\dagger] \exp[-\alpha^* a], \quad (7)$$

where the last equality comes from using a BCH relation. *The displacement-operator coherent states of the harmonic oscillator are obtained by applying the displacement operator $D(\alpha)$ on an extremal state, i.e., the ground state.* Specifically, this yields

$$D(\alpha)|0\rangle = \exp[\alpha a^\dagger - \alpha^* a]|0\rangle = \exp\left[-\frac{1}{2}|\alpha|^2\right] \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \equiv |\alpha\rangle, \quad (8)$$

where $|n\rangle$ are the number states. With the identifications

$$\mathrm{Re}(\alpha) = [m\omega/2]^{1/2} x_0, \qquad \mathrm{Im}(\alpha) = p_0/[2m\omega]^{1/2}, \quad (9)$$

these are the same as the minimum-uncertainty coherent states, up to an irrelevant phase factor.

Obtaining the displacement-operator squeezed states for the harmonic oscillator from the coherent states is more complicated than with the minimum-uncertainty method. One starts with the "unitary squeeze operator"

$$S(z) = \exp\left[z\frac{a^\dagger a^\dagger}{2} - z^*\frac{aa}{2}\right] \quad (10)$$

$$\equiv \exp\left[G_+\frac{a^\dagger a^\dagger}{2}\right] \exp\left[G_0\frac{(a^\dagger a + \frac{1}{2})}{2}\right] \exp\left[G_-\frac{aa}{2}\right] \quad (11)$$

$$= \exp\left[e^{i\phi}(\tanh r)\frac{a^\dagger a^\dagger}{2}\right] \left(\frac{1}{\cosh r}\right)^{(1/2 + a^\dagger a)} \exp\left[-e^{-i\phi}(\tanh r)\frac{aa}{2}\right], \quad (12)$$

where Eq. (12) is obtained from a BCH relation [31, 32] and $z \equiv re^{i\phi}$. A normal-ordered form for the second term in Eq. (12) is [10]

$$\left(\frac{1}{\cosh r}\right)^{(1/2+a^\dagger a)} = \left(\frac{1}{\cosh r}\right)^{1/2} \left[\sum_{n=0}^{\infty} \frac{(\mathrm{sech}\, r - 1)^n}{n!}(a^\dagger)^n (a)^n\right]. \quad (13)$$

The squeezed states equivalent to the ψ of Eqs. (4) and (5) are obtained by operating on the ground state by

$$T(\alpha, z)|0\rangle = D(\alpha)S(z)|0\rangle \equiv |(\alpha, z)\rangle, \quad (14)$$

$$z \equiv re^{i\phi}, \qquad r = \ln s. \quad (15)$$

Here ϕ is a phase which defines the starting time $t_0 = (\phi/2\omega)$ and s is the wave-function squeeze of Eq. (5). Note that $S(z)$ by itself can be considered to be the displacement operator for the group SU(1,1) defined by

$$K_+ = \frac{1}{2}a^\dagger a^\dagger, \quad K_- = \frac{1}{2}aa, \quad K_0 = \frac{1}{2}\left(a^\dagger a + \frac{1}{2}\right), \quad (16)$$

so that the $S(z)|0\rangle$ by themselves form SU(1,1) coherent states.

IV. SUPERCOHERENT STATES

The displacement-operator supercoherent states of the harmonic oscillator are obtained from the super Heisenberg-Weyl algebra, defined by

$$[a, a^\dagger] = I, \qquad \{b, b^\dagger\} = I. \tag{17}$$

Using lemma 1 of Ref. [26], one obtains that the superdisplacement operator is [14]

$$\mathcal{D}(A,\theta) = \exp[Aa^\dagger - \overline{A}a + \theta b^\dagger + \overline{\theta} b] \tag{18}$$

$$\equiv D_B(A)\, D_F(\theta), \tag{19}$$

where

$$D_B(A) = \left(\exp[-\tfrac{1}{2}|A|^2]\exp[Aa^\dagger]\exp[-\overline{A}a]\right) \tag{20}$$

and

$$D_F(\theta) = \left(\exp[-\tfrac{1}{2}\overline{\theta}\theta]\exp[\theta b^\dagger]\exp[\overline{\theta} b]\right). \tag{21}$$

The B and F subscripts denote the fact that our supersymmetric displacement operator can be written as a product of "bosonic" and "fermionic" (more properly, even and odd) displacement operators.

The variables θ and $\overline{\theta}$ are Grassmann odd. They are nilpotent (they only contain a "soul") and satisfy anticommutation relations among themselves and with the fermion operators b and $b^\dagger$. The variables A and $\overline{A}$ are Grassmann even. Explicit calculation yields

$$\mathcal{D}(A,\theta)|0,0\rangle = (1 - \tfrac{1}{2}\overline{\theta}\theta)|A,0\rangle + \theta|A,1\rangle. \tag{22}$$

The two labels A and ν, with $\nu = 0, 1$, in Eq. (22) represent the even (bosonic) and odd (fermionic) sectors. The bosonic state $|A\rangle$ is a superposition of the the number states $|n\rangle$ with the form of an ordinary coherent state given in Eq. (8). The fermionic displacement acting alone produces a Grassmann-valued linear combination of the states $|0,0\rangle$ and $|0,1\rangle$. We refer the reader to Ref. [14] for further details of this construction.

V. DIFFERENTIAL EQUATIONS FOR THE SUPERSQUEEZE OPERATOR

The preceding sections show that we desire the supersymmetric generalization of the SU(1,1) squeeze operator of Eqs. (10)–(12). The symmetry involved is the supergroup OSP(2/2). In addition to the su(1,1) algebra elements of Eq. (16), it has five more:

$$M_0 = \frac{1}{2}\left(b^\dagger b - \frac{1}{2}\right),$$
$$Q_1 = \frac{1}{2}a^\dagger b^\dagger, \qquad Q_2 = \frac{1}{2}ab, \tag{23}$$
$$Q_3 = \frac{1}{2}a^\dagger b, \qquad Q_4 = \frac{1}{2}ab^\dagger.$$

The graded commutation relations among the eight elements follow from Eq. (17). (See the Appendix.) Therefore, by using BCH relations for this supergroup, the supersqueeze operator can in principle be written as

$$\mathcal{S}(g) = \exp\left[\sum_{i=1}^{6}\alpha_i\hat{g}_i\right] = \prod_{i=1}^{8}\exp[\beta_i g_i], \tag{24}$$

where $\hat{g}$ is the factor algebra.

We next obtain the differential equations needed to solve Eq. (24), using the general method developed in Ref. [26]. Consider the following parametrization of the supersqueeze operator:

$$\begin{aligned}\mathcal{S}(Z,\theta_j,t,\hat{g}) &= \exp[t(ZK_+ - \overline{Z}K_- + \theta_1 Q_1 + \overline{\theta_1}Q_2 + \overline{\theta_2}Q_3 + \theta_2 Q_4)] \\ &= e^{\gamma_+K_+}e^{\gamma_0K_0}e^{\gamma_-K_-}e^{\beta_1Q_1}e^{\mu M_0}e^{\beta_4Q_4}e^{\beta_3Q_3}e^{\beta_2Q_2} \\ &\equiv \mathcal{S}_1(\mu,\gamma_i,\beta_k,t,g).\end{aligned} \tag{25}$$

Here, the Grassmann-valued variable Z and its complex conjugate $\overline{Z}$ are even, while the Grassmann-valued variables $\{\theta_j\} \equiv \{\theta_1, \theta_2, \overline{\theta_1}, \overline{\theta_2}\}$ are odd. In writing the product form of Eq. (25), we chose an ordering that yields relative ease of calculation as well as approximate normal ordering. Since the fermionic operators are nilpotent, we ordered them to the right of the purely bosonic operators. The position of M_0 was also chosen for calculational convenience.

Since μ, the γ_j, and the β_k are functions of t, by taking the derivative of Eq. (25) with respect to t and then multiplying on the right by $\mathcal{S}^{-1}$, one has

$$\left[\frac{d}{dt}\mathcal{S}\right]\mathcal{S}^{-1} = \left[\frac{d}{dt}\mathcal{S}_1\right]\mathcal{S}_1^{-1}. \tag{26}$$

This can explicitly be written as (a dot over a quantity signifies $\frac{d}{dt}$)

$$\begin{aligned}&[ZK_+ - \overline{Z}K_- + \theta_1Q_1 + \overline{\theta}_1Q_2 + \overline{\theta}_2Q_3 + \theta_2Q_4] \\ &\quad= \dot{\gamma}_+K_+ + [e^{\gamma_+K_+}]\dot{\gamma}_0K_0[e^{-\gamma_+K_+}] + [e^{\gamma_+K_+}e^{\gamma_0K_0}]\dot{\gamma}_-K_-[e^{-\gamma_0K_0}e^{-\gamma_+K_+}] + S_B\dot{\beta}_1Q_1S_B^{-1} \\ &\qquad + S_B[e^{\beta_1Q_1}]\dot{\mu}M_0[e^{-\beta_1Q_1}]S_B^{-1} + S_B[e^{\beta_1Q_1}e^{\mu M_0}]\dot{\beta}_4Q_4[e^{-\mu M_0}e^{-\beta_1Q_1}]S_B^{-1} \\ &\qquad + S_B[e^{\beta_1Q_1}e^{\mu M_0}e^{\beta_4Q_4}]\dot{\beta}_3Q_3[e^{-\beta_4Q_4}e^{-\mu M_0}e^{-\beta_1Q_1}]S_B^{-1} \\ &\qquad + S_B[e^{\beta_1Q_1}e^{\mu M_0}e^{\beta_4Q_4}e^{\beta_3Q_3}]\dot{\beta}_2Q_2[e^{-\beta_3Q_3}e^{-\beta_4Q_4}e^{-\mu M_0}e^{-\beta_1Q_1}]S_B^{-1},\end{aligned} \tag{27}$$

where

$$S_B = [e^{\gamma_+ K_+} e^{\gamma_0 K_0} e^{\gamma_- K_-}]. \tag{28}$$

Note that S_B has a structure analogous to the ordinary squeeze operator defined in Eq. (10).

All the terms on the right-hand side of Eq. (27) can be written in nonexponential form by using BCH formulas and the graded commutation relations. One has

$$[ZK_+ - \overline{Z}K_- + \theta_1 Q_1 + \overline{\theta}_1 Q_2 + \overline{\theta}_2 Q_3 + \theta_2 Q_4]$$
$$= \dot{\gamma}_+ K_+ + [\dot{\gamma}_0 K_0 - \dot{\gamma}_0 \gamma_+ K_+] + \cdots . \tag{29}$$

The terms left out of the above Eq. (29) become longer and longer, but can be calculated. When this is done, eight equations for the eight variables emerge by extracting the coefficients of each of the eight generators of osp(2/2), i.e., one equation for each factor multiplying K_+, K_0, etc. The coefficients yield the following equations:

$$0 = \dot{\mu} + \frac{1}{2}\dot{\beta}_3\beta_4 - \frac{1}{2}\dot{\beta}_2\beta_1 e^{-\mu/2}, \tag{30}$$

$$\begin{aligned} Z = {} & \dot{\gamma}_+ - \dot{\gamma}_0\gamma_+ + \dot{\gamma}_-\gamma_+^2 e^{-\gamma_0} \\ & + \dot{\beta}_3 \left\{ \frac{1}{2}\beta_1 e^{-\mu/2} F_-^2 - \frac{1}{2}\beta_4 [\gamma_+ e^{-\gamma_0/2}] F_- \right\} \\ & + \dot{\beta}_2 \left\{ -\frac{1}{2}\beta_1 e^{-\mu/2} [\gamma_+ e^{-\gamma_0/2}] F_- + \frac{1}{2}\beta_4 [\gamma_+ e^{-\gamma_0/2}]^2 \right\}, \end{aligned} \tag{31}$$

$$\begin{aligned} 0 = {} & \dot{\gamma}_0 - 2\dot{\gamma}_-\gamma_+ e^{-\gamma_0} + \dot{\beta}_3 \left\{ \beta_1 e^{-\mu/2} [\gamma_- e^{-\gamma_0/2}] F_- \right. \\ & \left. + \frac{\beta_4}{2} [1 - 2\gamma_-\gamma_+ e^{-\gamma_0}] \right\} \\ & + \dot{\beta}_2 \left\{ \frac{1}{2}\beta_1 e^{-\mu/2} [1 - 2\gamma_-\gamma_+ e^{-\gamma_0}] - \beta_4 [\gamma_+ e^{-\gamma_0}] \right\}, \end{aligned} \tag{32}$$

$$\begin{aligned} -\overline{Z} = {} & \dot{\gamma}_- e^{-\gamma_0} + \dot{\beta}_3 \left\{ \frac{1}{2}\beta_1 e^{-\mu/2} [\gamma_-^2 e^{-\gamma_0}] + \frac{1}{2}\beta_4 [\gamma_- e^{-\gamma_0}] \right\} \\ & + \dot{\beta}_2 \left\{ \frac{1}{2}\beta_1 e^{-\mu/2} [\gamma_- e^{-\gamma_0}] + \frac{1}{2}\beta_4 [e^{-\gamma_0}] \right\}, \end{aligned} \tag{33}$$

$$\begin{aligned} \theta_1 = {} & \left[\dot{\beta}_1 - \frac{1}{2}\dot{\mu}\beta_1 + \frac{1}{2}\dot{\beta}_3\beta_1\beta_4 \right] F_- \\ & - \left[\dot{\beta}_4 e^{\mu/2} + \frac{1}{2}\dot{\beta}_2\beta_1\beta_4 \right] [\gamma_+ e^{-\gamma_0/2}], \end{aligned} \tag{34}$$

$$\overline{\theta}_1 = \dot{\beta}_3 e^{-\mu/2} [\gamma_- e^{-\gamma_0/2}] + \dot{\beta}_2 e^{-\mu/2} [e^{-\gamma_0/2}], \tag{35}$$

$$\overline{\theta}_2 = \dot{\beta}_3 e^{-\mu/2} F_- - \dot{\beta}_2 e^{-\mu/2} [\gamma_+ e^{-\gamma_0/2}], \tag{36}$$

$\theta_2 = [\dot{\beta}$ [illegible]

$$+ \left[\dot{\beta}_4 e^{\mu/2} + \frac{1}{2}\dot{\beta}_2\beta_1\beta_4 \right] e^{-\gamma_0/2}, \tag{37}$$

where

$$F_\mp \equiv [e^{\gamma_0/2} \mp \gamma_-\gamma_+ e^{-\gamma_0/2}]. \tag{38}$$

Each of the equations (30)–(37) is linear in time derivatives, so with some algebra an equation can be found for each of the eight t derivatives:

$$\begin{aligned} \dot{\mu} = {} & \frac{1}{2}[\overline{\theta}_1 F_- - \gamma_- e^{-\gamma_0/2}\overline{\theta}_2]\beta_1 \\ & - \frac{1}{2} e^{\mu/2} e^{-\gamma_0/2} [\gamma_+\overline{\theta}_1 + \overline{\theta}_2]\beta_4, \end{aligned} \tag{39}$$

$$\dot{\gamma}_+ = +Z - \overline{Z}\gamma_+^2 - \frac{1}{2} e^{\gamma_0/2} [\gamma_+\overline{\theta}_1 + \overline{\theta}_2]\beta_1, \tag{40}$$

$$\begin{aligned} \dot{\gamma}_0 = {} & -2\overline{Z}\gamma_+ - \frac{1}{2}[F_+\overline{\theta}_1 + \gamma_- e^{-\gamma_0/2}\overline{\theta}_2]\beta_1 \\ & - \frac{1}{2} e^{\mu/2} e^{-\gamma_0/2} [\gamma_+\overline{\theta}_1 + \overline{\theta}_2]\beta_4, \end{aligned} \tag{41}$$

$$\dot{\gamma}_- = -\overline{Z} e^{\gamma_0} - \frac{1}{2} e^{\gamma_0/2}\overline{\theta}_1 [\gamma_-\beta_1 + e^{\mu/2}\beta_4], \tag{42}$$

$$\begin{aligned} \dot{\beta}_1 = {} & e^{-\gamma_0/2}[\theta_1 + \gamma_+\theta_2] \\ & + \frac{1}{4} e^{\mu/2} e^{-\gamma_0/2} [-\gamma_+\overline{\theta}_1 - \overline{\theta}_2]\beta_1\beta_4, \end{aligned} \tag{43}$$

$$\dot{\beta}_2 = e^{\mu/2}[F_-\overline{\theta}_1 - \gamma_- e^{-\gamma_0/2}\overline{\theta}_2], \tag{44}$$

$$\dot{\beta}_3 = e^{\mu/2} e^{-\gamma_0/2} [\overline{\theta}_1\gamma_+ + \overline{\theta}_2], \tag{45}$$

$$\begin{aligned} \dot{\beta}_4 = {} & e^{-\mu/2}[-\gamma_- e^{-\gamma_0/2}\theta_1 + F_-\theta_2] \\ & + \frac{1}{2}[-F_-\overline{\theta}_1 + \gamma_- e^{-\gamma_0/2}\overline{\theta}_2]\beta_1\beta_4. \end{aligned} \tag{46}$$

These are the differential equations whose solutions yield the group parameters for the supersqueezed states. Note that the boundary conditions needed for these equations are that the solutions must all be zero when $t = 0$. Then, the supersqueeze operator will be obtained when we set $t = 1$.

VI. SOLUTION FOR THE SUPERSQUEEZE OPERATOR

Equations (39)–(46) can be separated into twenty coupled differential equations. This can be seen by expanding the group parameters in powers of the odd variables θ_j, substituting into the eight equations, and collecting coefficients. First, the four even group parameters $\{\mu, \gamma_+, \gamma_0, \gamma_-\}$ can each be written as having three terms, containing products of zero, two, or four of the θ_j, respectively. Second, the four odd group parameters $\{\beta_k\}$ can be written as having two terms, containing products of one or three of the θ_j, respectively. We use a presubscript to denote this, e.g.,

$$\mu = ({}_0\mu) + ({}_2\mu) + ({}_4\mu), \qquad \beta_1 = ({}_1\beta_1) + ({}_3\beta_1). \tag{47}$$

One takes the eight equations (39)–(46) and expands all of the expressions in powers of the θ_j. The order-zero, -two, and -four pieces of the even equations are separated and, similarly, the order-one and -three pieces of the odd equations are separated. Note that the lower-order solutions are placed into the higher-order equations.

To solve the equations, one first observes that $({}_0\mu) = 0$. The equation for $({}_0\gamma_+)$ is a Ricatti equation that is solved by the usual procedure, e.g., as is done to obtain the left-hand term of the normal squeeze operator s in Eq. (12) [31]. This solution is substituted into the equation for $({}_0\gamma_0)$, whose solution is in turn put into the equation for $({}_0\gamma_-)$. Except as noted, e.g., for $({}_0\gamma_+)$ above, the differential equations are all simple in the sense that there is a t derivative of a group parameter on the left and only powers and hyperbolic functions of t on the right. Proceeding, and substituting all previous solutions into subsequent equations, one directly solves for $({}_1\beta_1)$, $({}_1\beta_2)$, $({}_1\beta_3)$, $({}_1\beta_4)$, and $({}_2\mu)$.

The equation for $({}_2\gamma_+)$ is an inhomogeneous first-order equation of the form

$$\dot{q}(t) = k(t)q(t) + f(t). \tag{48}$$

Its solution is obtained in a standard way:

$$q(t) = q^H(t)\int_0^t \frac{f(\nu)}{q^H(\nu)}d\nu, \tag{49}$$

where q^H is the solution to the homogeneous equation $(f = 0)$.

One can then proceed to solve directly for $({}_2\gamma_0)$, $({}_2\gamma_-)$, $({}_3\beta_1)$, $({}_3\beta_2)$, $({}_3\beta_3)$, $({}_3\beta_4)$, and $({}_4\mu)$. The equations become more complicated, but obtaining the solutions remains mainly a question of careful Grassmann-valued algebra. With $({}_4\gamma_+)$ one has another first-order inhomogeneous differential equation, whose solution is obtained as above. Finally, the solutions are completed with $({}_4\gamma_0)$ and $({}_4\gamma_-)$.

In presenting the solutions, we introduce the suggestive notation

$$r \equiv [Z\overline{Z}]^{1/2}, \quad e^{i\phi} \equiv [Z/\overline{Z}]^{1/2}, \tag{50}$$

where r and $e^{i\phi}$ are now understood to represent Grassmann-valued quantities. Then, one can make the replacements

$$Z \to re^{i\phi}, \quad \overline{Z} \to re^{-i\phi}. \tag{51}$$

Some care is needed because the quantity $e^{i\phi}$ is strictly defined only for $|\overline{Z}| \neq 0$ and $\overline{z} \neq 0$, where $\overline{z}$ is the body of $\overline{Z}$. However, the solutions given below are not affected by this. We also define

$$c \equiv \cosh y, \quad s \equiv \sinh y, \quad y \equiv rt, \tag{52}$$

$$\Phi \equiv \overline{\theta_2}\theta_2\overline{\theta_1}\theta_1 = \overline{\theta_2}\overline{\theta_1}\theta_1\theta_2. \tag{53}$$

Then, the complete solutions for the group parameters are

$$\mu = 0 + \frac{1}{2r^2}\{[\overline{\theta}_1\theta_1 - \overline{\theta}_2\theta_2](c-1) + [\overline{\theta}_2\theta_1 e^{-i\phi} - \overline{\theta}_1\theta_2 e^{i\phi}](s-y)\} + \frac{\Phi}{r^4}\left[c - 1 - \frac{1}{2}sy\right], \tag{54}$$

$$\begin{aligned}\gamma_+ = {} & \left[e^{i\phi}\frac{s}{c}\right] - \frac{e^{i\phi}}{4r^2c^2}[\overline{\theta}_1\theta_1(sc-y) + e^{i\phi}\overline{\theta}_1\theta_2(c-1)^2 + e^{-i\phi}\overline{\theta}_2\theta_1 s^2 + \overline{\theta}_2\theta_2(sc+y-2s)] \\ & + \frac{\Phi e^{i\phi}}{8r^4c^3}\left[(2y + sy^2 - s) + c\left(\frac{11}{8}y - 2s\right) + \left(-\frac{5}{8}sc^2 + \frac{1}{4}sc^4\right)\right],\end{aligned} \tag{55}$$

$$\begin{aligned}\gamma_0 = {} & [-2\ln c] + \frac{1}{2r^2}\left[\overline{\theta}_1\theta_1\left(\frac{-ys}{c} + c - 1\right) + e^{i\phi}\overline{\theta}_1\theta_2\left(\frac{s}{c} - s\right) + e^{-i\phi}\overline{\theta}_2\theta_1\left(-\frac{s}{c} + s\right) + \overline{\theta}_2\theta_2\left(\frac{2+ys}{c} - c - 1\right)\right] \\ & + \frac{\Phi}{8r^4c^2}\left[(y^2 - 1 - 2ys) - c\left(\frac{11}{4}ys + 4\right) + c^2\left(2\ln c + 8c - 3 - 4ys - \frac{1}{4}s^2\right)\right],\end{aligned} \tag{56}$$

$$\begin{aligned}\gamma_- = {} & \left[-e^{-i\phi}\frac{s}{c}\right] + \left(\frac{e^{-i\phi}}{4r^2c^2}\right)[\overline{\theta}_1\theta_1(sc-y) - e^{i\phi}\overline{\theta}_1\theta_2 s^2 - e^{-i\phi}\overline{\theta}_2\theta_1(c-1)^2 + \overline{\theta}_2\theta_2(sc+y-2s)] \\ & - \frac{\Phi e^{-i\phi}}{8r^4c^3}\left[(2y + sy^2 - s) + c\left(\frac{11}{8}y - 2s\right) + sc^2\left(\frac{15}{8} + 2\ln c\right) - \frac{9}{4}c^3 y\right],\end{aligned} \tag{57}$$

$$\beta_1 = \frac{1}{r}[s\theta_1 + (c-1)e^{i\phi}\theta_2] + \frac{1}{4r^3}[\overline{\theta}_2\theta_1\theta_2(y - 2cs + yc) + e^{i\phi}\overline{\theta}_1\theta_1\theta_2(2c(1-c) + ys)], \tag{58}$$

$$\beta_2 = \frac{1}{r}[s\overline{\theta}_1 + (c-1)e^{-i\phi}\overline{\theta}_2] + \frac{1}{4r^3}\left[\overline{\theta}_2\theta_1\theta_2\left(yc - s + \frac{1}{2}(sc-y)\right) + \overline{\theta}_2\overline{\theta}_1\theta_1 e^{-i\phi}\left(ys - 3(c-1) - \frac{1}{2}s^2\right)\right], \tag{59}$$

$$\beta_3 = \frac{1}{r}[(c-1)e^{i\phi}\overline{\theta}_1 + s\overline{\theta}_2] + \frac{1}{4r^3}\{e^{i\phi}\overline{\theta}_2\overline{\theta}_1\theta_1 2[ys - 2(c-1)] + \overline{\theta}_2\overline{\theta}_1\theta_1 2(yc-s)a\}, \tag{60}$$

$$\beta_4 = \frac{1}{r}[(c-1)e^{-i\phi}\theta_1 + s\theta_2] + \frac{1}{4r^3}[e^{-i\phi}\overline{\theta}_2\theta_1\theta_2(-4c^2 + 4c + 2ys) + \overline{\theta}_1\theta_1\theta_2(-4sc + 2s + 2yc)]. \tag{61}$$

Setting $t = 1$ yields the general supersqueeze group parameters. In zero and first order, one notices a symmetry among the parameters. Compare, for example, $({}_0\gamma_+)$ with $({}_0\gamma_-)$ and $({}_0\beta_1)$ with $({}_1\beta_4)$. The symmetry remains partial all the way up to fourth order; e.g., two of the three components of $({}_4\gamma_+)$ and $({}_4\gamma_-)$ are identical. It is even more evident in the $Z \to 0$ limit discussed in the next section. The symmetry would be modified if the position of group element $\exp[\mu M_0]$ were different, say one place to the right or left in Eq. (25).

VII. FERMIONIC SQUEEZED STATES

As with the superdisplacement operator, the supersqueeze operator can be separated into a product of bosonic and fermionic pieces:

$$\mathcal{S}(g) = S_B(g)\, S_F(g). \tag{62}$$

Therefore, the supersqueezed states are, in general, of the form

$$\begin{aligned}\mathcal{T}(g)|0,0\rangle &= \mathcal{D}\mathcal{S}|0,0\rangle \\ &= D_B(A)D_F(\theta)S_B(g)S_F(g)|0,0\rangle \\ &= [D_B(A)S_B(g)][D_F(\theta)S_F(g)]|0,0\rangle \\ &\equiv T_B(g)T_F(g)|0,0\rangle. \end{aligned}\tag{63}$$

The general operator produces a linear combination of states $|n,\nu\rangle$ with arbitrary $n = 0, 1, 2, \ldots$ and $\nu = 0$ or 1.

There is an interesting distinction between the superdisplacement and the supersqueeze operators. The superdisplacement operator can be written as

$$\mathcal{D}(A,\theta) = D_B(A)D_F(\theta). \tag{64}$$

Thus, the bosonic displacement operator D_B depends only on the even Grassmann variables A and $\overline{A}$, and the fermionic displacement operator D_F depends only on the odd Grassmann variables θ and $\overline{\theta}$. However, the same is not true of the bosonic and fermionic squeeze operators. There both operators depend on both even and odd Grassmann variables:

$$\mathcal{S}(Z,\theta_j) = S_B(Z,\theta_j)S_F(Z,\theta_j). \tag{65}$$

This is because the nonzero graded commutation relations of the supersqueeze algebra mix the even and odd elements of the algebra, something which does not happen in the case of the superdisplacement operator for the coherent states.

The distinction can be seen more clearly by taking limits. In the limit $\theta_j \to 0$, one is left with a bosonic squeeze that has form analogous to that of the ordinary squeeze operator in Eq. (12):

$$\mathcal{S}(Z,0) = S_B(Z,0) = S(Z). \tag{66}$$

However, when Z goes to zero the situation is quite different. For present purposes, this is equivalent to taking the limit $r = |Z\overline{Z}|^{1/2} \to 0$ in Eqs. (54)–(61). One finds

$$\mu = 0 + \frac{1}{4}[\overline{\theta}_1\theta_1 - \overline{\theta_2}\theta_2]t^2 - \frac{1}{24}\Phi t^4, \tag{67}$$

$$\gamma_+ = 0 - \frac{1}{4}\overline{\theta}_2\theta_1 t^2 + 0, \tag{68}$$

$$\gamma_0 = 0 - \frac{1}{4}[\overline{\theta}_1\theta_1 + \overline{\theta}_2\theta_2]t^2 - \frac{1}{16}\Phi t^4, \tag{69}$$

$$\gamma_- = 0 - \frac{1}{4}\overline{\theta}_1\theta_2 t^2 + 0, \tag{70}$$

$$\beta_1 = \theta_1 t - \frac{5}{24}\overline{\theta}_2\theta_1\theta_2 t^3, \tag{71}$$

$$\beta_2 = \overline{\theta}_1 t + \frac{1}{6}\overline{\theta}_2\overline{\theta}_1\theta_2 t^3, \tag{72}$$

$$\beta_3 = \overline{\theta}_2 t + \frac{1}{6}\overline{\theta}_2\overline{\theta}_1\theta_1 t^3, \tag{73}$$

$$\beta_4 = \theta_2 t - \frac{1}{3}\overline{\theta}_1\theta_1\theta_2 t^3. \tag{74}$$

Setting $t = 1$, one sees that

$$\mathcal{S}(0,\theta_j) = S_B(0,\theta_j)S_F(0,\theta_j). \tag{75}$$

Therefore, a fermionic squeeze ($\theta_j \neq 0$) is not defined only in terms of the fermionic squeeze operator S_F. Rather, as shown in Eq. (75), in addition it has a soul part from the bosonic squeeze operator S_B. A fermionic squeeze on $|0,0\rangle$ produces a Grassmann-valued linear combination of the states $|0,0\rangle, |1,1\rangle$, and $|2,0\rangle$. Specifically, from Eqs. (67)–(74) one finds

$$\begin{aligned}\mathcal{S}(0,\theta_i)|0,0\rangle = &\left[1 - \frac{1}{2}\left(\frac{\overline{\theta_1}\theta_1}{4}\right) - \frac{1}{12}\left(\frac{\Phi}{16}\right)\right]|0,0\rangle \\ &+ \left[\frac{\theta_1}{2} - \frac{1}{3}\left(\frac{\overline{\theta_2}\theta_1\theta_2}{8}\right)\right]|1,1\rangle \\ &- \frac{1}{\sqrt{2}}\left(\frac{\overline{\theta_2}\theta_1}{4}\right)|2,0\rangle. \end{aligned}\tag{76}$$

Note that we have associated a factor $\frac{1}{2}$ with each θ_j. This is due to the $\frac{1}{2}$ in the bivariant elements of the osp(2/2) algebra.

This result suggests the possibility of an extension of the above states in the context of field theory, just as the ordinary squeezed states can be extended and then interpreted as "two-photon" coherent states [33]. The squeezing operation involves products of two operators. Therefore, with the bosonic squeeze turned off ($Z = 0$) one expects to excite only two-particle states in the field theory, which, for example, might perhaps be either two photons or one photon and one photino [34].

Therefore, the general fermionic squeezed states, which are defined in the two limits $Z \to 0$ and $A \to 0$, are

$$\begin{aligned}\mathcal{D}(0,\theta)\mathcal{S}(0,\theta_i)|0,0\rangle = &\left[1 - \frac{1}{2}\left(\frac{\overline{\theta_1}\theta_1}{4}\right) - \frac{1}{12}\left(\frac{\Phi}{16}\right)\right]\left[\left(1 - \frac{1}{2}\overline{\theta}\theta\right)|0,0\rangle + \theta|0,1\rangle\right] \\ &+ \left[\frac{\theta_1}{2} - \frac{1}{3}\left(\frac{\overline{\theta_2}\theta_1\theta_2}{8}\right)\right]\left[\left(1 + \frac{1}{2}\overline{\theta}\theta\right)|1,1\rangle + \overline{\theta}|1,0\rangle\right] \\ &+ \left[-\frac{1}{\sqrt{2}}\left(\frac{\overline{\theta_2}\theta_1}{4}\right)\right]\left[\left(1 - \frac{1}{2}\overline{\theta}\theta\right)|2,0\rangle + \theta|2,1\rangle\right]. \end{aligned}\tag{77}$$

VIII. GENERAL SUPERSQUEEZED STATES

The general supersqueezed states are given by

$$\begin{aligned}\mathcal{T}(A,\theta;Z,\theta_j)|0,0\rangle &= D_B(A)D_F(\theta)S_B(Z,\theta_j)S_F(Z,\theta_j)|0,0\rangle \\ &\equiv |A,\theta;Z,\theta_j\rangle. \end{aligned} \quad (78)$$

The structure of these states can be seen as follows. From the definition of S_F, its action can be split into the action of five separate elements. Counting from the right, the first three group elements act as unity, the fourth just multiplies $|0,0\rangle$ by a constant, and the fifth yields a linear combination of the states $|0,0\rangle$ and $|1,1\rangle$. Similarly, S_B can be written as the product of three elements. The first element $\exp[\gamma_- K_-]$ acts as unity and the second element $\exp[\gamma_0 K_0]$ yields a new linear combination of $|0,0\rangle$ and $|1,1\rangle$. Next, one comes to

$$\exp[\gamma_+ K_+] = \exp\{[({}_2\gamma_+) + ({}_4\gamma_+)]K_+\}\exp[({}_0\gamma_+)K_+]. \quad (79)$$

Expand $\exp\{[({}_2\gamma_+) + ({}_4\gamma_+)]K_+\}$ and commute it until it is to the left of $D_B(A)$, using the relation

$$D(\alpha)f(a^\dagger, a) = f(a^\dagger - \alpha^*, a - \alpha)D(\alpha). \quad (80)$$

Write $|1,1\rangle$ as $a^\dagger|0,1\rangle$ and similarly commute $a^\dagger$ through $\exp[({}_0\gamma_+)K_+]$ and further until it also is to the left of $D_B(A)$. Next, commute $D_F(\theta)$ to the right, expand, and apply it to the linear combination of states $|0,0\rangle$ and $|0,1\rangle$. The result is a new linear combination of the states $|0,0\rangle$ and $|0,1\rangle$.

The remaining operator, which multiplies both these states, is $D_B(A)\ \exp[({}_0\gamma_+)K_+]$. This operator acting on a bosonic ground state has the same effect as $D_B(A)S_B(Z)$. That is, it produces a squeezed state, denoted by $|(A,Z)\rangle$, of the form of Eq. (14), but with $\alpha \to A$ and $z \to Z$. This state, combined with a ket in the fermion space, yields $|(A,Z),\nu\rangle$. Therefore, in this supersymmetric system we have produced a linear combination of the states $|(A,Z),0\rangle$ and $|(A,Z),1\rangle$.

Combining all the above together one finds

$$\begin{aligned}\mathcal{T}(A,\theta,Z,\theta_j)|0,0\rangle = |A,\theta;Z,\theta_j\rangle = \hat{\mu}\Gamma_- h_1(a^\dagger)\left[(1-\tfrac{1}{2}\bar{\theta}\theta)\,|(A,Z),0\rangle + \theta\,|(A,Z),1\rangle\right] \\ +\hat{\mu}\Gamma_+\frac{\beta_1}{2}h_2(a^+)\left[\bar{\theta}\,|(A,Z),0\rangle + (1+\tfrac{1}{2}\bar{\theta}\theta)\,|(A,Z),1\rangle\right],\end{aligned} \quad (81)$$

where

$$\hat{\mu} = 1 - \frac{1}{4}[({}_2\mu) + ({}_4\mu)] + \frac{1}{32}({}_2\mu)^2, \quad (82)$$

$$\Gamma_\pm = 1 + \frac{(2\pm 1)}{4}[({}_2\gamma_0) + ({}_4\gamma_0)] + \frac{(2\pm1)^2}{32}({}_2\gamma_0)^2, \quad (83)$$

$$h_1(a^\dagger) = 1 + \frac{1}{2}[({}_2\gamma_+) + ({}_4\gamma_+)](a^\dagger - \overline{A})^2 + \frac{1}{8}({}_2\gamma_+)^2(a^\dagger - \overline{A})^4, \quad (84)$$

$$h_2(a^\dagger) = \frac{(a^\dagger - \overline{A})}{c}\left[1 + \frac{1}{2}({}_2\gamma_+)(a^\dagger - \overline{A})^2\right]. \quad (85)$$

The products in Eq. (81) can be expanded and then, from Grassmann multiplication, reduced to

$$\begin{aligned}\hat{\mu}\Gamma_- h_1(a^\dagger) = \ & 1 - \frac{1}{4}[({}_2\mu) + ({}_4\mu)] + \frac{1}{32}({}_2\mu)^2 + \frac{1}{4}[({}_2\gamma_0) + ({}_4\gamma_0)] + \frac{1}{32}({}_2\gamma_0)^2 \\ & + \frac{1}{2}[({}_2\gamma_+) + ({}_4\gamma_+)](a^\dagger - \overline{A})^2 + \frac{1}{8}({}_2\gamma_+)^2(a^\dagger - \overline{A})^4 \\ & - \frac{1}{16}({}_2\mu)({}_2\gamma_0) + \frac{1}{8}[({}_2\gamma_0) - ({}_2\mu)]({}_2\gamma_+)(a^\dagger - \overline{A})^2,\end{aligned} \quad (86)$$

$$\hat{\mu}\Gamma_+\frac{\beta_1}{2}h_2(a^+) = \left[1 - \frac{1}{4}({}_2\mu) + \frac{3}{4}({}_2\gamma_0) + \frac{1}{2}({}_2\gamma_+)(a^\dagger - \overline{A})^2\right]\left(\frac{a^\dagger - \overline{A}}{c}\right)\frac{({}_1\beta_1)}{2} + (a^\dagger - \overline{A})\frac{({}_3\beta_1)}{2}. \quad (87)$$

The supersqueezed state of Eq. (81) shows one major similarity to the supercoherent state of Eq. (22) and one major difference. It is the same in that it can be written as a linear combination of factors times a squeezed or coherent state in the bosonic sector with occupation number zero or one in the fermionic sector. It is different in that for the supersqueezed state the factors multiplying these states are polynomials in up to four bosonic creation operators on the $|(A,Z),\nu\rangle$ states. Further, in the limit of no fermionic displacement, $\theta \to 0$, $|(A,Z),0\rangle$ is multiplied only by the polynomial h_1 of order four, while $|(A,Z),1\rangle$ is multiplied only by the polynomial h_2 of order three. In the limits $A \to 0$ and $Z \to 0$, the supersqueezed states of Eq. (81) reduce to the fermionic squeezed states of Eq. (77), as expected.

IX. DISCUSSION

Other approaches to coherent states exist, both for specific and for more general supersymmetric systems. There has even been a discussion of a type of squeezed state for the supersymmetric oscillator. One can identify at least three classes of ordinary coherent states: displacement-operator states, annihilation operator states, and minimum-uncertainty states. In this section, we use similar notions to discuss coherent and squeezed states for supersymmetric systems.

A. Displacement-operator states

In our earlier work on supercoherent states [14], we described the construction of a generalized unitary superdisplacement operator. The action of this operator on an extremal state creates supercoherent states with several attractive features. For example, in the special case of the supersymmetric oscillator these supercoherent states exhibit natural extensions of properties of the usual coherent states for the ordinary oscillator. The approach employs Rogers's definition [30] of supermanifolds and supergroups and provides a natural generalization of the group-theoretic approach to ordinary coherent states. We ask the reader to recall that the harmonic-oscillator supersqueezed states introduced in the present paper are defined by the product of the superdisplacement operator and the supersqueeze operator. The discussion in Secs. II and VIII shows that this definition provides a natural generalization of the corresponding construction for the ordinary oscillator. The unitary supersqueeze operator (24), by itself, defines osp(2/2) supercoherent states, just as the normal squeeze operator defines su(1,1) coherent states.

Another program that uses superunitary operators acting on an extremal state was introduced in [15] and subsequently applied in detail to a number of specific supergroups [16–19]. Note that Refs. [18] and [19] have also discussed the role of path integrals in this approach.

In the definition that is the first equality of Eq. (24), it is important to recognize that the coefficients α_i multiplying the bosonic generators of OSP(2/2) are even Grassmann-valued parameters while the coefficients α_i multiplying the fermionic generators are odd Grassmann-valued parameters. In the second equality of Eq. (24), the β_i are all Grassmann-valued: the even ones multiplying bosonic generators and the odd ones multiplying fermionic generators. However, all the β_i are superfunctions of *both* the odd and even α_i. This is analogous to the Baker-Campbell-Hausdorff relations for Lie groups where complex or real canonical coordinates of the second kind are analytic functions of the complex or real canonical coordinates of the first kind. The operators in Eq. (24) are therefore the most general ones within the context of Rogers's theory of supergroups. One might instead use displacement operators or supersqueeze operators that are products of supergroup operators *not* leaving the extremal state fixed and then obtain a normalization afterwards. Another possibility is to choose to omit the souls of the even Grassmann parameters. In either case, the general Grassmann structure is lost.

B. Annihilation-operator states

Among the interesting properties of the supercoherent states of Ref. [14] for the special case of the supersymmetric oscillator is that they are eigenstates of the supersymmetric annihilation operator. This is a generalization of the corresponding statement that the usual coherent states for the ordinary oscillator are eigenstates of the ordinary annihilation operator.

We are aware of one previous attempt to construct squeezed states for supersymmetric systems [35]. This work is based on the formulation of coherent states for supersymmetric systems presented in Ref. [20]. The method of Ref. [20] generates coherent states that are eigenstates of the supersymmetric annihilation operator, without using Grassmann-valued variables. Instead, coherent states are constructed as linear combinations with complex-number coefficients. Reference [35] presented squeezed states for supersymmetric systems constructed by applying the ordinary squeeze operator of Eq. (10) to the two towers of Ref. [20]. This represents an su(1,1) squeeze.

C. Minimum-uncertainty states

For the ordinary harmonic oscillator, coherent states minimize the physical uncertainty product and preserve it in time. Reference [14] shows that our supercoherent states also do this.

The situation in the general case remains a subject for research at the present time. A number of years ago, Ref. [36] proved the following result for compact Lie groups with algebras obeying the commutation relations

$$[J_r, J_s] = iG^t_{rs} J_t. \tag{88}$$

The quadratic Casimir $C_2 = g^{rs} J_r J_s$, constructed as usual from the positive-definite Cartan-Killing metric $g_{rs} = \frac{1}{2} G^q_{rp} G^p_{qs}$, can be used to define an uncertainty $(\Delta J)^2$ by

$$(\Delta J)^2 = \langle g^{rs}(J_r - \langle J_r \rangle)(J_s - \langle J_s \rangle)\rangle \tag{89}$$

$$= \langle C_2 \rangle - g^{rs}\langle J_r \rangle \langle J_s \rangle. \tag{90}$$

Then, $(\Delta J)^2$ is minimized by a maximum weight vector and vectors that are unitarily equivalent.

One can speculate that a generalization of this result holds for superalgebras in that one has a minimum uncertainty for an extremal state and states that are unitarily equivalent to it. This has been shown for some supersymmetric systems [14, 17, 19]. The connection to the annihilation operator method is that the associated operator

$$P = g^{rs}(J_r - \langle J_r \rangle) \tag{91}$$

can be considered the annihilation operator for these coherent states. That is, the supercoherent states, which minimize the Casimir operator quantity $(\Delta J)^2$, are also eigenstates of the operator P.

D. Physical interpretation

Obtaining a complete physical interpretation of supercoherent and supersqueezed states requires understanding in detail the role of Grassmann-valued quantities. In Ref. [14], two physical models were examined: an electron moving in a constant magnetic field and the electron-monopole system. It was shown that the supercoherent states involved superpositions of the number eigenstates with Grassmann-valued coefficients. Some insight into the physical content of these states was acquired by examining the expectation values of various operators. As shown in Eq. (81), the supersqueezed states presented in the present paper are also linear combinations of the eigenstates with Grassmann-valued coefficients. They are the product of a squeezed state in the bosonic sector labeled by the Grassmann-valued parameters A and Z with a linear combination of the kets in the fermionic space.

For both supercoherent and supersqueezed states and in more general contexts, a possible physical interpretation of Grassmann-valued variables remains a topic of speculation and investigation [34, 37–39].

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APPENDIX: SUPERALGEBRAS FOR SUPERCOHERENT AND SUPERSQUEEZED STATES

The super Heisenberg-Weyl algebra contains the odd generators b and $b^\dagger$ and the even generators a, $a^\dagger$, and I. They satisfy the nonzero graded commutation relations:

$$[a, a^\dagger] = I \ , \quad \{b, b^\dagger\} = I \ . \tag{A1}$$

From these operators, we will be able to define the supersymmetric generalization of the squeeze algebra.

The usual squeeze algebra contains the operators

$$K_+ = \frac{1}{2} a^\dagger a^\dagger \ , \quad K_- = \frac{1}{2} aa \ , \quad K_0 = \frac{1}{2}\left(a^\dagger a + \frac{1}{2}\right) \ . \tag{A2}$$

That these operators comprise an su(1,1) Lie algebra can be seen by calculating their commutation relations. They are

$$[K_0, K_\pm] = \pm K_\pm \ , \qquad [K_+, K_-] = -2K_0 \ . \tag{A3}$$

These are even elements of the supersqueeze algebra. There is another even operator and it is defined as

$$M_0 = \frac{1}{2}\left(b^\dagger b - \frac{1}{2}\right) \ . \tag{A4}$$

The operator M_0 commutes with the operators $K_\pm$ and K_0.

In addition to the even operators, there are four odd operators which are defined as follows:

$$Q_1 = \frac{1}{2} a^\dagger b^\dagger \ , \qquad Q_2 = \frac{1}{2} ab \ , \tag{A5}$$

$$Q_3 = \frac{1}{2} a^\dagger b \ , \qquad Q_4 = \frac{1}{2} ab^\dagger \ . \tag{A6}$$

These odd operators satisfy a set of anticommutation relations, namely

$$\begin{aligned} &\{Q_j, Q_j\} = 0 \ , \quad j = 1, \ldots, 4 \ , \\ &\{Q_1, Q_2\} = \frac{1}{2}K_0 - \frac{1}{2}M_0 \ , \quad \{Q_1, Q_3\} = \frac{1}{2}K_+ \ , \\ &\{Q_1, Q_4\} = \{Q_2, Q_3\} = 0 \ , \quad \{Q_2, Q_4\} = \frac{1}{2}K_- \ , \\ &\{Q_3, Q_4\} = \frac{1}{2}K_0 + \frac{1}{2}M_0 \ . \end{aligned} \tag{A7}$$

The remaining commutation relations, between the even and odd elements, are

$$\begin{aligned} &[K_+, Q_1] = 0 \ , \quad [K_+, Q_2] = -Q_3 \ , \\ &[K_+, Q_3] = 0 \ , \quad [K_+, Q_4] = -Q_1 \ , \\ &[K_-, Q_1] = Q_4 \ , \quad [K_-, Q_2] = 0 \ , \\ &[K_-, Q_3] = Q_2 \ , \quad [K_-, Q_4] = 0 \ , \\ &[K_0, Q_1] = \frac{1}{2}Q_1 \ , \quad [K_0, Q_2] = -\frac{1}{2}Q_2 \ , \\ &[K_0, Q_3] = \frac{1}{2}Q_3 \ , \quad [K_0, Q_4] = -\frac{1}{2}Q_4 \ , \\ &[M_0, Q_1] = \frac{1}{2}Q_1 \ , \quad [M_0, Q_2] = -\frac{1}{2}Q_2 \ , \\ &[M_0, Q_3] = -\frac{1}{2}Q_3 \ , \quad [M_0, Q_4] = \frac{1}{2}Q_4 \ . \end{aligned} \tag{A8}$$

From the above graded commutation relations we see that our supersqueeze algebra is the superalgebra osp(2/2).

To complete the formulation of the symmetry algebra for the supersymmetric oscillator, we take the semidirect sum of the super Heisenberg-Weyl algebra with the osp(2/2) superalgebra. The additional graded commutation relations among these elements are

$$\begin{aligned} &[K_+, a^\dagger] = 0 \ , \quad [K_+, a] = -a^\dagger \ , \\ &[K_-, a^\dagger] = a \ , \quad [K_-, a] = 0 \ , \\ &[K_0, a^\dagger] = \frac{1}{2}a^\dagger \ , \quad [K_0, a] = 0 \ , \\ &[M_0, b^\dagger] = \frac{1}{2}b^\dagger \ , \quad [M_0, b] = -\frac{1}{2}b \ , \\ &[Q_1, a^\dagger] = 0 \ , \quad [Q_1, a] = -\frac{1}{2}b^\dagger \ , \\ &\{Q_1, b^\dagger\} = 0 \ , \quad \{Q_1, b\} = \frac{1}{2}a^\dagger \\ &[Q_2, a^\dagger] = \frac{1}{2}b \ , \quad [Q_2, a] = 0 \ , \\ &\{Q_2, b^\dagger\} = \frac{1}{2}a \ , \quad \{Q_2, b\} = 0 \ , \\ &[Q_3, a^\dagger] = 0 \ , \quad [Q_3, a] = -\frac{1}{2}b \ , \\ &\{Q_3, b^\dagger\} = \frac{1}{2}a^\dagger \ , \quad \{Q_3, b\} = 0 \ , \\ &[Q_4, a^\dagger] = \frac{1}{2}b^\dagger \ , \quad [Q_4, a] = 0 \ , \\ &\{Q_4, b^\dagger\} = 0 \ , \quad \{Q_4, b\} = \frac{1}{2}a \ . \end{aligned} \tag{A9}$$

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