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Interpretation of geometric phase via geometric distance and length during cyclic evolution

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We cast the nonadiabatic geometric phase in terms of the geometric distance function and the geometric length of the curve for arbitrary cyclic evolution of the quantum states. An interpretation is given to the geometric phase as the value of the integral of the contracted length of the curve along which the system traverses. It is found that for arbitrary cyclic evolution of the quantum states the geometric phase $\beta(C)$ acquired by the system cannot be greater than the total length of the curve $l(C)$. We have argued that the *geometric phase* arises because of the fundamental inequality between the length of the curve and the distance function. Finally, we have illustrated the calculation of the geometric phase based on the geometric distance function and the geometric length of the curve.

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I. INTRODUCTION

The geometric phase plays an important role in understanding some of the outstanding enigmas of quantum mechanics. Originally the geometric phase was formulated for adiabatically changing environments [1], in which the system Hamiltonian $H(t)$ varies very slowly. The wave function in such systems is an associated instantaneous eigenstate of the Hamiltonian and after a cyclic evolution of the external parameters the state returns to itself apart from a phase factor. It was Berry who realized that the phase factor is not just the usual dynamical one, but contains another one which is purely geometric in nature. The dynamical phase provides information about the duration of the evolution of the system whereas the geometric phase reflects the geometry of the circuit and its magnitude depends on the path, but not on the rate of traversal of the path. But what significance do we attach to this geometric phase? As long as the wave function describes the whole system we know that the phase factors associated with the state do not matter, because the physical quantities are determined by the absolute value squared. Berry suggested that this is not so. If we divide the system into two parts and let the subsystem undergo a cyclic evolution, then the interference experiments between different parts allow the determination of the relative phases. In this way the geometric phase could lead to observable consequences.

Simon [2] interpreted the adiabatic Berry phase as a consequence of parallel transport of vectors in a curved space appropriate to the quantum system. He showed that the Berry phase is an early example of holonomy that was known to physicists long before differential geometry. In an important generalization Aharonov and Anandan [3] removed the adiabaticity condition and defined a geometric phase for arbitrary cyclic evolution of the quantum system. This nonadiabatic Berry phase (AA phase) is the holonomy transformation for parallel transport around a curve $\hat{C}$ in the projective Hilbert space $\mathcal{P}$, with respect to the natural connection given by the inner product in $\mathcal{H}$. It is also the "area" of any surface spanned by C with respect to the natural symplectic structure in $\mathcal{H}$ determined by this inner product. Soon after this Samuel and Bhandari [4] showed that the geometric phase is not akin to adiabatic and nonadiabatic cyclic evolutions, but it also appears in a more general context like nonunitary, noncyclic evolutions. In a further generalization the AA phase made its appearance in nonlinear equations governing the classical fields [5]. Geometric phases now abound in many areas of physics, such as in the Born-Oppenheimer approximation [6], in the Jahn-Teller effect [7], in the quantum Hall effect [8], and in understanding anomaly phenomena in quantum field theory [9]. Recently it was shown that the spin-orbit interaction arises as a Berry phase term in the adiabatic effective Hamiltonian for the orbital motion of a Dirac electron [10]. Many experiments have been performed and proposed to support the discovery of the geometric phase.

Turning to the aspect of calculation of the nonadiabatic Berry phases there are three main methods available so far in the literature. One is the operator decomposition method of Moore and Stedman [11], where they use the Floquet theorem for periodic Hamiltonians and decompose the unitary evolution operator into two parts. One part gives the total phase for cyclic choice of initial states and the other part, being a periodic one, gives the geometric phase. It provides a simple algorithm for calculating the nonadiabatic Berry phase. However, this method is inefficient and is not applicable for general nonperiodic and time-independent Hamiltonians. The other two methods employed are the geometrical approach and the Lie-algebraic approach [12]. In the geometrical approach the AA phase is expressed in terms of coordinates of the projective Hilbert space $\mathcal{P}$ and can be written as an integral over one form. In the Lie-algebraic approach the Hamiltonian of the system is a member of some Lie algebra and the Berry phase is calculable in a direct way from the Hamiltonian itself.

This paper concerns the interpretation and an efficient calculation of the AA phase using geometric concepts such as "distance function" and "length of the curve" during arbitrary cyclic evolutions. The Hamiltonian that describes the system may or may not be dependent on time and need not even be periodic. All we require is that the state should be cyclic—a fundamental feature of the evolution of the system where the final state differs from the initial state only by a multiplicative phase factor. We also give an explanation about how the geometric phase originates during the time evolution of the quantum system. The claim by Moore [13] that the Berry phases for time-dependent and time-independent Hamiltonians have a different origin seems to be unfounded. In our view the geometric phases for both types of Hamiltonians have the same origin, which has a deep topological relation to the quantum state space. In Sec. II we outline the formalism that will be used to calculate and interpret the nonadiabatic Berry phase. There we briefly define the quantities such as "geometric distance function," "geometric length of the curve," and, for the sake of completeness, also define the AA phase. In Sec. III we present three examples to illustrate the ready viability of this method for calculation of the geometric phase. The first example is that of a spin-$\frac{1}{2}$ particle precessing in a homogeneous magnetic field, the second example is that of a two-level atom interacting with external electromagnetic (em) field, and the third is that of a neutral spin-$\frac{1}{2}$ particle in the presence of a harmonic oscillator potential along with a nonuniform magnetic field. In each of these cases we calculate the "geometric distance," the "geometric length of the curve," and utilize these quantities to calculate the geometric phase. This is followed by some conclusions in Sec. IV.

II. FORMALISM

Consider a set of normalized vectors belonging to a Hilbert space $\mathcal{H}$ of dimension $N+1$. Then the set of rays of $\mathcal{H}$ forms a projective Hilbert space $\mathcal{P}$ with one dimension less than that of $\mathcal{H}$, i.e., N, where the rays are defined as the equivalence classes of states differing only in phase.

Let $|\Psi(t)\rangle$ be a quantum state that evolves according to the Schrödinger equation

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle \ , \tag{1}$$

and for all times, $|\Psi(t)\rangle \in \mathcal{H}$ with $\langle\Psi|\Psi\rangle = 1$. Define a cyclic vector for the evolution equation if there is a cycle time T such that an initial state and a final state differ by a multiplicative phase factor, i.e.,

$$|\Psi(T)\rangle = e^{i\phi}|\Psi(0)\rangle \ . \tag{2}$$

The existence of the cyclic state is assured by the very fact that it is in an eigenvector of the unitary evolution operator $U(T)$ with the corresponding eigenvalue $e^{i\phi}$. The existence of such cyclic initial states has been discussed in detail by Moore [12]. It should be noted that for the periodic Hamiltonians the cyclic state exists. For Hamiltonians of the form $H(t)=e^{-iAt}H(0)e^{iAt}$, the cyclic initial states are precisely the eigenvectors of the time-independent operator $B=H(0)-A$. Although a complete set of such cyclic states must exist for systems with finite-dimensional Hilbert space, this may not be true in general.

Let there be a natural projection map in $\mathcal{P}$, Π: $\mathcal{H}\rightarrow\mathcal{P}$ defined by $\Pi(|\Psi\rangle)=\{|\Psi\rangle' \ |\Psi\rangle'=c|\Psi\rangle$, for any complex number $c\}$. Then the cyclic evolution of the state describes a curve $\hat{C}, t\rightarrow\Psi(t)$ in $\mathcal{H}$ that begins and ends on the same ray. That is to say, C: $[0,T]\rightarrow\mathcal{H}$, with $\hat{C}=\Pi(C)$ being a closed curve in $\mathcal{P}$ and it is the image of the curve C under the projection map Π. If we want a "connection" to be defined in $\mathcal{P}$, we remove the dynamical phase factor from the state $|\Psi(t)\rangle$ and obtain the state $|\psi(t)\rangle$. Then the state vector $|\psi(t)\rangle$ will undergo a parallel transportation, where $|\psi(t)\rangle$ is given by

$$|\psi(t)\rangle = \exp\left[\frac{i}{\hbar}\int_0^t\langle\Psi|H|\Psi\rangle dt'\right]|\Psi(t)\rangle \ , \tag{3}$$

such that $\langle\psi(t)|d\psi(t)/dt\rangle=0$. Alternatively, we can see that $|\psi(t)\rangle$ keeps its phase unchanged to the extent possible for infinitesimal changes, i.e., the infinitesimal change in $|\psi(t)\rangle$ is orthogonal to $|\psi(t)\rangle$ itself. This can be expressed as $\langle\psi(t)|d\psi(t)\rangle=0$. Thus $|\psi(t)\rangle$ is phased with the parallel connection. During cyclic evolution the curve traced by $\psi(t)$ is not closed and $|\psi(T)\rangle=e^{i\beta}|\psi(0)\rangle$, where $e^{i\beta}$ is the holonomy transformation associated with the curve $\hat{C}$. An explicit expression for β is obtained [3] by choosing a single-valued state $|\tilde{\Psi}(t)\rangle=e^{-if(t)}|\Psi(t)\rangle$ with $\langle\tilde{\Psi}|\tilde{\Psi}\rangle=1$, and $f(t)$ is any smooth function of time satisfying $f(T)-f(0)=\phi$. It is then easy to show that $|\tilde{\Psi}(T)\rangle=|\tilde{\Psi}(0)\rangle$. These single-valued states do not depend on the redefinition of the phase of $|\Psi(t)\rangle$; hence single-valued vectors only depend on the shadow of the evolution of $|\Psi(t)\rangle$ on the projective Hilbert space. Thus the geometric phase is given in terms of these single-valued states as

$$\beta=i\int_0^T\langle\tilde{\Psi}(t)|\dot{\tilde{\Psi}}(t)\rangle dt \ , \tag{4}$$

for arbitrary cyclic evolution of the quantum state, implying that the nonadiabatic Berry phase depends only on the image of the evolution in $\mathcal{P}$. Therefore the AA phase is independent of the phase that relates the initial state to the final state. It is also independent of the Hamiltonian that causes the motion for a given projection of the evolution in $\mathcal{P}$, or in other words it is independent of the energy normalization. This is because if we add a scalar E to the Hamiltonian then it will give a different total phase factor to the evolving state; however, the extra phase will be absorbed in the dynamical phase and in no way affects the geometric phase. One can see also that if one changes the parameter t of $\hat{C}$ to t' with $dt/dt'>0$ then β is an invariant and is a geometric property of the unparametrized image of $\hat{C}$ in $\mathcal{P}$.

In addition to the geometric phase there are two more geometric objects in the projective Hilbert space $\mathcal{P}$. For an arbitrary quantum evolution (not necessarily cyclic) there exist (i) the geometric distance function and (ii) the geometric length of the curve. It has long [14] been recognized that the inner product of vectors in $\mathcal{H}$ gives a metric. This metric is the distance between quantum states. Provost and Valle [15] considered in particular the distance between two quantum states which are infinitesimally close, and found that it induces a Riemannian metric. If the quantum system evolves in time, then the minimum normed distance function between $|\Psi(t)\rangle$ and $|\Psi(t+dt)\rangle$ is given by

$$dD=[2-2|\langle\Psi(t)|\Psi(t+dt)\rangle|]^{1/2} . \quad (5)$$

This can be evaluated directly by Taylor expanding $|\Psi(t+dt)\rangle$ up to second order in time and using the expression

$$|\langle\Psi(t)|\Psi(t+dt)\rangle|=1-\tfrac{1}{2}\Delta E^2(t)dt^2/\hbar^2+O(dt^3) ,$$

where

$$\Delta E^2(t)=\langle\Psi|H^2|\Psi\rangle-\langle\Psi|H|\Psi\rangle^2 . \quad (6)$$

Thus for an arbitrary evolution the infinitesimal distance function as measured by the Riemannian metric is given by

$$dD=\Delta E(t)dt/\hbar . \quad (7a)$$

If $\Psi(t)$ is a curve C: $[0,T]\rightarrow\mathcal{H}$, then the distance traveled during the interval $[0,T]$ is

$$D=\int_0^T\Delta E(t)dt/\hbar . \quad (7b)$$

This distance function differs from that of the Fubini-Study metric [16] by a factor of 2. Anandan and Aharonov have given a geometric meaning to this quantity in Ref. [17]. This is geometric in the sense that it does not depend on the particular Hamiltonian used to transport the state along a given curve $\hat{C}$ in $\mathcal{P}$. It is also independent of the phases of $|\Psi(t)\rangle$ and $|\Psi(t+dt)\rangle$, which is a consequence of the two-point Bargmann invariant; and therefore depends only on the points to which they project. By changing the Hamiltonian we will have a different overall phase factor associated with the evolving state; however, they will give the same value for the dimensionless quantity D. According to Anandan and Aharonov the evolution of the system in $\mathcal{P}$ completely determines $\Delta E(t)$; no other information about the Hamiltonian is needed to determine $\Delta E(t)$. Also the geometric distance function is independent of the rate of evolution of the system and depends only on the unparametrized curve $\hat{C}$ in $\mathcal{P}$, that is determined by the evolution of the state vector.

The next geometric object in $\mathcal{P}$ is the "length of the curve" along which the quantum system moves. We study the transport of the state vector in the projective Hilbert space along a closed curve but earlier authors have not defined and calculated it explicitly for a given problem. Below we define it for arbitrary evolution of the quantum system and we will calculate it for various problems that will be treated subsequently in Sec. III. On a proper Riemannian manifold the existence of a metric allows the definition of the "length of the curve" $\hat{C}$ in $\mathcal{P}$ which is traced out by the normalized vector $|\tilde{\Psi}(t)\rangle$ [20].

Let $|\Psi(t)\rangle$ be a curve C: $[0,T]\rightarrow\mathcal{H}$. We choose a section of the curve as $\tilde{\Psi}$ which is differentiable along C such that the length of the $\tilde{\Psi}(t)$ along which the system evolves from point $\tilde{\Psi}(0)$ to a point $\tilde{\Psi}(T)$ is a number defined as

$$l(\tilde{\Psi})|_0^T=\int_0^T\langle\dot{\tilde{\Psi}}(t)|\dot{\tilde{\Psi}}(t)\rangle^{1/2}dt . \quad (8)$$

Here $|\dot{\tilde{\Psi}}(t)\rangle$ is the velocity vector in the projective Hilbert space $\mathcal{P}$ of the curve $\tilde{\Psi}$ at point t along the path of evolution of the state vector. It is the tangent vector to the curve $\tilde{\Psi}(t)$.

We would like to mention a few important geometric properties of the length of the curve. First the integral exists, since the integrand is continuous. The length of a broken C curve is defined as the finite sum of the length of its C pieces. The number $l(\tilde{\Psi})|_0^T$ is independent of the parametrization of its image set. If we change the parameter from t to t' with $dt/dt'>0$, then the length of the curve remains unchanged. Therefore it is a geometric property of the whole curve $\hat{C}$ in $\mathcal{P}$ and is a t-invariant quantity. Hence for arbitrary time evolution of the state, we can define the infinitesimal length of the curve during the infinitesimal time dt as

$$dl=\langle\dot{\tilde{\Psi}}(t)|\dot{\tilde{\Psi}}(t)\rangle^{1/2}dt . \quad (9)$$

If the parameter t is such that $\langle\dot{\tilde{\Psi}}(t)|\dot{\tilde{\Psi}}\rangle^{1/2}$ is constant then we may surmise that the length of the curve is parametrically proportional to the arc length. We emphasize here that as the existence of the Berry phase is measured experimentally, the total length of the curve could be measured for cyclic evolution of the quantum states.

Having defined the geometric distance function and geometric length of the curve we can easily see that for an arbitrary cyclic evolution of the quantum system any physical state traces a closed curve $\hat{C}$ in $\mathcal{P}$, such that at each instance of time, the length of the curve is greater than the distance traveled by the quantum system and the nonadiabatic Berry phase is given by

$$\beta=\int_0^T(dl^2-dD^2)^{1/2}=\int_0^T(1-v_{\mathcal{H}}^2/u_{\mathcal{H}}^2)^{1/2}dl , \quad (10)$$

or $\beta=\int_0^T dL$, where $dL=(1-v_{\mathcal{H}}^2/u_{\mathcal{H}}^2)^{1/2}dl$, is called the infinitesimal contracted length of the curve. Therefore the geometric phase is manifested as the integral of the contracted length of the curve C along which the system moves. The factor $(1-v_{\mathcal{H}}^2/u_{\mathcal{H}}^2)^{1/2}$ is called the length contraction factor because $v_{\mathcal{H}}<u_{\mathcal{H}}$, where $v_{\mathcal{H}}=dD/dt$ is called the speed of transportation in the projective Hilbert space $\mathcal{P}$ and $u_{\mathcal{H}}=dl/dt$ is called the magnitude of the rate of change of arc length of the curve C in $\mathcal{P}$. The speed of transportation in $\mathcal{P}$ gives the rate at which the state moves away from the original state determined by the inner product between them. Expression (10) gives a new way of looking into the geometric phase factor accompanying nonadiabatic evolution. It only depends on the two geometric objects in $\mathcal{P}$, the geometric length of the curve (a t-invariant quantity) and the geometric distance function [an $H(t)$-invariant quantity]. This also says how the geometric phase arises in a general situation. Thus from (10) it is quite clear that the geometric phase as introduced here originates from a different reason than realized so far. It arises because at each instant of time the "length" is greater than the "distance." The above fact does not depend on the Hamiltonian's

time dependence. As long as the system evolves in time, we have the concept of length and distance, and because of the fundamental inequality between the "length" and "distance" the system shows up a geometric phase. For cyclic evolution of the quantum system the excess length of the curve at each instant of time over the distance goes on accumulating, so that it finally appears as the geometric phase. However, if the system evolves along a shortest geodesic the length and the distance coincide and hence the geometric phase vanishes. Also it is evident that if the contraction factor approaches zero the system will not acquire any geometric phase. For no contraction, i.e., when the contraction factor approaches unity the geometric phase acquired by the system exactly coincides with the total length of the curve during the cyclic evolution. Thus our expression also sets a limit to the value of the geometric phase that a quantum system may acquire upon a cyclic excursion. Therefore $\beta(C)$ ranges from zero to total length of the curve, i.e., $0<\beta(C)<l(C)$.

In the main result (10), the integrand and therefore β is non-negative. But for a general $|\tilde{\Psi}\rangle, \beta$ can be negative and furthermore, even if β is non-negative, the integrand would take both positive and negative values. But it is always possible to do a guage transformation $|\tilde{\Psi}(t)\rangle \to e^{i\Lambda t}|\tilde{\Psi}(t)\rangle$, $\Lambda(0)=\Lambda(T)+2\pi n$, where n is an integer, so that for new $|\tilde{\Psi}\rangle, i\langle\tilde{\Psi}|\dot{\tilde{\Psi}}\rangle$ is non-negative in the interval $[0,T]$. The latter condition, which implies that beta is non-negative, is necessary and sufficient for the validity of (10).

In the following section we use (10) explicitly to study various examples and also to calculate the geometric phase via calculating the geometric distance function and the geometric length. Our expression not only provides a better understanding of the geometric phase but also provides a tractable algorithm for calculation of the nonadiabatic Berry phase. Our first example comprises a fermion system interacting with the homogeneous magnetic field. Secondly we focus our attention on a time-dependent Hamiltonian involving coupling between bosons and fermions, i.e., the Jaynes-Cummings model of quantum optics which describes the interaction of light with a two-level atom. The third example we consider is that of a time-independent Hamiltonian where the fermion is coupled to a bosonic (harmonic oscillator) potential along with a nonuniform magnetic field. In each of these cases the geometric distance function and geometric length of the curve are calculated. The latter quantity is found to be always greater than the former one and the integrated difference between them is attributed to the geometric phase.

III. EXAMPLES

A. Spin-$\frac{1}{2}$ particle precessing in a magnetic field

This is a simplest example in which the precession of a spin-$\frac{1}{2}$ particle in a homogeneous magnetic field **B** has been considered. The Hamiltonian in the rest frame is given by $H=-\mu B\sigma_z$, where $B=|\mathbf{B}|$ and σ_z is the Pauli spin matrix. The Hilbert space is spanned by two-dimensional vectors with components $\cos(\theta/2)$, $\sin(\theta/2)$ and $\theta\in[0,\pi]$. The initial state is given by

$$|\Psi(0)\rangle = \begin{bmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{bmatrix}, \tag{11a}$$

which evolves into $|\Psi(t)\rangle$, where

$$|\Psi(t)\rangle = \begin{bmatrix}\cos(\theta/2)e^{i\phi/2}\\ \sin(\theta/2)e^{-i\phi/2}\end{bmatrix}, \tag{11b}$$

and $\phi=2\mu Bt/\hbar$, and corresponds to the spin direction being always at an angle θ to the z axis. Due to precession, the wave function rotates by 2π radians about some axis which results in a cyclic motion of every state vector of $\mathcal{H}$. We now calculate the infinitesimal normed distance function dD^2. It can be easily seen that dD^2 is equal to

$$\mu^2B^2\sin^2(\theta)dt^2/\hbar^2=\sin^2(\theta)d\phi^2 . \tag{12}$$

This is nothing but $\frac{1}{4}$ of the usual metric defined on the sphere of a unit radius with a fixed θ. (The factor $\frac{1}{4}$ arises from our convention.) Then the total distance traveled by the state vector during a cyclic evolution is given by $D=\pi\sin\theta$.

For the calculation of the geometric length of the curve we have to choose the single-valued state $|\tilde{\Psi}(t)\rangle$. That can be done easily by setting

$$|\tilde{\Psi}(t)\rangle = \begin{bmatrix}\cos(\theta/2)\\ \sin(\theta/2)e^{-i\phi}\end{bmatrix}, \tag{13}$$

such that $|\tilde{\Psi}(T)\rangle=|\tilde{\Psi}(0)\rangle$. Then the rate of change of arc length of the curve, $u_{\mathcal{H}}$ is given by

$$\begin{aligned} u_{\mathcal{H}} &= \langle \dot{\tilde{\Psi}}(t)|\dot{\tilde{\Psi}}(t)\rangle^{1/2} \\ &= \{(2\mu^2B^2/\hbar^2)[1-\cos(\theta)]\}^{1/2} . \end{aligned} \tag{14}$$

Therefore the total length of the curve during the cyclic evolution is given by

$$l(C)=\{2[1-\cos(\theta)]\}^{1/2}\pi . \tag{15}$$

From this calculation it is clear that the length of the curve is also a geometric object of the motion of the system, in the sense that $l(C)^2$ is just π times the total solid angle subtended by a curve traced on a unit sphere, by the direction of the spin state, at the center. Therefore it is as "geometric" as the geometric phase acquired by the system.

Once we have calculated the dD^2 and dl^2, it is now easy to calculate the geometric phase β. Thus β is given by

$$\begin{aligned} \beta &= \int_0^{2\pi}\{\tfrac{1}{2}[1-\cos(\theta)]-\tfrac{1}{4}\sin^2(\theta)\}^{1/2}d\phi \\ &= \pi[1-\cos(\theta)] , \end{aligned} \tag{16}$$

which is half the solid angle subtended by the orbit of motion in a sphere of unit radius. This result matches exactly with the ones derived previously in different ways.

In addition to these geometric objects we can also calculate the contraction factor $(1-v_{\mathcal{H}}^2/u_{\mathcal{H}}^2)^{1/2}$. This is found to be $\sin\theta/2$ in this case. A necessary and sufficient condition for acquiring a geometric phase is that the contraction factor during a cyclic evolution should be nonzero and in this case it is indeed true. For $\theta=0$ the spin state traces a curve such that the contraction factor vanishes identically, and hence the system will not acquire any geometric phase. It is interesting to note that for $\theta=\pi$ the contraction factor is unity, i.e., there is no contraction of the length of the curve and hence the geometric phase coincides with the length of the curve during a cyclic traversal of the spin state.

B. The Jaynes-Cummings model

In this example we consider a system where bosons and fermions are coupled and the Hamiltonian that describes it is an explicit time-dependent one. The situation is that of interaction between light and a two-level atom, which is well represented by the Jaynes-Cummings model (JCM) [18]. For this problem we calculate the geometric distance function, the geometric length, and the geometric phase using the former two quantities. Consider an atom in a strong laser beam that is nearly in resonance with one of the atomic transitions, for example, the transition between the ground state and the excited state. This is approximated by a two-level atom with a combined atom-electromagnetic field interaction Hamiltonian developed by Jaynes and Cummings [18]. The semiclassical Hamiltonian for this system is given by

$$H(t)=\omega S_z+g(e^{-i\omega t}S_+ + e^{i\omega t}S_-) , \tag{17}$$

where ω is the transition frequency of the atom, g is the atom-field coupling coefficient, and $S_\pm$, S_z are the usual atomic operators. The system is initially in a pure state which is factorizable and given by

$$|\Psi(0)\rangle=|\Psi_a(0)\rangle\otimes|\Psi_f(0)\rangle , \tag{18}$$

where $|\Psi_a(0)\rangle$ is a general atomic state and $|\Psi_f(0)\rangle$ is a coherent state. The total wave function at a later time can be written as

$$|\Psi(t)\rangle=c_+|\Psi_+(t)\rangle+c_-|\Psi_-(t)\rangle . \tag{19}$$

Now the cyclic state is chosen by assuming either $c_+=1$ and $c_-=0$ or $c_+=0$ and $c_-=1$. In the first case we choose $c_+=1$ and $c_-=0$. Then the states $|\Psi_\pm(t)\rangle$ are given by

$$\begin{aligned}|\Psi_+(t)\rangle&=\frac{1}{\sqrt{2}}e^{-igt}(e^{-i\omega t/2}|1\rangle+e^{i\omega t/2}|0\rangle) ,\\ |\Psi_-(t)\rangle&=\frac{1}{\sqrt{2}}e^{igt}(e^{i\omega t/2}|1\rangle-e^{-i\omega t/2}|0\rangle) ,\end{aligned} \tag{20}$$

where $|0\rangle$ and $|1\rangle$ are ground and excited states of the two-level atom. We can easily see that this is cyclic with an overall phase factor $\phi_+=(\pi-gT)$ for a period of $T=2\pi/\omega$, i.e.,

$$|\Psi_+(T)\rangle=e^{i\phi_+}|\Psi_+(0)\rangle . \tag{21}$$

We now calculate the distance function by evaluating ΔE. For this problem it is simple, because $\langle\Psi_+|H|\Psi_+\rangle=g$ and $\langle\Psi_+|H^2|\Psi_+\rangle=(\omega^2/4)+g$. Hence

$$dD^2=(\omega^2/4)dt^2 . \tag{22}$$

This shows that the speed of transportation $v_{\mathcal{H}}$ is constant for a two-level atom. Therefore the total distance traveled by the state during a cyclic evolution is just π.

To calculate the geometric length of the curve, we have to choose the single-valued state $|\tilde{\Psi}_+(t)\rangle$ corresponding to the state $|\Psi_+(t)\rangle$. This is achieved by writing $|\tilde{\Psi}_+(t)\rangle=e^{-if_+(t)}|\Psi_+(t)\rangle$, such that $f_+(T)-f_+(0)=\phi_+$. Therefore

$$|\tilde{\Psi}_+(t)\rangle=\frac{1}{\sqrt{2}}(e^{-i\omega t}|1\rangle+|0\rangle) . \tag{23}$$

Hence the infinitesimal length of the curve is given by

$$dl=\frac{1}{\sqrt{2}}\omega dt , \tag{24}$$

which again shows that the arc length changes at a constant rate. Therefore the total length of the curve during a cyclic evolution is equal to $\sqrt{2}\pi$.

The calculation of the geometric phase is now trivial, i.e.,

$$\beta(C)=\int_0^T(1-v_{\mathcal{H}}^2/u_{\mathcal{H}}^2)^{1/2}dl=\pi . \tag{25}$$

This matches exactly with the result that is obtained in Refs. [14,19]. In this example also we can calculate the contraction factor and it is found to be $1/\sqrt{2}$. However, there is no way of making the contraction factor zero and the system will always acquire a geometric phase. In this case the phase acquired by the system is $1/\sqrt{2}$ times the total length of the curve. Thus in the time-dependent problem considered here the length is greater than that of the distance and because of this there is a net phase π in the final state of the system.

On the other hand if $|\Psi_-(t)\rangle$ undergoes a cyclic evolution, then the total phase acquired will be $\phi_-=(\pi+gT)$, but the geometric phase will be the same, i.e., π. It has been shown by Moore [13] that the well-known Rabi oscillation frequency and the splitting of the Mollow triplet is related to the difference in the total phase for the two cyclic $(|\Psi_+\rangle,|\Psi_-\rangle)$ states.

C. Spin-$\frac{1}{2}$ particle in a harmonic oscillator potential with magnetic field

In this last example what we consider is a time-independent situation where our system is a fermion coupled with a bosonic potential together with a nonuniform magnetic field. The fermion interacts with the magnetic field via its magnetic moment. For such a system we calculate the geometric distance function, geometric length of the curve, and the geometric phase by using (10).

Let us consider an uncharged, spin-$\frac{1}{2}$ particle in a harmonic oscillator potential with a very weak, nonuniform magnetic field. For simplicity we consider the motion of the particle to be confined to one dimension, say along the z axis. The Hamiltonian of the one-dimensional system is given by

$$H=-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+\tfrac{1}{2}m\omega^2z^2+\mu B(z)\sigma_z \ , \tag{26}$$

where μ is the magnetic moment of the spinning particle, σ_z is the 2×2 Pauli spin matrix. It has been chosen that the nonuniform magnetic field has a minimum at $z=0$, therefore $B(z)=B_0z^2$, where B_0 is a constant having dimension of G/cm^2. Thus the Hamiltonian takes the form

$$H=-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+\tfrac{1}{2}m\hat{\omega}^2z^2 \ , \tag{27}$$

where $\hat{\omega}$ is a frequency operator with eigenfrequencies $\omega_\pm$, where

$$\omega_\pm=(\omega^2\pm B_0\mu/m)^{1/2} \ . \tag{28}$$

Physically this shows that in the presence of the magnetic field the particle with spin polarized in the positive z direction will oscillate with a frequency ω_+ and the one with its spin polarized in the negative z direction will oscillate with a frequency ω_-.

The state with spin up (spin down) can be chosen to be a cyclic one with a period $T=2\pi/\omega_+(T=2\pi/\omega_-)$. The cyclic state with spin up is

$$|\chi_+(z,t)\rangle=|\Psi_+(z,t)\rangle\otimes\begin{bmatrix}1\\0\end{bmatrix} \ , \tag{29}$$

where $|\Psi_+(z,t)\rangle$ is a wave packet oscillating at frequency ω_+ about $z=0$, with an amplitude z_0. Initially we choose a Gaussian shape of the wave function and after the summation is carried out $|\Psi_+(z,t)\rangle$ is given by

$$|\Psi_+(z,t)\rangle=(\alpha_+^{1/2}/\pi^{1/4})\exp\{-\tfrac{1}{2}\alpha_+^2[z-z_0\cos(\omega_+t)]^2 -i[\tfrac{1}{2}\omega_+t+\alpha_+^2zz_0\sin(\omega_+t)-\tfrac{1}{4}\alpha_+^2z_0^2\sin(2\omega_+t)]\} \ . \tag{30}$$

It can be easily seen that $|\chi_+(z,t)\rangle$ is cyclic, i.e.,

$$|\chi_+(z,T)\rangle=e^{i\phi_+}|\chi_+(z,0)\rangle \ , \tag{31}$$

with $\phi_+=\pi$, the total phase acquired in one complete cycle.

We now calculate the geometric distance function by evaluating $\langle\chi_+|H|\chi_+\rangle$ and $\langle\chi_+|H^2|\chi_+\rangle$:

$$\begin{aligned}\langle\chi_+|H|\chi_+\rangle&=\tfrac{1}{2}h\omega_++\tfrac{1}{2}h\omega_+\alpha_+^2z_0^2 \ ,\\ \langle\chi_+|H^2|\chi_+\rangle&=\tfrac{1}{4}h^2\omega_+^2+\tfrac{1}{4}h^2\omega_+^2\alpha_+^4z_0^4+h^2\omega_+^2\alpha_+^2z_0^2 \ .\end{aligned} \tag{32}$$

Hence the speed of transportation v is given by

$$v_{\mathcal{H}}=\Delta E/\hbar=\frac{1}{\sqrt{2}}\alpha_+\omega_+z_0 \ . \tag{33}$$

During the cyclic evolution of the system the total distance is

$$D=\sqrt{2}\pi\alpha_+z_0 \ . \tag{34}$$

To calculate the geometric length of the curve we have to define the single-valued wave function. This is done again by defining $|\tilde{\chi}_+(z,t)\rangle=e^{-if_+(t)}|\chi(z_+t)\rangle$ in such a way that $f_+(T)-f_+(0)=\phi_+$. Hence

$$|\tilde{\chi}_+(z,t)\rangle=(\alpha_+^{1/2}/\pi^{1/4})\exp\{-\tfrac{1}{2}\alpha_+^2[z-z_0\cos(\omega_+t)]^2-i[\alpha_+^2zz_0\sin(\omega_+t)-\tfrac{1}{4}\alpha_+^2z_0^2\sin(2\omega_+t)]\}\begin{bmatrix}1\\0\end{bmatrix} \ , \tag{35}$$

and $|\tilde{\chi}_+(z,T)\rangle=|\tilde{\chi}_+(z,0)\rangle$.

Therefore the rate of change of arc length of the curve is given by

$$\begin{aligned}u_{\mathcal{H}}&=\langle\dot{\tilde{\Psi}}_+(z,t)|\dot{\tilde{\Psi}}_+(z,t)\rangle^{1/2}\\&=\{\tfrac{1}{4}\omega_+^2\alpha_+^4z_0^4+\tfrac{1}{2}\omega_+^2\alpha_+^2z_0^2\}^{1/2} \ .\end{aligned} \tag{36}$$

Now the total length curve, a geometric property of the curve, is

$$l(C)=[\tfrac{1}{4}\alpha_+^2z_0^2+\tfrac{1}{2}]^{1/2}2\pi\alpha_+z_0 \ . \tag{37}$$

Once again one can see here that total length of the curve is greater than that of the geometric distance function during a cyclic evolution of the wave packet.

Once the geometric distance and length is calculated it is easy to calculate the geometric phase using (10). Thus $\beta(C)$, during one cycle, is

$$\beta(C)=\pi\alpha_+^2z_0^2 \ , \tag{38}$$

which can otherwise be calculated and matched with the result obtained here.

Lastly we calculate the contraction factor. It is found to be

$$\frac{dL}{dl}=[\tfrac{1}{2}\alpha_+^2z_0^2/(1+1/2\alpha_+^2z_0^2)]^{1/2} \ , \tag{39}$$

which is independent of time and is finite. There is no way to make the contraction factor equal to zero thereby making the geometric phase vanish identically. However, when the particle oscillates with a resonant frequency $\omega^2 = \beta_0 \mu / m$, then only may we have zero contraction factor. But then the geometric distance and length also vanish for resonant oscillation and hence it is of no physical importance. It is important to note that we cannot make the contraction factor equal to unity and hence there is always a contraction of the length in this problem which asserts that the geometric phase acquired by the system is the integral of the contracted length of the curve during the cyclic evolution.

IV. CONCLUSIONS

In conclusion we have shown in this paper that the geometric objects such as "phase," "distance," and "length of the curve" in the projective Hilbert space are intimately related. We could cast the expression for the geometric phase in terms of the geometric distance function and geometric length of the curve. This enables us to interpret it as an integral of the contracted length of the curve which the system traverses. In addition to this we are able to answer an important question concerning the geometric phase, viz., given a cyclic evolution of the quantum system, what would the maximum value of the nonadiabatic geometric phase acquired by the system during that period be? The answer we found is that it is just the total length of the curve $\hat{C}$ in $\mathcal{P}$, during the evolution. Thus in any case the geometric phase cannot be greater than the total length of the curve. Furthermore our expression provides a new way of calculating the nonadiabatic geometric phase, which we have illustrated by studying three examples and calculating explicitly the geometric objects.

As a final remark we would like to mention that the length of curve $l(C)$ is as geometric as the geometric phase. Since the geometric phase is a physically observed quantity, and the geometric distance function can be estimated by measuring the uncertainty in the energy of the system; hence the length of the curve should in principle be measurable.

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