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Phase measurement and Q function

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Generalizing a recent theoretical result obtained by Freyberger and Schleich [Phys. Rev. A **47**, R30 (1993)], we show that the phase-measurement scheme proposed and realized by Noh, Fougères, and Mandel [Phys. Rev. Lett. **67**, 1426 (1991); Phys. Rev. A **45**, 424 (1992)] amounts to measuring the Q function for the light under investigation, provided the reference beam (used for homodyne detection) is a very strong coherent field so that it can be described classically. The desired phase distribution follows from the Q function by averaging over the field amplitude. Since an analysis of an earlier proposal by Bandilla and Paul [Ann. Phys. (Leipzig) **23**, 323 (1969)] to measure phase distributions via amplification led to just the same result, a perfect physical equivalence of those two approaches has thus been established.

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I. INTRODUCTION

The quantum-mechanical description of phase by introducing a Hermitian phase operator [1] or, equivalently, phase states with correct orthogonality properties suffers, apart from the fact that this problem could be solved in a perfectly satisfactory way only by resorting to a finite-dimensional Hilbert space [2], from the lack of any prescription for an actual phase measurement, even in the form of a gedanken experiment. So, in order to make contact with reality, one will have to turn the tables: One will first devise an experimental scheme for measuring phase properties, thus giving an operational definition of phase, and afterwards search for the proper quantum-mechanical description of the experimental procedure. In 1969, such a program was first carried out successfully by Bandilla and Paul [3], who proposed to amplify, with the help of a laser (or parametric) amplifier, the microscopic field to be investigated to a macroscopic level, where classical phase measurement techniques could readily be applied. Fifteen years later Shapiro and Wagner [4] analyzed a heterodyne-detection scheme, which allows simultaneous, however noisy, observations of both the phase and the squared amplitude of a signal field.

Only recently, Noh, Fougères, and Mandel [5] proposed and, moreover, realized a different experimental scheme based on homodyne detection. The basic idea, well known from classical optics, is to "duplicate" the signal beam by means of a semitransparent mirror and to measure $\sin\phi$ and $\cos\phi$ separately on the reflected and the transmitted beam [6].

It has been shown [7,4] that both the proposals by Bandilla and Paul [3] and Shapiro and Wagner [4] amount to measuring the Q function for the signal field. Moreover, Freyberger and Schleich [8] succeeded in demonstrating that the same holds true for Mandel's scheme [5], provided the reference beam used for homodyne detection is very strong, for the special case of coherent signal fields. In the following we will show that this result is, in fact, quite generally valid.

II. WHAT IS MEASURED?

Following Mandel's proposal [5], we study the experimental scheme sketched in Fig. 1. The signal beam is divided, with the help of a lossless 50:50 beam splitter, into two parts. On each of them a homodyne measurement is carried out, whereby the reference beams differ in their phases by $\pi/2$. We will assume that the reference beams are very strong, compared to the signal field, coherent fields, so that the observed phase properties actually reflect features of the signal field not distorted by the reference fields.

This assumption allows us to describe the reference fields classically, which drastically reduces the mathematical effort needed. In fact, the homodyne measurements on the split beam, in those circumstances, are nothing but measurements of the two field quadrature components x and p, respectively. So the essence of Mandel's scheme is as follows. Since x and p do not commute, it is impossible to measure them simultaneously on the original field. However, one can measure them separately on the two beams emerging from the beam splitter. The price to be paid for this is the introduction of undesired additional noise in the process of beam splitting: Formally speaking, vacuum fluctuations are entering the unused port of the beam splitter.

Let us now describe the experiment in the quantum-mechanical formalism. Since the description of the action of a (lossless) beam splitter takes its simplest form in the x representation, we will use the latter. We thus characterize the incident signal field by a wave function $\psi(x_1)$ and the vacuum field entering the second port, as is well known, by the wave function

$$\psi_{\text{vac}}(x_2)=\pi^{-1/4}\exp(-\tfrac{1}{2}x_2^2) \; . \tag{1}$$

The wave function for the outgoing fields $\psi_{\text{out}}(x_3,x_4)$ is then obtained from that for the incoming fields $\psi_{\text{in}}(x_1,x_2)=\psi(x_1)\psi_{\text{vac}}(x_2)$ by rotating the (x_1,x_2) plane by $\pi/4$ [9], i.e., by substituting

$$x_1=2^{-1/2}(x_3+x_4), \quad x_2=2^{-1/2}(-x_3+x_4) \tag{2}$$

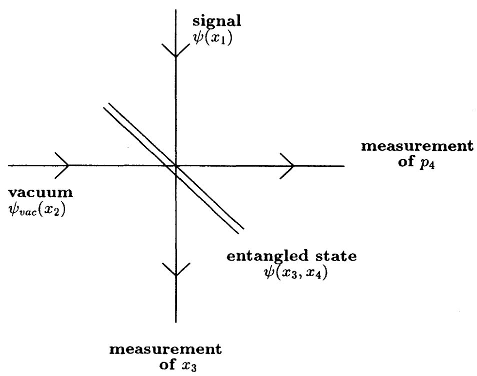

FIG. 1. Experimental scheme for a simultaneous, however noisy, measurement of the two quadrature components of the field via homodyne detection with strong coherent fields.

in $\psi_{\rm in}(x_1,x_2)$. This means that $\psi_{\rm out}$ is simply given by

$$\psi_{\rm out}(x_3,x_4)=\psi(2^{-1/2}[x_3+x_4])\times\psi_{\rm vac}(2^{-1/2}[x_4-x_3]) . \quad (3)$$

Note that this wave function describes an entangled state in general.

Since we want to measure x_3 and p_4, it is advantageous to Fourier transform the wave function (3) with respect to x_4, i.e., to form the integral

$$\Phi_{\rm out}(x_3,p_4)=\pi^{-1/4}(2\pi)^{-1/2}\int\psi(2^{-1/2}[x_3+x_4])\times\exp(-\tfrac{1}{4}[x_3-x_4]^2)\times\exp(-ip_4x_4)dx_4 , \quad (4)$$

where Eq. (1) has been observed.

Now, it is well known from the statistical interpretation of quantum mechanics that the modulus squared of the wave function (4) gives us just the probability distribution for the joint measurement of x_3 and p_4,

$$w(x_3,p_4)=|\Phi_{\rm out}(x_3,p_4)|^2 . \quad (5)$$

Introducing here polar coordinates

$$x_3=\rho\cos\varphi , \quad p_4=\rho\sin\varphi , \quad (6)$$

and averaging over ρ, we find the phase distribution

$$w(\varphi)=\int_0^\infty w(x_3=\rho\cos\varphi,\, p_4=\rho\sin\varphi)\rho\, d\rho , \quad (7)$$

the knowledge of which enables us to calculate any expectation value that reflects phase properties, e.g., $\langle\cos\varphi\rangle$, $\langle\sin\varphi\rangle$, $\langle\cos^2\varphi\rangle$, etc.

Almost needless to say, the experimentally determined "phase-space distribution" (5), in fact, contains more information than the phase distribution (7); in particular, the amplitude distribution is found from Eq. (5) by averaging over the phase.

Let us now have a closer look at the right-hand side of Eq. (4). It is easy to show that, apart from a normalization constant and an irrelevant phase factor, it is equal to the scalar product of the Glauber state $|\alpha\rangle$ and the signal field ψ for $\alpha=x_3+ip_4$. In fact, in the x representation this scalar product reads [cf. Eq. (17) for $s=1$]

$$\langle\alpha|\psi\rangle=\pi^{-1/4}\int\psi(x)\exp(-\tfrac{1}{2}[x-2^{1/2}\xi]^2-i2^{1/2}\eta x)dx , \quad (8)$$

where we have put

$$\alpha=\xi+i\eta . \quad (9)$$

Substituting here

$$X=2^{1/2}x \quad (10)$$

gives us

$$\langle\alpha|\psi\rangle=\pi^{-1/4}2^{-1/2}\int\psi(2^{-1/2}X)\exp(-\tfrac{1}{4}[X-2\xi]^2-i\eta X)dX , \quad (11)$$

and the new substitution

$$X'=X-\xi \quad (12)$$

leads to

$$\langle\alpha|\psi\rangle=\pi^{-1/4}2^{-1/2}\exp(-i\eta\xi)\times\int\psi(2^{-1/2}[X'+\xi])\times\exp(-\tfrac{1}{4}[X'-\xi]^2-i\eta X')dX' . \quad (13)$$

With the identifications $\xi=x_3$, $\eta=p_4$ the integral occurring here is indeed identical to that on the right-hand side of Eq. (4). Thus we arrive at the following relation:

$$\Phi_{\rm out}(x_3,p_4)=\pi^{-1/2}\exp(ip_4x_3)\langle\alpha|\psi\rangle . \quad (14)$$

Observing Eq. (5) and noticing the definition of the Q function for a pure state ψ

$$Q(\alpha)=\pi^{-1}|\langle\alpha|\psi\rangle|^2 , \quad (15)$$

we thus find from Eq. (14) the simple relation

$$w(x_3,p_4)=Q(\alpha=x_3+ip_4) , \quad (16)$$

which, in fact, has already been derived for the special case of an incident coherent field [8].

One learns from the quite general result (16) that Mandel's measuring procedure [5], when not restricted from the beginning to the study of phase properties, amounts to measuring the Q function of the signal field. Since the same holds true for the measuring schemes proposed by Bandilla and Paul [3,10] and by Shapiro and Wagner [4], we can state that all three approaches are perfectly equivalent. In particular, they lead, according to Eq. (7), to identical (measured) phase distributions, despite the large differences in their experimental setup and the physical processes involved. It should be noticed, however, that they actually share the common feature that undesired additional noise is introduced either by beam splitting or by amplification.

At first sight, one might find it surprising that those different sources of noise should give rise to precisely the same enhancement of phase (as well as amplitude) fluctuations, compared to the "true" fluctuations present in the original field according to the phase operator concept [2]. It should be noted, however, that formally amplifier and beam-splitter noise share a common property: Both of them enter, in the form of Langevin forces, the quantum-mechanical equations of motion in just such a way as to preserve the quantum-mechanical commutation relations for the photon creation and annihilation operators in the course of interaction, either with the beam splitter or the amplifying medium. From this argument, one may actually find the equivalence of all operational definitions of phase discussed so far quite natural.

III. EXAMPLES OF MEASURABLE PHASE DISTRIBUTIONS

A. Squeezed states

As is well known, in this case the wave function reads [11]

$$\psi_{\rm sq}(x)=(s/\pi)^{1/4}\exp\left\{-\frac{s}{2}[x-2^{1/2}\xi]^2+i2^{1/2}\eta x\right\}, \tag{17}$$

where ξ and η characterize the displacement and s is the squeezing parameter. From Eq. (17) the Q function is readily calculated,

$$Q_{\rm sq}(\alpha=q+ip)=2\pi^{-1}s^{1/2}(s+1)^{-1}\times\exp\left\{-\frac{2s}{s+1}(q-\xi)^2-\frac{2}{s+1}(p-\eta)^2\right\}. \tag{18}$$

Introducing now polar coordinates

$$q=\rho\cos\varphi\ , \quad p=\rho\sin\varphi\ , \tag{19}$$

and forming the average (7) over ρ, we arrive at the result

$$w(\varphi)=\pi^{-1}s^{1/2}(s+1)^{-1}u^{-2}\times\exp\{-4s(s+1)^{-2}u^{-2}(\xi\sin\varphi-\eta\cos\varphi)^2\}\times f(v/u)\ , \tag{20}$$

where the following abbreviations have been introduced:

$$u^2=2(s+1)^{-1}(s\cos^2\varphi+\sin^2\varphi)\ , \tag{21}$$

$$v=2(s+1)^{-1}(s\xi\cos\varphi+\eta\sin\varphi)\ , \tag{22}$$

$$f(x)=\exp(-x^2)+\pi^{1/2}x[1+\mathrm{erf}(x)]\ . \tag{23}$$

Equation (20) includes as special cases the squeezed vacuum ($\xi=0$, $\eta=0$) and the Glauber state ($s=1$).

B. Fock states

For these states the Q function reads

$$Q_{\rm Fock}(\alpha)=\pi^{-1}|\langle\eta|\alpha\rangle|^2=\pi^{-1}\exp(-|\alpha|^2)|\alpha|^{2n}/n!\ . \tag{24}$$

Obviously, it is independent of the phase; hence the corresponding phase distribution is constant. This is just what one expects, since there are no distinguished phases in a Fock state.

IV. DISCUSSION

The Q distribution is more strongly smeared out than the Wigner distribution (both are Gaussian convolutions of Glauber's P function, the width of the Gaussian, however, being greater by a factor of $2^{1/2}$ in case of the Q distribution [12]). As a result, the widths of the two marginal distributions for x_3 and p_4, respectively, following from the Q function (16), satisfy the inequality [13]

$$\Delta x_3\Delta p_4\geq 1\ , \tag{25}$$

instead of Heisenberg's sharper inequality

$$\Delta x_1\Delta p_1\geq\tfrac{1}{2} \tag{26}$$

valid for the signal field. Accordingly, the phase distribution obtained from the Q function by averaging over ρ will be broader than the "true" one following from the phase operator concept [2].

Finally, we would like to mention that the problem of simultaneously measuring a pair of canonically conjugate variables has, in fact, a longer history. It was first tackled by Arthurs and Kelly [14], and recently their approach was followed up by Braunstein, Caves, and Milburn [15]. It is to the credit of Stenholm [13] to have elucidated the physical content of those papers and to have given the mathematical formalism an admirably transparent form, thereby establishing the connections with related work by Husimi [16] and himself [17]. The proposed measurement process, thus far discussed only theoretically, consists in instantaneously coupling, via a suitably chosen nonlinear interaction, the system under investigation to two "meter systems," which are read out after the interaction. It turns out [13] that also with such a setup the Q function of the signal is measured, provided optimal conditions (corresponding to equal resolution in the measurement of the two conjugate variables) are chosen. Moreover, the Q function was recently shown [18] to be a special case of the quantum-mechanical propensity [19], corresponding to the choice of the reference state as the vacuum state and of the phase-space "motion" as Glauber's displacement operator.

So the result found in the present paper fits very well into a, as it seems, quite general context. What can be measured, at best, in any attempt to obtain information on the simultaneous values of two conjugate variables, is apparently the Q function.

To summarize, we have demonstrated that the experimental scheme for phase measurement recently proposed and realized by Noh, Fougères, and Mandel [5] quite generally amounts to measuring the Q function for the signal field under investigation, provided the coherent reference fields are very strong compared to the signal field. We have thus succeeded in generalizing a special result by Freyberger and Schleich [8].

Note added. Recently we became aware of a paper by Y. Lai and H. A. Haus in Quantum Opt. **1**, 99 (1989), in which beam splitting, combined with homodyne detection, was shown to be a practical means of measuring the Q function. However, those authors did not establish a connection between their result and phase measurement.

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[10] Actually, in the measuring scheme [3] based on amplification the stretched distribution $Q(G^{-1/2}\alpha)$, where $Q(\alpha)$ denotes the Q function for the original field and G the amplification factor, is measured directly. $Q(\alpha)$ is then readily obtained from $Q(G^{-1/2}\alpha)$ by rescaling.

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