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Experimental characterization of unstable periodic orbits by controlling chaos

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A modification of the method proposed by Ott, Grebogi, and Yorke to control chaos [Phys. Rev. Lett. **64**, 1196 (1990)] has allowed us to characterize unstable orbits of a Nd-doped fiber laser in their whole domain of existence. Stabilization could be achieved both on orbits embedded inside the chaotic attractor and on orbits located in other parts of the phase space. Transient regimes provide direct experimental information on the unstable Floquet multipliers.

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Lasers like other nonlinear dynamical systems exhibit deterministic chaos when some instability conditions are met. As the laser applications rely mainly on their ability to provide clean "coherent" radiation, this chaos appears as a serious drawback. For instance, it was shown that chaos limits the possibilities of intracavity modulated CO_2 lasers [1] and one practical interest of studies of chaos in lasers is to circumvent the parameter domain in which chaos is likely to appear in order to avoid it.

Recently several techniques have been proposed to inhibit the chaotic behavior in a nonlinear system. They are based on feedback techniques in which information from the system output is used to modify the value of a control parameter so as to stabilize the behavior in a previously unstable state. In the original method proposed by Ott, Grebogi, and Yorke (OGY) [2], deviations from the exact position of the chosen unstable orbit are corrected via small shifts of the control parameter to maintain the system as close as possible of the unstable periodic orbit. With this method, the stabilized orbit is theoretically identical to the unstable periodic orbit, except for its stability. This was applied in periodically forced systems such as chemical reactions [3], a gravitationally buckled magnetostrictive ribbon [4], and a diode resonator [5]. Recently, this method has also been applied on autonomous laser systems by Roy *et al.* [6]. The OGY method has been modified to create nonexistent periodic orbits, as shown by Hunt, who used larger corrections of the control parameter [5]. It has also been used to stabilize an autonomous chaotic water flow in a stationary state [7]. Methods derived from OGY have recently been introduced to generate controlled aperiodic orbits [8] and to take advantage of chaos to direct trajectories to targets [9]. Chaos can also be suppressed by using a small periodic modulation of a control parameter [10]. The aim of the work presented here is to investigate the properties of unstable orbits. As will be shown, the advantage of a control method over the well-known close return technique is that it is not restricted to the parameter domain for which the orbit is embedded in the chaotic attractor.

A reliable characterization of unstable orbits requires that the considered unstable orbit not be modified by the control method. This is realized if the correction on the control parameter vanishes when the trajectory lies on the stabilized cycle. Therefore, a feedback method analogous to that of OGY is preferred to a modulation technique. OGY suggested stabilizing a chaotic flow $\mathbf{X}(t)$ using values $\xi_n(t)$ taken from a Poincaré section. Let $\xi_F=\mathbf{0}$ denote the position of the unstable orbit for a control parameter $p=0$. They showed that if the unstable orbit has only one Floquet multiplier λ_u with modulus larger than 1, a correction p_n of the control parameter may be calculated such that, after the correction, the system lies on the stable manifold of $\xi=\mathbf{0}$. The correct feedback factor does not need to be exactly known since there exists a parameter domain in which the orbit remains stabilized [4], but the OGY method requires a knowledge of the dynamics near the unstable orbit and, in particular, its location. We have modified the OGY technique and the new method has been checked on autonomous chaos of the Nd-doped optical fiber laser (OFL). We propose here to use only the differences $(\xi_n-\xi_{n-1})$ to determine the required correction. Correcting the control parameter by an amount proportional to $(\xi_n^u-\xi_{n-1}^u)$ instead of $\xi_n^u-\xi_F^u$, as suggested by Ott, Grebogi, and Yorke [2], allows one to stabilize the unstable orbit even when a control parameter is swept (ξ_n^u is the component along the unstable direction).

Another difference with the OGY method is that the correction is applied during a small part of the period. In the usual case where the Poincaré section is reconstructed by the time delay technique, Dressler and Nitsche [11] have shown that the OGY method with a correction during all the period can lead to instabilities, because the plane of the Poincaré section depends on the value of the correction. The instabilities can be suppressed if the correction is applied during a small part of the period and the measure is always with a correction on the control parameter equal to zero. Moreover, a well-chosen correction (duration and timing) may be more efficient than a quasicontinuous one [6], especially in systems such as the OFL for which a quasicontinuous correction shifts the orbit location in the Poincaré section essentially along the stable manifold.

In the case of small corrections and in the vicinity of the unstable orbit, the response of the system may be linearized in ξ and μ_n [2]: $\xi_{n+1}=\underline{\lambda}\xi_n+\mu_n\mathbf{h}$, where $\underline{\lambda}$ is the Floquet matrix of the unstable orbit at $\xi=0$ and $\mathbf{h}$ characterizes the response of the system to the correction μ_n, taking into account its shape and its duration. The unstable orbit is supposed to have only one Floquet multiplier λ_u with modulus larger than one associated with h_u, the component of $\mathbf{h}$ along the unstable direction. If the proposed correction is applied at each period, i.e., $\mu_n=\alpha(\xi_n^u-\xi_{n-1}^u)$, where α is the feedback factor, the stability analysis reveals that it is not possible to obtain a value of α stabilizing the orbit when $\lambda_u>1$ or $\lambda_u<-3$. The instabilities are due to the fact that we do not know the location ξ_F of the unstable orbit and they can be suppressed if the feedback takes the previous value into account. This is similar to the modification of the OGY method by Dressler and Nitsche for experimental systems in which time delay coordinates are used. Their method requires the simultaneous adjustment of two feedback coefficients, which is difficult to obtain in an experimental system whose dynamical equations are unknown. The stabilization of the system is obtained here by the application of the feedback only every other period, i.e., $\mu_{2n}=\alpha(\xi_{2n}^u-\xi_{2n-1}^u)$ and $\mu_{2n+1}=0$. In this case, the stability analysis reveals that there exists a range of values of α leading to the stabilization of the unstable orbit for any unstable Floquet multiplier λ_u. This is achieved when α belongs to the $[(\lambda_u^2-1)/(\lambda_u-1)h_u,(\lambda_u^2+1)/(\lambda_u-1)h_u]$ interval. Figure 1 represents the suitable range of values of α versus λ_u. A single value of the feedback coefficient α leads to stabilization for a large range of the Floquet multiplier λ_u. Thus, it is possible to keep α constant and to stabilize a particular unstable orbit, even when a control parameter is swept in a wide range. By this method it is possible to track an unstable orbit in its whole domain of existence. This feedback system automatically finds the unstable orbit but, *a priori*, several different orbits could be stabilized for given values of feedback parameters, depending on the initial conditions. In fact, if the periodicity of the correction is adjusted to stabilize an nT unstable periodic orbit, the system can stabilize (i) different nT orbits if they exist in the chaotic attractor, (ii) a $2nT$ unstable orbit if its unstable Floquet multiplier verifies $-3<\lambda_u<+1$, (iii) an mT unstable orbit if n is a multiple of m. However, in our experiments, this phenomenon has not been observed.

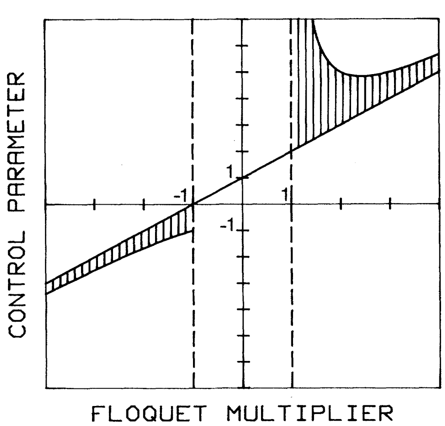

FIG. 1. Stability domains of the control parameter α (in h_u units) vs the unstable Floquet multiplier λ_u. In the hatched region the chaotic system can be stabilized on the unstable periodic orbit.

Our experimental setup essentially consists of a 4.5-m-long Nd-doped silica fiber pumped by a diode laser with index-matched plane mirrors. The design of this laser is the same as that reported in our previous paper [12] and the powers (P_1,P_2) emitted by the laser along the two polarization eigendirections are monitored with the pump power as the main control parameter. Other parameters such as the coupling of the fiber with the mirrors or of pump power into the fiber provide additional control of the laser behavior. Laser oscillation occurs on many longitudinal modes, but the laser behavior appears to be governed by a small number of dynamical variables [13] in which light polarization plays a crucial role [12,14]. This laser is known to present chaos when subjected to pump modulation [12,15], but we have recently found conditions under which it spontaneously destabilizes to chaos through a cascade of period-doubling bifurcations. In general, the stabilization method requires a Poincaré section in which the unstable direction corresponding to the unstable orbit should be known in order to measure ξ_n^u. This could be obtained by the time delay technique [11]. In the particular case of our OFL, it is possible to use P_{1n} defined by the sampled values of P_1 instead of ξ_n^u. With this purpose, we detect the maxima of the power P_1. Then the sampling is performed after an adjustable delay, and one of every m samples is kept if the unstable-m orbit is to be stabilized. The correction is applied immediately after the sampling. In fact, P_{1n} is a linear combination of the components of ξ_n. For a well-chosen delay, the first return map of P_{1n} looks to be quasiunidimensional, like in previous experiments on the control of chaos [4–7]. In this case, the contribution of the stable components does not prevent the stabilization. This provides the data required for stabilization, which, in our experiment, are sent to an analog system delivering the suitable corrections to the driving current of the pump laser diode if the difference between two successive sampled values is less than ϵ (a few percent of P_{1n}). In this case, the trajectory can stay in an ϵ-ball neighborhood of an unstable orbit.

Stabilization has been achieved with the method presented here. The duration of the correction has been adjusted to optimize the efficiency. It is typically a small part of the period (5–20 %). When the ϵ-ball condition mentioned above is met, the feedback control switches on and stabilizes the trajectory. The correction on the pump power decreases and amounts to a very small value (less than 1% of the average pump power). It displays random amplitudes because once the system is set on the unstable orbit, the feedback corrects only technical noise originating from the laser. In particular, it does not present the periodicity of the stabilized cycle. Therefore, we can conclude that the stabilized trajectory is not measurably affected by the stabilization. The chaotic OFL could be stabilized on $2T$ and $4T$ unstable orbits in a wide range of pump power. For example, Fig. 2 displays the output signal $P_1(t)$ of the laser power corresponding to a $4T$ cycle and the correction applied to the pump power.

When an orbit has been stabilized, we can slowly vary a control parameter (here the pump power) without modifying any parameters of the feedback system. Due to the high stability of the system with feedback the orbit remains stabilized in a wide range of the control parameter. Moreover, during the sweep, the correction applied to the system remains small and random, showing that for each value of the parameter, the observed orbit is identical to the unstable orbit existing without stabilization.

The feedback technique proposed here allows us to follow an unstable orbit not only when it is embedded in the chaotic attractor but in its whole domain of existence. For instance, the $2T$ unstable orbit is embedded in the chaotic attractor after the $C4$-$C2$ transition of the inverse cascade [16]. This suggests that this $2T$ unstable orbit comes from the destabilization of the $2T$ stable orbit through the $2T$-$4T$ bifurcation [17]. We propose here to show experimentally that this hypothesis is true. The chaotic laser can be stabilized on the $2T$ cycle above the $C4$-$C2$ transition since the attractor will visit the vicinity of this orbit. Then by decreasing the control parameter, the system stays on this unstable $2T$ orbit in a parameter domain where this orbit is no longer embedded in the attractor and even where the chaotic attractor no longer exists. More precisely, as shown on Fig. 3(b), the laser could be stabilized on the $2T$ unstable orbit at $P=5.6$ mW, and it could be tracked down to the $2T$-$4T$ bifurcation ($P=4.4$ mW) where this orbit becomes unstable. The fact that the stable and unstable orbits continuously merge into each other at the bifurcation point is another proof that the stabilized trajectory is the same as the unstable cycle.

Automatic following of a particular orbit allows us to study its properties all through its domain of existence, while other methods, like the close return technique or the OGY method, are restricted to the parameter domain where the unstable orbit is embedded in the chaotic attractor. As an example, the Floquet multiplier of the $2T$ unstable orbit has been measured on the whole domain of existence of this orbit (Fig. 4). It could be measured by switching off the control of system and monitoring the transient regime out of the unstable orbit. The fact that at the $2T$-$4T$ bifurcation the Floquet multiplier is equal to -1 and the continuity of the bifurcation diagram [Fig. 3(b)] at this point clearly indicates that the orbit on which the system is controlled results from the destabilization of the $2T$ cycle at the second period-doubling bifurcation. The method is quite general and could also be applied to the $4T$ unstable orbit, which destabilizes at the

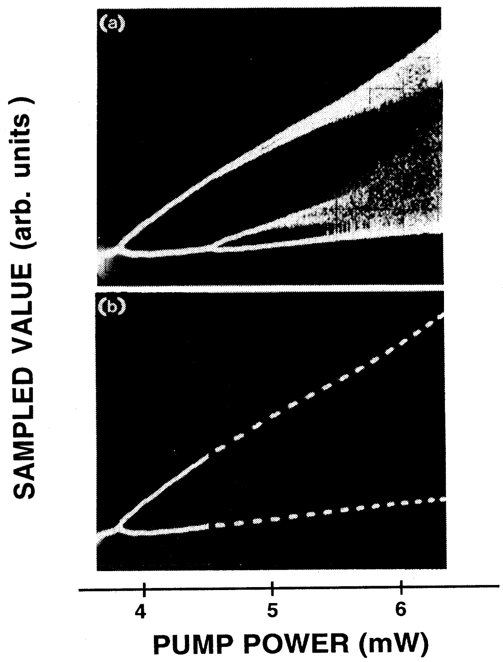

FIG. 3. Bifurcation diagrams without (a) and with (b) stabilization of the unstable $2T$ orbit. The dashed line of (b) corresponds to samples of the unstable $2T$ cycle. The $2T$ stable orbit is destabilized at $P=4.4$ mW and the unstable orbit is embedded in the chaotic attractor for $P>5.6$ mW.

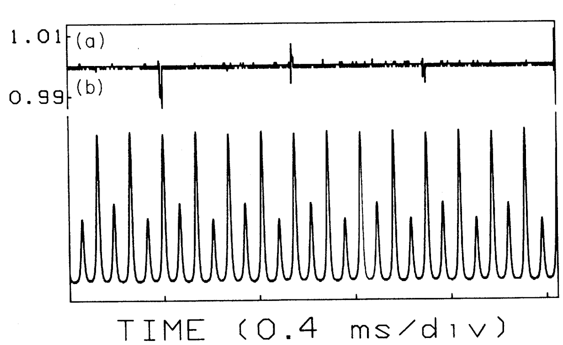

FIG. 2. Stabilization of the $4T$ unstable orbit: (a) the upper trace represents the correction applied to the pump power in units of the average pump power (the narrow spikes are the applied corrections, the rest corresponds to the technical noise). (b) The lower trace is the laser output in the $4T$ stabilized cycle in arbitrary units.

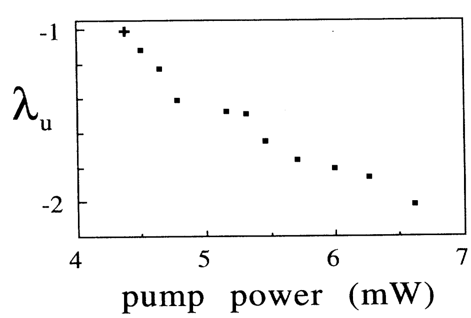

FIG. 4. Evolution of the Floquet multiplier λ_u with the pump power for the $2T$ unstable orbit. The cross indicates the $2T$-$4T$ bifurcation that occurs at $P=4.4$ mW, and the $2T$ unstable orbit is embedded in the attractor for $P>5.6$ mW.

$4T$-$8T$ period-doubling bifurcation.

To conclude, chaos could be suppressed in a laser system by locking it to unstable orbits. The method proposed here makes it possible even when no information on the location of these orbits is available. Moreover, automatic locking allows us to follow these orbits when the control parameters are varied. In particular, an orbit can be tracked even in a parameter region in which it is no longer embedded inside the chaotic attractor. For instance, the $2T$ unstable orbit could be analyzed up to the $2T$-$4T$ bifurcation where the $2T$ orbit destabilizes. The information on unstable orbits embedded in the chaotic attractor can be used to derive topological properties of this attractor [18]. On the other hand, stabilization of unstable orbits created in other regions of the phase space is also possible by the method proposed here and has been achieved on a $3T$ unstable cycle. This provides direct experimental information on the limits of the basin of attraction, which are usually difficult to obtain.

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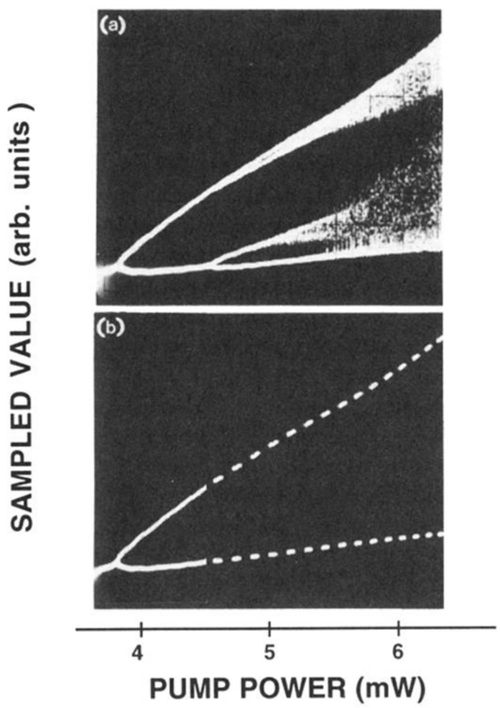

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