Relativistic extension of the Kay-Moses method for constructing transparent potentials in quantum mechanics

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For the Dirac equation in one space dimension with a potential of the Lorentz scalar type, we present a complete solution for the problem of constructing a transparent potential. This is a relativistic extension of the Kay-Moses method which was developed for the nonrelativistic Schrödinger equation. There is an infinite family of transparent potentials. The potentials are all related to solutions of a class of coupled, nonlinear Dirac equations. In addition, it is argued that an admixture of a Lorentz vector component in the potential impairs perfect transparency.

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I. INTRODUCTION

In their classic paper of 1956, Kay and Moses gave a complete solution for the problem of finding all possible transparent potentials for the Schrödinger equation in one dimension [1]. In the present paper we are interested in developing a relativistic version of the Kay-Moses method [1], i.e., in finding all possible transparent potentials for the Dirac equation in one dimension. The transparent potentials for the Schrödinger equation later found many interesting applications in nonlinear problems such as those in soliton physics. The relativistic extension of the Kay-Moses method may also find applications in nonlinear problems.

It is understood that the potentials are localized. A potential is said to be transparent or reflectionless if a wave of any shape incident on the potential, say, from the left, is transmitted to the right without any reflection. The shape of the wave may be altered through transmission. If the incident wave is a plane wave, the transparency should hold for any wave number or energy. The phase of the wave may be shifted. In the relativistic case negative-energy waves have to be included.

Let us define the problem more explicitly. We consider the Dirac equation in one space dimension,

$$[\alpha p+\beta m+\beta S(x)+V(x)]\psi(x)=E\psi(x) , \tag{1.1}$$

where $c=\hbar=1$ and $p=-id/dx$. We will be confined to one space dimension throughout, hence we will not repeat the phrase "one dimension" hereafter. In the potential $\beta S(x)+V(x)$ that we assume, S is a Lorentz scalar and V is the zeroth component of a Lorentz vector. It is understood that S and V both vanish as $x\rightarrow\pm\infty$. The relativistic energy E includes the rest mass m that we assume to be nonzero. The wave function ψ is a two-component spinor. α and β are 2×2 Pauli matrices; we use $\alpha=\sigma_y$ and $\beta=\sigma_z$. The problem is to find $\beta S(x)+V(x)$, which is transparent. As we will discuss in the last section there are reasons to suspect that V is not allowed for a transparent potential. Therefore let us assume that

$$V(x)=0 , \tag{1.2}$$

and focus on the Lorentz scalar type. In Ref. [2] we found, in a heuristic manner, an example of $S(x)$ that is transparent. In the present paper we give a systematic method for constructing such potentials. The method is a complete one in the sense that it exhausts all possibilities for the Lorentz scalar type.

It would facilitate having a perspective of the problem if we present the basic idea that has guided us throughout. Rather than directly dealing with Eq. (1.1), consider bound-state solutions of the nonlinear Dirac (NLD) equation,

$$\left[\alpha p+\beta m-\beta\sum_{j=1}^{N}g_j(\phi_j^\dagger\beta\phi_j)\right]\phi_i(x)=E_{Bi}\phi_i(x) , \tag{1.3}$$

where the subscript i $(=1,2,\ldots,N)$ refers to N unknown wave functions, each of which is a two-component spinor. The g_i's are positive constants. The ϕ_i's are normalized as

$$\int_{-\infty}^{\infty}\phi_i^\dagger\phi_i dx=1 . \tag{1.4}$$

When the ϕ_i's are found, define $S(x)$ by

$$S(x)=-\sum_{i=1}^{N}g_i\phi_i^\dagger(x)\beta\phi_i(x) . \tag{1.5}$$

We suspected that, when used in Eq. (1.1), $S(x)$ of Eq. (1.5) would be a transparent potential. Note that, unlike Eq. (1.3), Eq. (1.1) is a *linear* equation for ψ (with an "external" potential S). Equation (1.1) with $S(x)$ of Eq. (1.5) has N bound-state solutions with eigenvalues E_{Bi} and eigenfunctions ϕ_i. For the transmission (scattering) problem, energy E can be chosen at will ($|E|>m$).

The above idea was hinted at by a similar situation for the Schrödinger equation,

$$\left[-\frac{1}{2m}\frac{d^2}{dx^2}+U(x)\right]\psi(x)=\varepsilon\psi(x) . \tag{1.6}$$

All possible transparent potentials for the Schrödinger equation can be constructed by the Kay-Moses method [1]. Furthermore, there is one-to-one correspondence between the Kay-Moses transparent potentials and the bound-state solutions of the nonlinear Schrödinger (NLS) equation [3,4],

$$\left[-\frac{1}{2m}\frac{d^2}{dx^2}-\sum_{j=1}^{N} g_j\phi_j^2\right]\phi_i(x)=\varepsilon_i\phi_i(x) , \tag{1.7}$$

where $\phi_i(x)$ is normalized as $\int_{-\infty}^{\infty}\phi_i^2dx=1$ and

$$g_i=2\kappa_i/m , \quad \varepsilon_i=-\kappa_i^2/2m . \tag{1.8}$$

Any of the Kay-Moses potentials can be expressed as [3–5]

$$U(x)=-\sum_{i=1}^{N} g_i\phi_i^2(x) . \tag{1.9}$$

Thus the problem of constructing a transparent potential for Eq. (1.6) is exactly equivalent to solving Eq. (1.7). There is an infinite variety of transparent potentials. The positive integer N can be chosen arbitrarily. For each value of N, there is a family of potentials with $2N$ parameters. In the present paper we will find exact parallelism between the Dirac and the Schrödinger cases. What we have done in Ref. [2] is to confirm this parallelism for the case of $N=1$ through numerical calculations. In the process off finding transparent potentials, we also find exact solutions for the NLD equation (1.3) for any value of N.

In Sec. II we prove that $S(x)$ defined by Eq. (1.5) is indeed a transparent potential. We do this by relating the NLD equation (1.3) to an auxiliary NLS equation. In Sec. III we present a method for explicitly constructing $S(x)$ by means of solutions of the auxiliary NLS equation. In Sec. IV we argue that a transparent potential cannot contain a Lorentz vector term. We also discuss some features of the time-dependent NLD equation.

II. PROOF THAT $S(x)$ OF EQ. (1.5) IS A TRANSPARENT POTENTIAL

We are going to prove that $S(x)$ of Eq. (1.5) is a transparent potential. We begin by reducing the Dirac equation (with $V=0$) to a Schrödinger equation. Equation (1.1) can be written as

$$\psi'_+-(m+S)\psi_+=-E\psi_- , \tag{2.1}$$

$$\psi'_-+(m+S)\psi_-=E\psi_+ , \tag{2.2}$$

where $\psi'\equiv d\psi/dx$ and $\psi_\pm$ are related to the two components of $\psi=\binom{u}{v}$ by

$$\psi_\pm=u\pm v . \tag{2.3}$$

When ψ is normalized, so is each of $\psi_\pm$ [6]. Equations (2.1) and (2.2) can be reduced to

$$\left[-\frac{1}{2m}\frac{d^2}{dx^2}+U_\pm\right]\psi_\pm=\frac{(E^2-m^2)}{2m}\psi_\pm , \tag{2.4}$$

where

$$U_\pm(x)=S+\frac{1}{2m}(S^2\pm S') . \tag{2.5}$$

Equation (2.4) is of the form of the Schrödinger equation [7]. In the above, S can be any function of x. *If S is a transparent potential, then $U_\pm$ are both transparent potentials, and vice versa.* This is because if u and v are both free from reflected waves, so are ψ_+ and ψ_-. If S is a given external potential (independent of ψ), we are dealing with a linear equation for ψ. In this case the two equations for $\psi_\pm$ of Eq. (2.4) are decoupled.

Now let us turn to the NLD equation (1.3). We will focus on the case of $N=1$. Extension to the general case of $N>1$ is straightforward. When we emphasize the distinction between the linear and nonlinear situations, we denote the solution for the linear case with ψ and that for the nonlinear case with ϕ. In the NLD equation S is related to ϕ by

$$S(x)=-g\phi^\dagger\beta\phi=-g(u^2-v^2)=-g\phi_+\phi_- , \tag{2.6}$$

where $\phi=\binom{u}{v}$. We are using the same notation, u and v, for both ϕ and ψ, but this should not be very confusing. The NLD equation has a bound-state solution with [2]

$$E=E_B=m[1+(g^2/4)]^{-1/2} . \tag{2.7}$$

Another form of Eq. (2.7) is $2\kappa=gE_B$, where $\kappa=(m^2-E_B^2)^{1/2}$.

Equations (2.1)–(2.5) also hold for the NLD equation. For example, the nonlinear version of Eq. (2.4) is

$$\left[-\frac{1}{2m}\frac{d^2}{dx^2}+U_\pm\right]\phi_\pm=-\frac{\kappa^2}{2m}\phi_\pm , \tag{2.8}$$

where $U_\pm$ are now related to S of Eq. (2.6), or more generally to S of Eq. (1.5), through Eq. (2.5). When we emphasize the linkage between the NLD equation and Eq. (2.8), we refer to the latter as the auxiliary NLS equation. Note that S of Eq. (2.6) and hence each of $U_\pm$ depends on both of $\phi_\pm$. In this sense the auxiliary NLS equations for $\phi_\pm$ are coupled. In the bound-state problem, however, we will show that $U_\pm$ are related to $\phi_\pm$ by

$$U_\pm=-\frac{gE_B}{m}\phi_\pm^2=-\frac{2\kappa}{m}\phi_\pm^2 . \tag{2.9}$$

Hence the coupling mentioned above is only deceptive. Now the crucial point is that, according to Refs. [3,4], $U_\pm$ of the form of Eq. (2.9) are both transparent potentials for the Schrödinger equation. This proves that $S(x)$ of Eq. (2.6) is a transparent potential for the Dirac equation.

Equation (2.9) can be derived as follows. By operating d/dx on both sides of Eq. (2.6) and using Eqs. (2.1) and (2.2), we find

$$S'=-E_Bg(\phi_+^2-\phi_-^2) . \tag{2.10}$$

Combining this with Eq. (2.5), we thus obtain

$$U_+ - U_- = \frac{S'}{m} = -\frac{gE_B}{m}(\phi_+^2 - \phi_-^2) \ . \tag{2.11}$$

Next, start with another equation implied by Eq. (2.5), i.e.,

$$U_+ + U_- = 2\left[S + \frac{S^2}{2m}\right] . \tag{2.12}$$

Operating d/dx on both sides of Eq. (2.12) and using Eqs. (2.1), (2.2), and (2.10), we find

$$\frac{d}{dx}\left[U_+ + U_- + \frac{gE_B}{m}(\phi_+^2 + \phi_-^2)\right] = 0 \ . \tag{2.13}$$

This means that the quantity in the square brackets is a constant. Since $U_\pm$ and $\phi_\pm$ all vanish as $x \to \pm\infty$, the constant must be zero. This result combined with Eq. (2.11) leads to Eq. (2.9). Let us emphasize that this feature of decoupling between ϕ_+ and ϕ_- is peculiar to the S of the form of $\phi^\dagger\beta\phi$. It does not hold, for example, if we add a term like $(\phi^\dagger\beta\phi)^2$.

In order to have a feel for what we have done, it would be useful to see how the results found in Ref. [2] fit into this scheme. In Ref. [2] we solved the NLD equation (1.3) for $N=1$ directly, found ϕ explicitly with E_B of Eq. (2.7), and derived

$$S(x) = \frac{-2\kappa^2}{m + E_B\cosh(2\kappa x)} \ . \tag{2.14}$$

By substituting this S into Eq. (2.5), we obtained

$$U_\pm = -\frac{2\kappa^2 E_B[E_B + m\cosh(2\kappa x) \mp \kappa\sinh(2\kappa x)]}{m[m + E_B\cosh(2\kappa x)]^2} \ . \tag{2.15}$$

Note that $U_+(x) = U_-(-x)$. We left $U_\pm$ as such in Ref. [2], but they can be reduced to the following simple form [8]:

$$U_\pm(x) = -\frac{\kappa^2}{m}\mathrm{sech}^2(\kappa x \pm \lambda) \ , \tag{2.16}$$

where

$$e^{2\lambda} = (m+\kappa)/E_B \ . \tag{2.17}$$

The $U_\pm$ of Eq. (2.16) belong to the two-parameter family ($N=1$) of the Kay-Moses potentials. Normalized solutions $\phi_\pm$ of Eq. (2.8) are given by

$$\phi_\pm = (\sqrt{\kappa}/2)\mathrm{sech}(\kappa x \pm \lambda) \ , \tag{2.18}$$

which satisfy Eq. (2.9). The $\phi_\pm$ of Eq. (2.18) satisfy Eqs. (2.1) and (2.2) together with $S(x)$ of Eq. (2.14).

Now let us consider the general case of $N > 1$. The proof given above can easily be extended. All that we have to do is to implement $\sum_i$ in the expressions for S and $U_\pm$. Equations (2.6) and (2.9), respectively, become

$$S(x) = -\sum_i g_i\phi_{i+}\phi_{i-} \tag{2.19}$$

and

$$U_\pm = -\frac{1}{m}\sum_i g_i E_{Bi}\phi_{i\pm}^2 = -\frac{2}{m}\sum_i \kappa_i\phi_{i\pm}^2 \ . \tag{2.20}$$

Again, according to Refs. [3,4], the above $U_\pm$ are transparent potentials, which proves that S of Eq. (2.19) or equivalently Eq. (1.5) is a transparent potential for the Dirac equation.

III. EXPLICIT CONSTRUCTION OF $S(x)$

The solutions $\phi_{i\pm}$ of the auxiliary equations (2.8) can be obtained as in Refs. [2–4]. Once $\phi_{i\pm}$ are determined, $S(x)$ can be constructed by means of Eq. (2.19). As explained below, however, there is a crucial, subtle point in determining $\phi_{i\pm}$. Again, let us first illustrate the procedure for the case of $N=1$. Start with the two-parameter Kay-Moses wave functions,

$$\phi_\pm = -\sqrt{A_\pm}e^{\kappa x}/[1 + (A_\pm e^{2\kappa x}/2\kappa)] \ , \tag{3.1}$$

where $A_\pm$ and κ are positive constants [9]. κ is common between ϕ_+ and ϕ_-. Each of the auxiliary equations (2.8) combined with Eq. (2.9) is satisfied by each $\phi_\pm$ of Eq. (3.1). This is so irrespective of the values of $A_\pm$ and κ. Since κ is essentially the "coupling constant" g in the sense that $2\kappa = gE_B = gm[1+(g^2/4)]^{-1/2}$, let us regard κ as a predetermined parameter.

As far as each of the auxiliary equations (2.8) combined with Eq. (2.9) is concerned, each of the $A_\pm$ of Eq. (3.1) can be chosen arbitrarily. This is so because the equations (2.8) for $\phi_\pm$ are decoupled as we pointed out below Eq. (2.9). In the Dirac equation for $\phi_\pm$, i.e., Eqs. (2.1) and (2.2) with $\psi_\pm$ replaced by $\phi_\pm$, however, ϕ_+ and ϕ_- are coupled. In fact, in order for $\phi_\pm$ of Eq. (3.1) to satisfy the Dirac equation, $A_\pm$ have to be chosen such that

$$\sqrt{A_+/A_-} = (m+\kappa)/E_B \ . \tag{3.2}$$

In this sense the $\phi_\pm$ are not totally dissociated from each other. This is the crucial, subtle point that we mentioned above Eq. (3.1). This can be seen as follows. When $x \to \infty$, $\phi_\pm$ behave like $\phi_\pm \sim -e^{-\kappa x}/\sqrt{A_\pm}$. On the other hand, Eq. (2.1) with $S(x)=0$ requires $\phi_+' - m\phi_+ = -E_B\phi_-$, which leads to Eq. (3.2). Equation (2.2) leads to the same. If we write $A_\pm$ as

$$A_\pm = Ae^{\pm 2\lambda} \ , \tag{3.3}$$

Eq. (3.2) leads to Eq. (2.17). The remaining parameter A is related to the choice of the origin ($x=0$). With $A=2\kappa$, we obtain Eq. (2.18). Now that the $\phi_\pm$ are known, we know $S = g\phi_+\phi_-$. Let us note that, irrespective of the value of λ, the $\phi_\pm$ of Eq. (2.18) satisfy Eq. (2.8) with $U_\pm$ of Eq. (2.16). However, they do not satisfy Eqs. (2.1) and (2.2) unless λ is chosen according to Eq. (2.17).

Let us briefly examine the transmission problem. Start with Eq. (2.4) with $U_\pm$ of Eq. (2.16). Note that $U_\pm$ in this case is a given potential and is not related to the transmission wave function. Equation (2.4) is satisfied by

$$f_\pm(k,x) = \frac{e^{ikx}}{ik+\kappa}[ik - \kappa\tanh(\kappa x \pm \lambda)] \ , \tag{3.4}$$

where $k^2 = E^2 - m^2$. Equation (3.4) is essentially the same as Eq. (4.9) of Ref. [2]. Each $f_\pm(k,x)$ represents a situation such that the incident wave (for $x \to -\infty$) is e^{ikx}, and the transmission amplitude is

$T=(ik-\kappa)/(ik+\kappa)$. It is tempting to take $f_\pm$ for $\psi_\pm$, but this is incorrect. The correct $\psi_\pm$ are given by

$$\psi_\pm(k,x)=c_\pm f_\pm(f,x) , \tag{3.5}$$

where $c_\pm$ are phase factors such that

$$c_+/c_-=(m+ik)/E . \tag{3.6}$$

This can be seen again by examining the asymptotic behavior of $\psi_\pm$. The $\psi_\pm$ satisfy the Dirac equation but $f_\pm$ do not.

Next let us turn to the case of $N=2$. The Kay-Moses wave functions for $N=2$ are given by

$$\phi_{i\pm}=N_{i\pm}/D_\pm , \quad i=1 \text{ or } 2 , \tag{3.7}$$

where

$$N_{1\pm}=-2\sqrt{A_{1\pm}}\kappa_1(\kappa_1+\kappa_2)e^{\kappa_1 x} \times[A_{2\pm}(\kappa_1-\kappa_2)e^{2\kappa_2 x}+2\kappa_2(\kappa_1+\kappa_2)] , \tag{3.8}$$

$$N_{2\pm}=N_{1\pm} \quad \text{with } (1\rightleftarrows 2) , \tag{3.9}$$

$$D_\pm=A_{1\pm}A_{2\pm}(\kappa_1-\kappa_2)^2 e^{2(\kappa_1+\kappa_2)x} +2(\kappa_1+\kappa_2)^2[A_{1\pm}\kappa_2 e^{2\kappa_1 x}+A_{2\pm}\kappa_1 e^{2\kappa_2 x}+2\kappa_1\kappa_2] . \tag{3.10}$$

Again, by examining the asymptotic behavior of the wave functions we find that A_{i+} and A_{i-} are related:

$$A_{i\pm}=A_i e^{\pm 2\lambda_i} , \quad e^{2\lambda_i}=(m+\kappa_i)/E_{Bi} . \tag{3.11}$$

We regard κ_1 and κ_2 as predetermined parameters. A_1 and A_2 can be chosen arbitrarily. The number of the parameters is the same as that of the corresponding Kay-Moses case.

We can write $S(x)$ in terms of the $\phi_{i\pm}$ that we found. With the S so constructed, we have confirmed that the NLD equation (1.3) for $N=2$ is indeed satisfied. We have done this by means of the symbolic computation software MAPLE. We have also checked the consistency among S, $U_\pm$, and Eq. (2.5). An interesting special choice of the parameter set is

$$\kappa_1=\kappa , \quad \kappa_2=2\kappa , \quad A_1=6\kappa , \quad A_2=12\kappa . \tag{3.12}$$

In this case, $S(x)$ is an even function of x and, in the nonrelativistic limit where $E_{Bi}\to m$ and $\lambda_i\to 0$, this $S(x)$ is reduced to the $N=2$ case of the potential of the Pöschl-Teller type [10],

$$V(x)=-N(N+1)(\kappa^2/2m)\mathrm{sech}^2(\kappa x) . \tag{3.13}$$

For arbitrarily chosen values of A_1 and A_2, $S(x)$ is not an even function of x in general, and $U_+(x)\neq U_-(-x)$.

Extension to the cases of $N>2$ should be obvious. There are $2N$ parameters, κ_i's and A_i's. Equation (3.11) holds as such. We confirmed, by means of symbolic computation, that the extension works. Thus the exact parallelism between the Dirac case and the Schrödinger case has been established. Let us add that the relativistic counterpart of the $N=3$ case of Eq. (3.13) is obtained by choosing $\kappa_1=\kappa$, $\kappa_2=2\kappa$, $\kappa_3=3\kappa$, $A_1=12\kappa$, $A_2=60\kappa$, and $A_3=60\kappa$.

Finally, let us point out that we have exhausted all possibilities for the transparent potential $S(x)$ of the Lorentz scalar type. Suppose there is a transparent potential $S(x)$ which may not belong to Eq. (1.5). Equations (2.1)–(2.5) hold and the Dirac equation with the S can be reduced to Eq. (2.4) with $U_\pm$ of Eq. (2.5). Since the S is transparent, $U_\pm$ must both be transparent. Recall that the Kay-Moses method exhausts all possible transparent potentials for the Schrödinger equation. Hence $U_\pm$ must be those of the Kay-Moses type and, according to Refs. [3,4], $U_\pm$ are of the form of Eq. (2.20) in which $\phi_{i\pm}$ are the corresponding Kay-Moses wave functions. The method described in this section then allows us to construct S of Eq. (2.19). The so-constructed S together with $U_\pm$ satisfy Eq. (2.5), and hence this S is identical with the S that we started with. Therefore, any transparent potential S can be identified with one of those given by Eq. (2.19) or (1.5). In this connection let us note two aspects of Eq. (2.5). (i) When U_+ (or U_-) is given, Eq. (2.5) with U_+ (U_-) can be regarded as a differential equation for S. For a specified boundary condition, $S\to 0$ as $x-\pm\infty$ in our case, S is determined uniquely. (ii) When Eq. (3.11) is implemented in defining $U_\pm$, the two equations of Eq. (2.5), one with U_+ and the other with U_-, lead to the same S.

IV. DISCUSSIONS

So far we have assumed that the potential of Eq. (1.1) is a pure Lorentz scalar, i.e., $V=0$. In this case there is symmetry between positive- and negative-energy states. This is because, if ψ is a solution with eigenvalue E, then $\alpha\beta\psi$ is a solution with $-E$. If S is transparent for positive energies, it is also transparent for negative energies. Let us emphasize the relevance of the negative-energy states. Imagine that the incident wave is in the form of a wave packet. In order to express the wave packet as a superposition of plane waves, we need to include negative-energy states.

If $V\neq 0$, the symmetry between positive- and negative-energy states is broken. This probably impairs perfect transparency. Recall that a transparent nonrelativistic potential for the Schrödinger equation must have a bound state at threshold [2,11]. For the Dirac equation, it can be shown that a transparent potential must have a bound state at $E=m$ and also at $E=-m$. This requirement severely restricts the choice of V. Furthermore, since effects of V on positive- and negative-energy states are asymmetric, it is extremely unlikely that a potential with V would give the same transmission probability for E and $-E$, and that for all values of E. The symmetry between positive- and negative-energy states in the absence of V is related to the supersymmetry exhibited through Eqs. (2.4) and (2.5) [7]. This supersymmetry disappears if $V\neq 0$. We will discuss supersymmetry aspects of the Dirac equation with a pure Lorentz scalar interaction in a separate paper [12].

In Ref. [2] w examined the $V(x)$ constructed through

$$V(x)=-g_v\phi^\dagger(x)\phi(x) . \tag{4.1}$$

Here ϕ is the bound-state solution of the NLD equation with $\beta S+V$, where S is that of Eq. (1.5) with $N=1$ and V is that of Eq. (4.1). We found that this $\beta S+V$ is not transparent.

Having completed the time-independent problem, we are naturally interested in the more difficult time-dependent problem. For the time-dependent NLS equation,

$$i\frac{\partial\phi_i}{\partial t}=\left[-\frac{1}{2m}\frac{\partial^2}{\partial x^2}-\sum_j g_j|\phi_j|^2\right]\phi_i(x,t)\ , \tag{4.2}$$

soliton solutions can easily be constructed as shown in Refs. [3,4]. The solitons that appear in those solutions are true solitons in the following sense. When they collide, they come out with the same shape and speed with which they entered. There is no inelastic collision.

We attempted to solve the time-dependent NLD equation with the Lorentz scalar interaction,

$$i\frac{\partial\phi}{\partial t}=[\alpha p+\beta m-g\beta(\phi^\dagger\beta\phi)]\phi(x,t)\ . \tag{4.3}$$

Here we are considering the $N=1$ case for simplicity. A solution of the form of a single solitary wave can be obtained by "Lorentz-boosting" the stationary solution for $N=1$, but this is not particularly interesting. We tried to obtain, in a manner similar to that of Refs. [3,4], solutions which describe collisions of two solitary waves, but we quickly realized that this is not possible. The time-dependent version of Eq. (2.8) is

$$\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+m^2+2mU_\pm\right]\phi_\pm(x,t)=-i\frac{\partial S}{\partial t}\phi_\mp\ , \tag{4.4}$$

where $U_\pm$ are related to $S=-g\phi^\dagger\beta\phi=-(g/2)(\phi_+^*\phi_-+\phi_-^*\phi_+)$ through Eq. (2.5). Unlike in Eq. (2.8), ϕ_+ and ϕ_- are directly coupled through the term with $\partial S/\partial t$. Moreover, Eq. (4.4) is of the form of the Klein-Gordon equation rather than the Schrödinger equation. For these reasons the method of Refs. [3,4] fails for Eq. (4.4). The parallelism that we have found between the Dirac and Schrödinger models seems to be restricted to the time-independent case.

Alvarez and Carreras carried out numerical experiments for Eq. (4.3) [13]. They found a number of interesting features in the collision process of two solitary waves. When the collision speed is relatively large, the collision is elastic. For a smaller speed, however, various inelastic processes take place. In particular, the two solitary waves can merge to form a quasibound state, which undergoes breathing oscillations and eventually decays. As Alvarez and Carreras noted, these complex phenomena indicate that the time-dependent NLD equation describes a nonintegrable system. This would mean that the time-dependent NLD and NLS systems are intrinsically different. This would be so even at very "low energies" for which relativistic effects are normally thought to be unimportant.

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[6] The $\psi_\pm$ are not spinors; rather, each $\psi_\pm$ represents a spinor component. A similar remark applies to $\phi_\pm$ that appear later. If we took $\alpha=-\sigma_y$ and $\beta=\sigma_x$ (instead of $\alpha=\sigma_y$ and $\beta=\sigma_z$, which we are using and which we used in Ref. [2]), the upper and lower components of ψ would be ψ_+ and ψ_-, respectively. Regarding the normalization of $\psi_\pm$ for a bound state, note that $\int_{-\infty}^{\infty}uv\,dx=0$. This is based on $2(E-V)uv=uu'-vv'$, which follows from the Dirac equation. We have included V for generality, but $V=0$ in the case under consideration. In Eq. (22) of Ref. [2] we used $\psi_\pm=(\psi_1\pm\psi_2)/\sqrt{2}$, where ψ_1 and ψ_2 are, respectively, the u and v of the present paper. Hence these $\psi_\pm$ differ from $\psi_\pm$ of Eq. (2.3) by a factor of $\sqrt{2}$. In Ref. [2], however, Eq. (22) as such was a linear equation for $\psi_\pm$ with the potential U explicitly specified by Eq. (24), and hence the normalization of $\psi_\pm$ was unimportant there.

[7] F. Cooper, A. Khare, R. Musto, and A. Wipf, Ann. Phys. (N.Y.) **187**, 1 (1985). Actually, Eq. (2.4) is a time-independent Klein-Gordon equation. In this regard, see Eq. (4.4).

[8] In Ref. [2] we failed to recognize $U_\pm$ of Eq. (2.15) as those of the Kay-Moses type. See the *Note added in proof* of Ref. [2].

[9] This is Eq. (4.2) of Ref. [2] divided by $\sqrt{A}$. In Ref. [2] there are functions $f_i(x)$ and $g_i(x)$ that are related by $g_i(x)=f_i(x)/\sqrt{A_i}$; $g_i(x)$ is normalized as $\int_{-\infty}^{\infty}g_i^2(x)dx=1$. The $g_i(x)$ is denoted by $f_i(x)$ in Ref. [3] and by $-\phi_i(x)$ in Ref. [4]. This is rather confusing. Our $\phi_{i\pm}(x)$ exactly correspond to $g_i(x)$ of Ref. [2]. Fortunately, the κ's and A's are the same throughout Refs. [2–4] and the present paper, but $2m=1$ in Refs. [2,3]. There is a misprint in Eq. (4.8) of Ref. [2]. It should read $V(x)=-2\kappa_1^2\mathrm{sech}^2[\kappa_1(x_1-x_0)]$.

[10] This $V(x)$ does not mean a Lorentz vector. In nonrelativistic quantum mechanics there is of course no distinction between the Lorentz scalar and vector. The Schrödinger equation with potential $V(x)$ of Eq. (3.13) has eigenstates with (nonrelativistic energy) eigenvalues

$\varepsilon = -(n\kappa)^2/2m$ where $n = 0, 1, 2, \ldots, N$. See S. Flügge, *Practical Quantum Mechanics* (Springer-Verlag, Berlin, 1974), pp. 94–100. The $V(x)$ of Eq. (3.13) can also be regarded as a special case of the Bargmann potentials. V. Bargmann, Rev. Mod. Phys. **21**, 488 (1949); R. G. Newton, *Scattering Theory of Waves and Particles,* 2nd ed. (Springer-Verlag, Berlin, 1982), pp. 433–441.

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