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Background level and counter efficiencies required for a loophole-free Einstein-Podolsky-Rosen experiment

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An analysis is made of the background level and counter efficiencies actually necessary to perform a loophole-free Einstein-Podolsky-Rosen experiment. Both requirements are correlated. Photon counters do not absolutely have to have more than 82.8% efficiency if the signal-over-noise ratio is very high.

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I. INTRODUCTION

In this paper, limits are set for the amount of background that can be tolerated in a loophole-free Einstein-Podolsky-Rosen (EPR) experiment [1–3], as a function of η, the efficiency of the counters used. The experiment is assumed to be performed on entangled states of two photons and involves polarization measurements on them. It is possible to make a loophole-free experiment with $\eta < 82.8\%$, but it requires the background level to be very low.

The initial state is assumed to be prepared as a superposition of states of two photons, a and b, with correlated planes of polarization. One state is defined as $|\leftrightarrow\updownarrow\rangle$, i.e., photon a polarized horizontally and photon b vertically; and another state as $|\updownarrow\leftrightarrow\rangle$, i.e., a polarized vertically and b horizontally. For both photons, the polarization measurements are made with Nicol prisms set in such a way that the ordinary trajectory applies to a photon polarized in the horizontal plane and the extraordinary trajectory to a photon polarized vertically. In front of either Nicol prism, devices are disposed that rotate the plane of polarization of the photons. The angle by which the plane of polarization of a is rotated will be called α and, for b, β.

There are demonstrations showing that a limit exists for the amount of possible violation of a Bell inequality by predictions of quantum mechanics for a two-particle system [4–6]. That maximum can be reached, for example, with the experiment that we describe in this paper, if the initial state is given by the state vector:

$$\psi_0 = (1/\sqrt{2})\left(|\leftrightarrow\updownarrow\rangle + |\updownarrow\leftrightarrow\rangle\right) \ , \tag{1}$$

using experimental setups with values of α:

$$\alpha_1 = -78.75^\circ \ , \tag{2}$$

$$\alpha_2 = 56.25^\circ \ , \tag{3}$$

and values of β

$$\beta_1 = 11.25^\circ \ , \tag{4}$$

$$\beta_2 = -33.75^\circ \ . \tag{5}$$

Then, to take the example of the Clauser and Horne inequality, the maximum violation, $(\sqrt{2}-1)/2$, is reached, in one of the inequalities (4) of Ref. [7], if the following identifications are made :

$$a' = \alpha_1 \ , \tag{6}$$

$$a = \alpha_2 \ , \tag{7}$$

$$b = \beta_1 \ , \tag{8}$$

$$b' = \beta_2 \ . \tag{9}$$

The demonstrations that a maximum exists make use of an operator called the "Bell operator," $\mathcal{B}$, which is related to the expectation value $\mathcal{J}_\mathcal{B}$ of a Bell inequality and any initial state ψ by the relation

$$\mathcal{J}_\mathcal{B} = \psi^\dagger \mathcal{B}\, \psi \ . \tag{10}$$

In these demonstrations, the inequality $\mathcal{J}_\mathcal{B}$ and the operator $\mathcal{B}$ are written for the case of a 100% efficiency. It is possible to modify the inequality to take into account the case of a less than 100% efficiency. If this is done but the initial state ψ and the values of α_1, α_2, β_1, and β_2 are kept at the values of Eqs. (1)–(5), i.e., if the optimization of $\mathcal{J}_\mathcal{B}$ is made *before* setting η to a value less than 100%, it can be shown [8] that Bell's inequality in these particular conditions requires an efficiency of the counters

$$\eta > 2\left(\sqrt{2}-1\right) \approx 82.8\% \ . \tag{11}$$

However, if $\mathcal{J}_\mathcal{B}$ is optimized, changing ψ, α_1, α_2, β_1, and β_2 *after* introducing the correction for $\eta < 100\%$, a lower requirement for the efficiency may be expected. This is the subject of this paper. The Bell operator $\mathcal{B}$ is first modified to take into account values of η less than 100%. Then all parameters ψ, α_1, α_2, β_1, and β_2 are changed to optimize $\mathcal{J}_\mathcal{B}$.

II. BELL INEQUALITIES FOR $\eta < 100\%$

Bell inequalities concern expectation values of quantities that can be measured in four different experimental setups, defined by specific values α_1, α_2, β_1, and β_2 of α and β. The setups will be referred to by the symbols (α_1,β_1), (α_1,β_2), (α_2,β_1), and (α_2,β_2), where the first index designates the value of α and the second index the value of β.

In each setup, the *fate* of the photon a and the *fate* of photon b is referred to by an index (o) for photon detected in the ordinary beam, (e) for photon detected in the extraordinary beam, or (u) for photon undetected. Therefore there are nine types of events: (o,o), (o,u), (o,e), (u,o), (u,u), (u,e), (e,o), (e,u), and (e,e), where the first index designates the fate of photon a and the second index the fate of photon b. Table I shows a display of boxes corresponding to the nine types of event in each setup. The value of α and the fate of photon a

TABLE I. Possible results expected in the four setups.

α \ β		β_1			β_2		
		o	u	e	o	u	e
	o	$\bullet$	$*$	$*$		$\oplus$	$\oplus$
α_1	u	$*$	$*$	$*$	$*$	$*$	$*$
	e	$*$	$*$	$*$	$*$	$*$	$*$
	o		$*$	$*$		$+$	$+$
α_2	u	$\otimes$	$*$	$*$	$\times$	$\times$	$\times$
	e	$\otimes$	$*$	$*$	$\times$	$\times$	$\times$

designate a row. The value of β and the fate of photon b designate a column. Any event obtained in one of the setups corresponds to one box in Table I.

For a given theory, we consider all the possible sequences of N events that can occur in each setup. N is the same for the four setups and arbitrarily large. As in [9] and [10], a theory is defined as being "local" if it predicts that, among these possible sequences of events, one can find four sequences (one for each setup) satisfying the following conditions:

(i) The fate of photon a is independent of the value of β, i.e., is the same in an event of the sequence corresponding to setup (α_1, β_1) as in the event with the same event number k for (α_1, β_2); also same fate for a in (α_2, β_1) and (α_2, β_2); this is true for all k's for these carefully selected sequences.

(ii) The fate of photon b is independent of the value of α, i.e., is the same in event k of sequences (α_1, β_1) and (α_2, β_1); also same fate for b in sequences (α_1, β_2) and (α_2, β_2).

(iii) Among all sets of four sequences that one has been able to find with conditions (i) and (ii) satisfied, there are some for which all averages and correlations differ from the expectation values predicted by the theory by less than, let us say, ten standard deviations.

These conditions are fulfilled by a deterministic *local* hidden-variable theory, i.e., one where the fate of photon a does not depend on β and the fate of b does not depend on α. For such a theory, these four sequences could be just four of the most common sequences of events generated by the same values of the hidden variables in the different setups. Conditions (i)–(iii) are also fulfilled by probabilistic *local* theories, which assign probabilities to various outcomes in each of the four setups and assume no "influence" of the angle β on what happens to a and no "influence" of α on b. With such theories, one can generate sequences of events having properties (i) and (ii) by Monte Carlo, using an algorithm that decides the fate of a without using the value of β and, for b, without using the value of α. If the same random numbers are used for the four different setups, the sequences of events will automatically have properties (i) and (ii), and the vast majority of them will have property (iii).

Let us follow an argument first used in Ref. [11]. When four sequences are found satisfying conditions (i) and (ii), the four events with the same event number k in the four different sequences will be called "conjugate events." Because of condition (i), two conjugate events in setups (α_1, β_1) and (α_1, β_2) fall into two boxes on the same row in Table I. The same thing applies for conjugate events in setups (α_2, β_1) and (α_2, β_2). Because of (ii), conjugate events for setups (α_1, β_1) and (α_2, β_1) lie in boxes in the same column; and so do conjugate events for (α_1, β_2) and (α_2, β_2). Let us select all the $n_{oo}(\alpha_1, \beta_1)$ events that fall into the box marked with a $\bullet$ in the section of Table I reserved for setup (α_1, β_1). None of these events falls into any other box for setup (α_1, β_1). Because of condition (i), their conjugate events in setup (α_1, β_2) fall into boxes on row o. Because of condition (ii), the conjugate events in setup (α_2, β_1) lie in boxes in column o. Therefore none of the boxes marked with a $*$ contains any of the events of this sample or any of their conjugates.

Now, from that sample, let us remove events with conjugates falling in one of the boxes marked with a $\otimes$ in setup (α_2, β_1). The number of events subtracted is smaller than or equal to the total number $n_{uo}(\alpha_2, \beta_1) + n_{eo}(\alpha_2, \beta_1)$ of events of all categories contained in those two boxes. Therefore the remaining sample contains $n_{oo}(\alpha_1, \beta_1) - n_{uo}(\alpha_2, \beta_1) - n_{eo}(\alpha_2, \beta_1)$ events or *more*. None of the events in the remaining sample has a conjugate falling in a box on rows u or e in setup (α_2, β_1); thus, because of condition (i), none falls in setup (α_2, β_2) either. None of the conjugate events falls in a box marked with an $\times$.

Let us further restrict the sample by removing events with conjugates in sequence (α_1, β_2) falling in boxes marked with a $\oplus$ in Table I. Using the same argument as in the preceding paragraph, the number of events left must be *more* than or equal to

$$n_{oo}(\alpha_1, \beta_1) - n_{uo}(\alpha_2, \beta_1) - n_{eo}(\alpha_2, \beta_1) - n_{ou}(\alpha_1, \beta_2) - n_{oe}(\alpha_1, \beta_2),$$

where $n_{ou}(\alpha_1, \beta_2) + n_{oe}(\alpha_1, \beta_2)$ is the total number of events of all categories falling into the boxes marked with a $\oplus$. None of the events in that restricted sample falls in column u and e in setup (α_1, β_2); therefore, because of condition (ii), none falls in setup (α_2, β_2) either; therefore, none falls in any box marked with a $+$.

All events belonging to the latter sample must have conjugates in sequence (α_2, β_2) falling into the only remaining box for that setup, i.e., box (o, o). That is possible only if that most restricted sample contains a number of events *less* than or equal to the total number $n_{oo}(\alpha_2, \beta_2)$ of events of all categories in that box. Thus conditions (i) and (ii) can be satisfied by our four sequences only if

$$n_{oo}(\alpha_1, \beta_1) - n_{uo}(\alpha_2, \beta_1) - n_{eo}(\alpha_2, \beta_1) - n_{ou}(\alpha_1, \beta_2) - n_{oe}(\alpha_1, \beta_2) \le n_{oo}(\alpha_2, \beta_2) \; ; \tag{12}$$

i.e.,

$$\begin{aligned} \mathcal{J}_B &= n_{oe}(\alpha_1, \beta_2) + n_{ou}(\alpha_1, \beta_2) \\ &+ n_{eo}(\alpha_2, \beta_1) + n_{uo}(\alpha_2, \beta_1) \\ &+ n_{oo}(\alpha_2, \beta_2) - n_{oo}(\alpha_1, \beta_1) \ge 0 \; . \end{aligned} \tag{13}$$

For condition (iii) to be true no matter how large the number of events N is, inequality (13) also has to apply to the expectation values of these numbers. It is a form of the Bell inequality, which Eqs. (6)–(9) make equivalent to inequality (4) of Ref. [7]. That is the Bell inequality that will be used in this paper.

An inequality similar to (13) can be derived where all o 's are changed into e 's and *vice versa*. In principle, one could average this new inequality with inequality (13) to improve statistics. (By doing so, one arrives directly at an inequality almost identical to the Bell–Clauser-Horne-Shimony-Holt inequality [3]). However, optimizing the averaged inequality leads to the conditions of Eqs. (1)–(5) regardless of the efficiency η. The minimum efficiency required is then 82.8%. The optimizing procedure of Sec. III actually makes an improvement on only one of the inequalities of the average, at the expense of the other inequality.

III. PREDICTIONS OF QUANTUM MECHANICS

The goal of a loophole-free experiment is to find results in a case where the predictions of quantum theory contradict the predictions of all *local* theories, i.e., where the predictions of quantum mechanics violate a form of the Bell inequality. If the predictions of quantum theory are upheld, the existence of nonlocal effects in nature will thus be proven.

To compute the predictions of quantum mechanics, let us use a representation where the helicities of the two photons are diagonal operators and the $|+\rangle$ and $|-\rangle$ helicity states are

$$|\pm\rangle \;= (1/\sqrt{2})\left(|\leftrightarrow\updownarrow\rangle \pm\, i\,|\updownarrow\leftrightarrow\rangle\right)\;. \tag{14}$$

Given an initial state ψ, a value for η, and a set of angles α_1, α_2, β_1, and β_2, predictions for the number of events involved in inequality (13) can be computed. For N pairs of photon emitted in the superposition state and assuming an ideal case where there is no background, these predictions are

$$n_{oo}^{\rm ideal}(\alpha_1,\beta_1) = N\frac{\eta^2}{4}\psi^\dagger\left[I+\sigma(\alpha_1)\right]\left[I+\tau(\beta_1)\right]\psi, \tag{15}$$

$$n_{oe}^{\rm ideal}(\alpha_1,\beta_2) = N\frac{\eta^2}{4}\psi^\dagger\left[I+\sigma(\alpha_1)\right]\left[I-\tau(\beta_2)\right]\psi, \tag{16}$$

$$n_{ou}^{\rm ideal}(\alpha_1,\beta_2) = N[\eta(1-\eta)/2]\psi^\dagger\left[I+\sigma(\alpha_1)\right]\psi, \tag{17}$$

$$n_{eo}^{\rm ideal}(\alpha_2,\beta_1) = N\frac{\eta^2}{4}\psi^\dagger\left[I-\sigma(\alpha_2)\right]\left[I+\tau(\beta_1)\right]\psi, \tag{18}$$

$$n_{uo}^{\rm ideal}(\alpha_2,\beta_1) = N[\eta(1-\eta)/2]\psi^\dagger\left[I+\tau(\beta_1)\right]\psi, \tag{19}$$

$$n_{oo}^{\rm ideal}(\alpha_2,\beta_2) = N\frac{\eta^2}{4}\psi^\dagger\left[I+\sigma(\alpha_2)\right]\left(I+\tau(\beta_2)\right)\psi. \tag{20}$$

The angles α and β are counted as positive when the angle of polarization of the system is rotated in the direction of a positive helicity. In our helicity representation, the phases are chosen in such a way that the elements of $\sigma(\alpha_1)$ and of $\tau(\beta_1)$ are real:

$$\sigma(\alpha) = \begin{vmatrix} 0 & e^{2i(\alpha-\alpha_1)} & 0 & 0 \\ e^{-2i(\alpha-\alpha_1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{2i(\alpha-\alpha_1)} \\ 0 & 0 & e^{-2i(\alpha-\alpha_1)} & 0 \end{vmatrix}, \tag{21}$$

$$\tau(\beta) = \begin{vmatrix} 0 & 0 & e^{2i(\beta-\beta_1)} & 0 \\ 0 & 0 & 0 & e^{2i(\beta-\beta_1)} \\ e^{-2i(\beta-\beta_1)} & 0 & 0 & 0 \\ 0 & e^{-2i(\beta-\beta_1)} & 0 & 0 \end{vmatrix}, \tag{22}$$

$$\begin{aligned}\mathcal{J}_{\mathcal{B}}{}^{\rm ideal} \;&= n_{oe}^{\rm ideal}(\alpha_1,\beta_2) + n_{ou}^{\rm ideal}(\alpha_1,\beta_2) \\ &\quad + n_{eo}^{\rm ideal}(\alpha_2,\beta_1) + n_{uo}^{\rm ideal}(\alpha_2,\beta_1) \\ &\quad + n_{oo}^{\rm ideal}(\alpha_2,\beta_2) - n_{oo}^{\rm ideal}(\alpha_1,\beta_1)\;.\end{aligned} \tag{23}$$

The above computation assumes that only the N photons in the entangled state contribute to the counting rates and that the polarization analyzers are perfect. A correction has to be made to formula (23) to take into account deviations from that ideal case. The sample of events of type (o,e), (o,u), (e,o), and (u,o) actually counted in the experiment and introduced in inequality (13) will include not only the $n_{oe}^{\rm ideal}(\alpha_1,\beta_2)$, $n_{ou}^{\rm ideal}(\alpha_1,\beta_2)$, $n_{eo}^{\rm ideal}(\alpha_2,\beta_1)$, and $n_{uo}^{\rm ideal}(\alpha_2,\beta_1)$ events of Eqs. (16)–(19), but other events with a less sharp dependence on α and β. It is a *background.* We will take that background into account by an α- and β-independent term, $N\zeta$, to be added to the $n_{oe}^{\rm ideal}(\alpha_1,\beta_2) + n_{ou}^{\rm ideal}(\alpha_1,\beta_2)$ events of Eqs. (16) and (17), and to the $n_{eo}^{\rm ideal}(\alpha_2,\beta_1) + n_{uo}^{\rm ideal}(\alpha_2,\beta_1)$ events of Eqs. (18) and (19). In principle there is also a background in samples of type (o,o) events in setup (α_1,β_1) and (α_2,β_2). However, since we assume no dependence of the background on α and β, the effect of the type (o,o) background cancels in inequality (13). After correction for background, we write

$$\mathcal{J}_{\mathcal{B}} \;= \mathcal{J}_{\mathcal{B}}^{\rm ideal} \;+ 2N\zeta\;. \tag{24}$$

Equation (13) stipulates that local theories predict

$$\mathcal{J}_{\mathcal{B}} \;\geq 0\;, \tag{25}$$

while quantum theory predicts

$$\mathcal{J}_{\mathcal{B}} \;= \psi^\dagger \mathcal{B}\,\psi\;, \tag{26}$$

where

$$\mathcal{B} \;= N\frac{\eta}{2}\begin{vmatrix} 2-\eta+\xi & 1-\eta & 1-\eta & A^*B^*-\eta \\ 1-\eta & 2-\eta+\xi & AB^*-\eta & 1-\eta \\ 1-\eta & A^*B-\eta & 2-\eta+\xi & 1-\eta \\ AB-\eta & 1-\eta & 1-\eta & 2-\eta+\xi \end{vmatrix} \tag{27}$$

and

$$A = (\eta/2)\;(e^{2i(\alpha_1-\alpha_2)}-1)\;, \tag{28}$$

$$B = \quad e^{2i(\beta_1-\beta_2)}-1\;, \tag{29}$$

$$\xi = 4\zeta/\eta\;. \tag{30}$$

To perform a loophole-free experiment, we need experimental conditions in which the prediction for $\mathcal{J}_{\mathcal{B}}$ of Eq. (26) is negative. That is possible as long as the operator $\mathcal{B}$ has a negative eigenvalue. That is impossible if all eigenvalues are positive, even if one uses incoherent mixtures of pure states. The maximum amount of background that can be tolerated corresponds to that value of ζ that makes the last negative eigenvalue of $\mathcal{B}$ turn from negative to positive, i.e., when the determinant of $\mathcal{B}$ of Eq. (27) ceases to be negative to become zero.

TABLE II. Extreme conditions for a loophole-free experiment.

η (%)	ζ (%)	r	ω (deg)	$\alpha_1 - \alpha_2$ (deg)
66.7	0.00	0.001	0.0	2.2
70	0.02	0.136	3.4	21.4
75	0.31	0.311	9.7	32.0
80	1.10	0.465	14.9	37.9
85	2.48	0.608	18.6	41.5
90	4.50	0.741	20.9	43.6
95	7.12	0.871	22.1	44.7
100	10.36	1.000	22.5	45.0

A computer program was written to compute the determinant of $\mathcal{B}$ of Eq. (27), for any given value of the efficiency η. The program varied $\alpha_1 - \alpha_2$, $\beta_1 - \beta_2$, and ζ to find the maximum value of the background ζ that kept the determinant negative. For $\eta < \frac{2}{3}$, there is none. For $\eta > \frac{2}{3}$ there are negative values of the determinant for small values of ζ, increasing from 0 to $\frac{\sqrt{2}-1}{4}$ as η increases from $\frac{2}{3}$ to 1. The maximum value of ζ as a function of η is given in Table II. It is plotted in Fig. 1, as well as the maximum affordable value of ζ if the conditions are not the optimum ones, but those of Eqs. (1)–(5) instead.

The program also recorded the values of $\alpha_1 - \alpha_2$ and $\beta_1 - \beta_2$ and computed the relevant eigenvector ψ, i.e., the conditions that make $\mathcal{J}_{\mathcal{B}}$ of Eq. (26) equal to zero for the maximum ζ. There were degeneracies in the solutions. The two angles $\alpha_1 - \alpha_2$ and $\beta_1 - \beta_2$ could always be taken to be the same, or the opposite of one another, as can be understood from an analytic study of Eqs. (10) and (27). The vector ψ turned out to be of the form

$$\psi = \frac{1}{2\sqrt{1+r^2}} \begin{vmatrix} (1+r)e^{-i\omega} \\ -(1-r) \\ -(1-r) \\ (1+r)e^{i\omega} \end{vmatrix} , \tag{31}$$

which can be reached in the two-photon experiment considered in this paper by first superposing states $| \leftrightarrow \updownarrow >$ and $| \updownarrow \leftrightarrow >$ in unequal amounts,

$$\psi_0 = (1/\sqrt{1+r^2}) \left(| \leftrightarrow \updownarrow > + r \, | \updownarrow \leftrightarrow > \right) , \tag{32}$$

then rotating the planes of polarization of a and of b in setup (α_1, β_1) by the angles

$$\alpha_1 = (\omega/2) - 90^\circ , \tag{33}$$

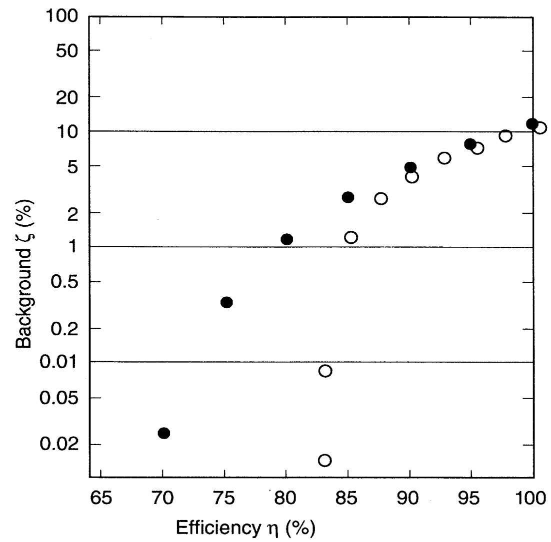

FIG. 1. Maximum affordable background vs efficiency: •, optimized conditions; ∘, conditions of Eqs. (1)–(5).

$$\beta_1 = \omega/2 , \tag{34}$$

respectively, and using the values of r, ω, and $\alpha_1 - \alpha_2$ ($\equiv \beta_1 - \beta_2$) given in Table II. Note that, for $\eta = 1$, the vector ψ_0 reduces to the value given by Eq. (1), and the angles α_1, α_2, β_1, and β_2 reduce to the values given by Eqs. (2)–(5).

In conclusion, it is possible to perform a loophole-free experiment if the efficiency η of the photon counters is higher than 66.7% and the background is less than the value indicated on Fig. 1 for that value of η. For small background levels, it is possible to perform a loophole-free EPR experiment with a less than 82.8% counter efficiency.

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