One-logarithmic recoil correction in muonium hyperfine splitting

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(Received 16 June 1992)

A source of recoil corrections to muonium hyperfine splitting which are linear in the logarithm of the electron-muon mass ratio is detected. The contribution is induced by the simultaneous insertion of muon and electron polarization loops in the external photon lines. This contribution is calculated analytically and is given by the numerical result $\delta E=(\alpha^2 Z\alpha/\pi)E_F\{0.6455[m/M]\ln[M/m]+0.9212[m/M]\}=0.011\,37$ kHz.

PACS number(s): 36.10.Dr, 35.10.Fk

Current theoretical work on corrections to muonium hyperfine splitting is concentrated on the calculation of contributions of order $\alpha^2 Z\alpha E_F$. Nonrecoil corrections of this order are produced by the six gauge-invariant sets of diagrams in Fig. 1 considered in the external field approximation [1], where the shaded block in Fig. 1(c) designates the sum of all one-loop renormalized dressings of two-photon emissions from the electron line, and the shaded block in Fig. 1(f) designates dressing by two radiative photons. All these diagrams may be obtained by different dressings from the skeleton diagram, which contains two exchanged photons between electron and muon lines. Contributions induced by polarization-operator insertions in external photon lines (we call exchanged photons external if the diagram is calculated in the positive-energy muon-pole approximation) and by the simultaneous insertion of a radiative photon in the electron line and a one-loop polarization operator in the external photon line have been calculated in analytic form [1]. Corrections produced by electron-loop polarization-operator insertions in radiative photons have been calculated in semianalytic form as a one-dimensional integral, where the integrand is itself a complete elliptic integral [2]. Penultimate nonrecoil contributions induced by light-scattering insertions were recently obtained numerically [3], and work on the last, as yet uncalculated, set of diagrams is now in progress.

Leading recoil contributions to hyperfine splitting of order $\alpha^2(Z\alpha)(m/M)E_F$, which are parametrically suppressed by the small mass ratio m/M, contain logarithms of the electron-muon mass ratio and are produced by the same six gauge-invariant sets of graphs in Fig. 1, but this time not in the external-field approximation. Logarithm-cubed and logarithm-squared recoil terms were also obtained previously [4]. Although they are of negligible phenomenological importance for contemporary experiments, these terms have some nice theoretical features. Logarithm-cubed terms are simply given by the effective charge renormalization in QED and thus may be obtained without calculation. Logarithm-squared contributions induced by diagrams of the type presented in Fig. 1(c) are connected with the high-frequency asymptote of the one-loop electron factor in electrodynamics with massless electrons, and illustrate nicely the nonsmooth nature of the transition between theory with massive and massless electrons. The logarithm-squared term connected with the light-scattering insertion between exchanged photons is also rather interesting from the theoretical point of view. It has been shown in [5] that this term directly measures anomalous renormalization of the axial current and may be easily deduced from Adler's classical result [6]. Hence one can, at least in principle, check the anomalous axial-current renormalization by simply measuring the ground-state muonium hyperfine splitting. There have been no attempts previously to attack the calculation of one-logarithm recoil contributions, which are contained abundantly in the nonpole contributions induced by the diagrams in Fig. 1. However, it has been a common belief that there are no recoil logarithms in the external-field approximation.

We would like to demonstrate below that the external-field contribution also contains recoil one-logarithm terms, and we will present an analytic calculation of the coefficient before this logarithm. The contribution to hyperfine splitting produced by the muon-pole residue in

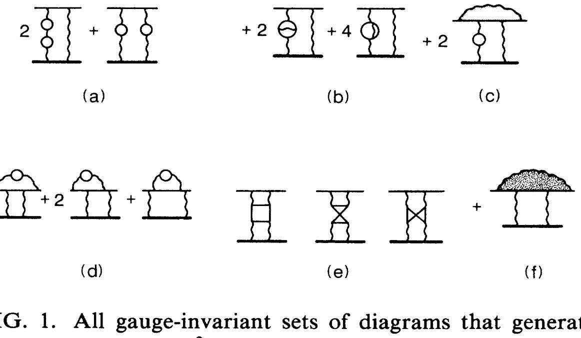

FIG. 1. All gauge-invariant sets of diagrams that generate corrections of order $\alpha^2(Z\alpha)E_F$ to the hyperfine splitting of muonium.

any diagram in Fig. 1 is given by the expression [1,7]

$$\delta E=\frac{8Z\alpha}{\pi}E_F\frac{m}{\left[1+\frac{m}{M}\right]}\int_0^\infty \frac{dk\,F(k)}{k^2}\,, \tag{1}$$

where $k=|\mathbf{k}|$ is the spatial momentum of the external photons, E_F is the Fermi hyperfine splitting energy, and the function $F(k)$ describes radiative insertions that are specific for each set of diagrams in Fig. 1. We are going to consider diagrams with polarization insertions in Fig. 1, and our main point is to perform insertions, not only of electron loops, but also of muon loops. First we want to mention that if one inserts two loops of the same nature in Fig. 1(a) (i.e., both loops are simultaneously electronic or muonic), or both electron and muon loops in Fig. 1(b), then all recoil factors cancel in the sum of muon and electron-loop contributions exactly. One can easily deduce this with the help of a simple argument of dimensional nature. Really, each polarization insertion in Figs. 1(a) and 1(b) is described by the replacement in the integrand in Eq. (1)

$$\frac{F(k)}{k^2}\to\left[\frac{\alpha}{\pi}\right]I(k,m_i)\,, \tag{2}$$

where m_i is either the electron or muon mass, and $I(k)$ is the respective one- or two-loop polarization operator. (For the explicit formula for one-loop polarization, see below.) The cancelation is based on the simple observation that the polarization operator is a second-degree homogeneous function of momentum, and one may get rid of all dimensional factors in the integrand by simply scaling all momenta by the respective (electron or muon) mass. Then the integral (1) with electron-loop insertions acquires a factor $1/m$, and the integral (1) with muon-loop insertions acquires a factor $1/M$. These factors are exactly the ones that are needed to cancel the factor $m/(1+m/M)$ in Eq. (1). We should like to mention that the direct proof of this statement, connected with the calculation of all integrals without scaling of the integration variable, is also accessible, but much more difficult. It can easily be seen that this argument works for any insertion in external photon lines where we first put the electron in all bubbles, and then the muon in all bubbles, and add to obtain the total contribution.

Our main observation is connected with the simultaneous insertion of an electron and a muon loop in Fig. 1(a). In this situation, one cannot make a simple scaling of the integration variable anymore because there are two scales in the problem. Moreover, it turns out that there is a region of integration over momentum when the electron loop enters into the asymptotic regime. The polarization operator contains a logarithm of the integration momentum in this region, and it leads to a logarithm of the mass ratio in the contribution to the energy shift.

Let us consider the calculation of the mixed bubble contribution. It is given by Eq. (1), where the following replacement is performed:

$$\frac{F(k)}{k^2}\to\left[\frac{\alpha}{\pi}\right]^2 k^2 I_1(k,m)I_1(k,M)\,, \tag{3}$$

where

$$I_1(k,m)=\int_0^\infty \frac{dv\,v^2(1-v^2/3)}{4m^2+k^2(1-v^2)}\,,$$

and where m is the electron mass and M is the muon mass.

After a tedious calculation, we obtain

$$\delta E=\frac{24\alpha^2(Z\alpha)E_F T}{\pi^3\left[1+\frac{m}{M}\right]}\,,$$

where

$$T=B_1\left[\frac{m}{M}\right]\ln\left[\frac{M}{m}\right]+B_2\left[\frac{m}{M}\right]\,,$$

and

$$B_1=\tfrac{3}{16}\ln^2\left[\frac{\sqrt{5}-1}{2}\right]+\frac{\sqrt{5}}{36}\ln\left[\frac{\sqrt{5}-1}{2}\right]+\frac{\pi^2}{48}+\frac{5}{108}\,,$$

$$B_2=\tfrac{5}{16}\ln^3\left[\frac{\sqrt{5}-1}{2}\right]+\left[\frac{\ln 2}{4}-\frac{5}{64}\right]\ln^2\left[\frac{\sqrt{5}-1}{2}\right]$$
$$-\left[\frac{7\pi^2}{160}+\frac{25\sqrt{5}}{432}+\frac{1}{4}\mathrm{Li}_2\left[\frac{3-\sqrt{5}}{4}\right]\right]\ln\left[\frac{\sqrt{5}-1}{2}\right]$$
$$+\tfrac{7}{16}\mathrm{Li}_3\left[\frac{\sqrt{5}-1}{2}\right]-\tfrac{23}{80}\zeta(3)-\left(\tfrac{5}{576}-\tfrac{11}{240}\ln 2\right)\pi^2$$
$$-\frac{A}{4}-\frac{89}{1296}\,,$$

where

$$\mathrm{Li}_3(x)=\int_0^x\frac{\mathrm{Li}_2(y)}{2}dy\,,$$

where Li_2 is the dilogarithm, and

$$A=\sum_{n=2}^{\infty}\frac{(3-\sqrt{5})^n}{2^n n}\sum_{k=1}^{n-1}\frac{1}{2^k k^2}=0.051\,384\,155\,,$$

and so

$$\delta E=\frac{\alpha^2(Z\alpha)}{\pi}E_F\left[0.6455\left[\frac{m}{M}\right]\ln\left[\frac{M}{m}\right]+0.9212\left[\frac{m}{M}\right]\right]$$
$$=0.011\,37\text{ kHz}\,.$$

In conclusion, we would like to emphasize that, besides the unexpected recoil logarithm obtained above, there are a number of recoil one-logarithm terms produced by the graphs in Fig. 1 in the recoil regime. We hope to report on their contribution to muonium hyperfine splitting in the near future.

One of us (M.I.E.) would like to thank the Physics Department of Oklahoma State University for its kind hospitality. This work was supported by the U.S. Department of Energy under Grant No. DE-FG0584ER40215.

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