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Tunneling-time probability distribution

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Localized wave-packet tunneling through a one-dimensional potential barrier is investigated with respect to the distribution of tunneling times. Specifically, we construct a quantity— termed the "tunneled flux"—which has properties of a tunneling-time probability distribution. The tunneled flux is determined completely in terms of the time evolution of an initially localized wave packet. The tunneled flux and the corresponding tunneling-time distributions are investigated analytically via a semiclassical approximation. Additionally, numerical studies are performed with both the semiclassical and quantal versions of the tunneled flux. The semiclassical calculation qualitatively reproduces numerical tunneled fluxes, converging quantitatively as $\hbar \to 0$. It also provides a wealth of insight into the nature of tunneling processes. Our principle conclusions are as follows: (i) There is an essential disparity between the time- and energy-domain pictures of wave-packet tunneling; (ii) the tunneling-time distribution can be modeled as an exponential of a skewed Lorentzian function, with a width governed primarily by the phase-space asymmetry of the initial wave packet; and (iii) tunneling is faster than simple classical motion from one side of the barrier to another, even assuming an instantaneous transit between the turning points. Effectively, the barrier acts as a filter for the high-momentum components of the initial wave packet.

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I. INTRODUCTION

Along with providing utterly accurate predictions regarding the behavior of the physical world, quantum theory has raised many conceptual difficulties, most of which center on reconciling quantum phenomenology with classical sensibilities. While the founders of quantum mechanics resolved many of these issues, some, such as the "quantum-measurement problem," continue to invite debate and investigation to the present day. This paper examines another such long-standing conundrum, the paradoxical "tunneling time" for classically forbidden quantum process. In recent years the definition of tunneling time has received considerable theoretical attention [1], without any clear consensus arising [1a]. Thus, the question may be regarded as open, and a reexamination seems particularly appropriate in light of new (and ongoing) *experimental* efforts in this area [2] which could eventually provide a criterion for resolving this vexing issue.

As with many of the other conceptual questions of quantum mechanics, much of the difficulty on the subject of tunneling time concerns formulating a clear impression of exactly what physical quantity is sought. It is thus worthwhile to begin with a caricature of the underlying experimental process. To wit, given a situation (i.e., either below-barrier scattering or a metastable state) in which particles will (at some time) tunnel, it is in theory possible to measure that a tunneling event has occurred and to record the time at which it does so. Repeating this process a large number of times will yield some probability distribution of "tunneling times." This is a concrete form of the simple but subtle question which has been the focus of much current attention to tunneling. Simply stated, the question takes the following form:

(A) How much time does a tunneling particle spend under the barrier?

Quantum mechanically, question (A), which seeks a "dwell time" for the underbarrier region, has no meaning as it requires the simultaneous measurement of incompatible observables. Specifically, one asks (i) did the particle tunnel? (ii) If so, how long did it take to traverse the barrier region? As is demonstrated below, the two observables in question are associated with noncommuting operators and, as a result, a joint probability distribution for the pair of eigenvalues does not exist. Thus, we believe that this form of the tunneling-time question is unanswerable.

However, the prototypical experiment described above also invites a somewhat more general question about the tunneling process:

(B) What is the distribution of the measured tunneling times?

The purpose of this paper is to demonstrate that question (B) can indeed be answered in a physically meaningful way [3]. In particular, we present a "transit-time" operator which generates a probability distribution of "arrival times" for particles on the far side of the barrier (i.e., for particles which must have tunneled). This operator is entirely developed within the quantum formalism, but the corresponding semiclassical quantity is easily evaluated via a steepest-descent approximation and the results compare favorably. In particular, there is close agreement between the semiclassical and quantum distributions as to what constitutes the "most probable" tunneling time.

The semiclassical analysis also provides insight into the nature of tunneling processes and the tunneling-time distribution. Two features of particular interest are the "quantum-speed-up effect" in which tunneling particles actually "travel faster" than the corresponding classical ones, and the imaginary part of the "tunneling time" which enters as a parameter in the steepest-descent calculation [4]. We demonstrate that the quantum speed-up occurs generically in tunneling processes because the barrier acts as a "filter" for the high-momentum components of the wave function. Similar conclusions have been drawn previously from both theoretical [5] and experimental [6] viewpoints. The imaginary part of the tunneling time plays a related role, as it contributes to the "shift" to higher energies of the classical particle, which allows the particle to surmount the barrier and arrive sooner on the far side. The imaginary time also plays a role in the width of the tunneling-time distribution.

Those familiar with recent treatments of tunneling time will no doubt be aware that there are two "schools of thought" on this subject [7]. One casts itself in the energy domain of quantum mechanics and focuses on tunneling at specific energies [8], while the other approach utilizes the time domain and is based on wave-packet propagation [9]. Our method falls squarely in the second camp, and is particularly influenced by the previous work of Pollok [10]. In the course of our investigation we find there is an essential disparity between the time- and energy-domain pictures of wave-packet tunneling.

We proceed as follows: Section II explicitly demonstrates the noncommutativity of the "transit or dwell-time" and "tunneling-flag" operators, and thus shows that question (A) is unanswerable. Rather than providing a completely null result, this analysis suggests examining a new quantity, the "forward-transit-time" operator, which is developed in Sec. III and results in a quantity with the properties of a tunneling-time-distribution operator. Section IV utilizes a steepest-descent approximation to develop the corresponding semiclassical tunneling-time distribution. As is well known, the semiclassical mechanics of tunneling generates various nonintuitive quantities such as complex position, time, and/or momenta. Similarly, these quantities arise in the steepest-descent approximation to the tunneling-time operator, and Sec. V is devoted to delineating the nature of these various quantities, as well as presenting numerical studies of both the quantum and semiclassical tunneling-time distributions. Section VI summarizes our findings and discusses the goals of future work.

II. NONEXISTENCE OF THE JOINT PROBABILITY DISTRIBUTION

The formalism of time-dependent quantum mechanics provides a straightforward means of addressing question (A) above. First, a simple transit-time operator, $\hat{\tau}$, can be associated with any fixed interval of space $D=[a,b]$ [11], where the eigenvalues of $\hat{\tau}$ represent the possible times a particle can spend in D. An explicit form for $\hat{\tau}$ can be constructed in terms of the projector, $\hat{d}$, which projects onto states supported in D.

$$(\hat{d}\psi)(x)=\begin{cases}\psi(x), & x\in D\\ 0, & \text{otherwise}\end{cases} \tag{1}$$

or

$$\hat{d}=\theta_D(\hat{x}) , \tag{2}$$

where $\theta_D(x)$ is the characteristic function of interval D.

Next, consider the probability that a particle, initially in state (a density matrix) $\hat{\rho}$, can be observed in D at time t

$$C(t)=\mathrm{Tr}[\hat{d}\hat{\rho}_t]=\mathrm{Tr}[\hat{d}_{-t}\hat{\rho}] , \tag{3}$$

where $\hat{\rho}_t\equiv\exp(-i\hat{H}t/\hbar)\hat{\rho}\exp(i\hat{H}t/\hbar)$ is the time-evolved state of the system and the second form follows from cyclic invariance of the trace. Integrating $C(t)$ over all time determines the mean time, τ, spent in D by a particle which passes through the state $\hat{\rho}$ at some point in its history [12]. Specifically,

$$\tau=\int_{-\infty}^{\infty}dt\, C(t)=\mathrm{Tr}[\hat{\tau}\hat{\rho}] , \tag{4}$$

where

$$\hat{\tau}\equiv w\int_{-\infty}^{\infty}dt\, \hat{d}_{-t} \tag{5}$$

is the "transit-time operator." The notation for the infinite time integral in (5) draws attention to the fact that it is defined as a "weak" operator limit, i.e., $\hat{\tau}$ is defined by (5) only in the sense that its inner product with arbitrary state, $\hat{\rho}$, is defined by (4).

The transit-time operator is best understood by expressing $\hat{d}_{-t}$ in terms of the energy representation of scattering states. To this end we choose the basis states $|El\pm\rangle$, where $E>0$ is the energy, $l=1$ and 2 denotes the channels at $x=-\infty$ and $+\infty$, respectively, and + or − signs which accompany the l index are included to distinguish between the "incoming specific" and "outgoing specific" energy eigenstates. For example, $|El+\rangle$ represents the stationary scattering state seeded with an incoming plane wave in the lth channel, and

$$\langle x|E1+\rangle\sim\left[\frac{m}{2\pi\hbar p(E)}\right]^{1/2}\times\begin{cases}e^{ip(E)x/\hbar}+R(E)e^{-ip(E)x/\hbar}, & x\to-\infty\\ T(E)e^{ip(E)x/\hbar}, & x\to\infty\end{cases} \tag{6}$$

is normalized such that $\langle El+|E'l'+\rangle=\delta(E-E')\delta_{ll'}$. Here $R(E)$ and $T(E)$ are the reflection and transmission coefficients and $p(E)\equiv\sqrt{2mE}$.

In this representation the transit-time operator is given by

$$\langle E_1 l_1 + |\hat{\tau}| E_2 l_2 + \rangle = \int_{-\infty}^{\infty} dt\, e^{iE_1 t/\hbar} \langle E_1 l_1 + |\hat{d}| E_2 l_2 + \rangle e^{-iE_2 t/\hbar}$$
$$= 2\pi\hbar \langle E_1 l_1 + |\hat{d}| E_1 l_2 + \rangle \delta(E_1 - E_2) , \tag{7}$$

i.e., it is simply related to the energy representation of $\hat{d}$. Equation (7) demonstrates that $\hat{\tau}$ is diagonal in this basis (with respect to energy), a consequence of the infinite time integration of (4), which makes $\hat{\tau}$ invariant with respect to time translation. This time-translation invariance permits the decomposition of $\hat{\tau}$ into a set of energy-specific 2×2 transit-time matrices, $\tau(E)$, by expressing the mean transit time, τ, in the energy representation. In particular,

$$\tau = \int_0^{\infty} dE\, \mathrm{Tr}_E[\tau(E)\rho(E)] , \tag{8}$$

where Tr_E denotes an ordinary finite matrix trace (i.e., the trace in the two-dimensional subspace corresponding to energy E). The (l_1+, l_2+) elements of the 2×2 matrices $\tau(E)$ and $\rho(E)$ are given by

$$\tau_{l_1+l_2+}(E) = 2\pi\hbar \langle E l_1 + |\hat{d}| E l_2 + \rangle$$
$$= 2\pi\hbar \int_a^b dx\, \langle x | E l_1 + \rangle^* \langle x | E l_2 + \rangle \tag{9}$$

and

$$\rho_{l_1+l_2+}(E) = \langle E l_1 + |\hat{\rho}| E l_2 + \rangle , \tag{10}$$

respectively.

The energy-specific transit-time matrix, $\tau(E)$, represents transit time associated with the two-dimensional Hilbert space of energy equal to E specific states. A corresponding normalized state in this Hilbert space would be provided by $\rho(E)/\mathrm{Tr}_E[\rho(E)]$. In this context $\mathrm{Tr}_E[\rho(E)]$ is viewed as the energy-dependent weight of the system state.

Our goal is to find the transit time specific to a particle tunneling through the barrier from left to right. The first step towards such a transit time is the restriction of the initial state, $\hat{\rho}$, to the $l=1$ incoming channel; i.e., we consider only particles incoming from the left (see Fig. 1). In this case, $\rho(E)/\mathrm{Tr}_E[\rho(E)]$ projects onto the first of the two-channel basis functions; that is, the basis chosen above. Only the $(1+,1+)$ element is nonzero, and as a result τ is the average of only one of the energy-specific transit-time matrix elements, $\tau_{1+1+}(E)$, the mean energy-specific transit time for a particle incoming from the left. In the semiclassical limit, this mean transit time takes the form

$$\tau_{1+1+}(E) = 2\int_a^b dx\, |\langle x | E1+ \rangle|^2$$
$$\sim 2\int_a^{x_1(E)} \frac{dx}{\sqrt{2[E-V(x)]/m}} + \int_{x_1(E)}^{b} \frac{e^{2\,\mathrm{Im}[s(x,E)]/\hbar}}{\sqrt{2|E-V(x)|/m}} , \tag{11}$$

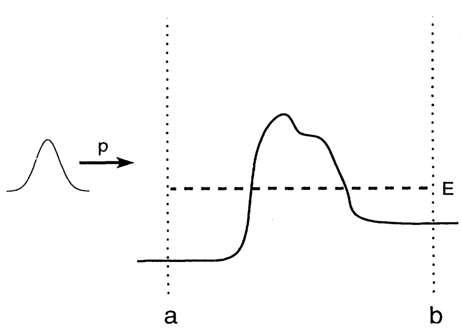

FIG. 1. Schematic representation of a generic tunneling process for wave-packet scattering. The packet is incoming from the left with momentum $p>0$, and with mean energy E below the height of the barrier, as depicted by the dashed line which also shows the classical turning points. We consider the dwell time in the interval $[a,b]$, where in most of the following we take a (b) far to the left (right) of the barrier, where "far" means at a point where the derivative of the potential is essentially zero.

where

$$s(x,E) = \int_a^x dx \sqrt{2m[E-V(x)]} \tag{12}$$

is the classical action at x measured from the initial point a. The first term of the second line of (11) is just the total time spent in (a,b) by a classical particle of energy E which reflects on reaching the left-turning point, $x_1(E)$.

The second term in (11) merits some discussion. It represents an exponentially damped accumulation of the time spent by a fictitious classical particle traveling under and beyond the barrier with velocity $\sqrt{2|E-V(x)|/m}$. A strict determination of the asymptotic form in (11) would omit the second term, which has significant asymptotic contribution only in the neighborhood of $x_1(E)$. Moreover, that contribution is $O(\hbar^{1/4})$, i.e., the same order as the reflected and incoming wave interference term [which was omitted in the derivation of (11)]. Nonetheless, we include the second term here to demonstrate how one might decompose the mean transit time into reflected and transmitted (or tunneled) contributions. In general, $\tau_{1+1+}(E)$ is the mean of transit times for both reflected and transmitted components of the initial wave packet. We seek a transit time specific to the tunneling component of the energy eigenstate. Because of the asymptotically small character of the transmitted component in the expansion of $\tau_{1+1+}(E)$, the second line of (11) does not achieve this goal.

An exact quantum-mechanical attempt to extract a tunneling transit time is provided by the representation of $\tau_{1+1+}(E)$ in terms of the outgoing channel specific basis, $|El-\rangle$. The outgoing basis functions are specific to the reflection and transmission channels and thus should provide the desired decomposition of the mean transit time. The change of representation is accomplished by standard means in terms of the appropriate unitary transformation matrix, which for the case at hand is just the energy equal to E scattering matrix. Thus, in terms of the incoming channel basis,

$$\begin{bmatrix} \psi_{1-} \\ \psi_{2-} \end{bmatrix} = \begin{bmatrix} R & T \\ T & -R^*T/T^* \end{bmatrix} \begin{bmatrix} \psi_{1+} \\ \psi_{2+} \end{bmatrix} , \tag{13}$$

where ψ is an arbitrary state in the two-dimensional energy-specific Hilbert space. The corresponding transformation of the transit-time matrix yields the following decomposition,

$$\tau_{1+1+} = |R|^2\tau_{1-1-} + 2\mathrm{Re}[RT^*\tau_{1-2-}] + |T|^2\tau_{2-2-} , \tag{14}$$

which expresses the mean transit time entirely in terms of "outgoing specific" quantities. [Note the suppression of the E dependence in Eqs. (13) and (14).]

Expression (14) illustrates the general problem encountered in attempting to extract a tunneling-specific transit time. The first and third terms are transit times weighted by the reflection and transmission probabilities at energy E, and are the sort of terms that we hoped to find in the decomposition of τ_{1+1+}. If they provided the only contributions the transit times τ_{1-1-} and τ_{2-2-} could be interpreted as the desired reflection and transmission-specific transit times. However, there is, in general, a cross term. Moreover, τ_{1-1-} and τ_{2-2-} cannot have the desired interpretation because they are not specific to an initial state incoming from the left. Instead τ_{1-1-} and τ_{2-2-} are only specific to "outgoing to the left" and "right," respectively. They are not specific to either incoming channel.

Relation (14) fails to provide the desired decomposition of the transit time due to the cross term [13], $2\,\mathrm{Re}[RT^*\tau_{1-2-}]$, which appears because $\tau(E)$ is not, in general, diagonal in either the incoming or outgoing channel specific representation. As a consequence of this fact the transit-time matrix does not commute with projectors onto left-going or right-going states. These projection operators represent observables which identify reflection or transmission events, i.e., "reflection-" and "tunneling-flag" observables. Thus, the inability to provide the desired decomposition of the transit-time results from a classic problem of incompatible observables, specifically, the transit-time and tunneling-flag observables.

III. THE FORWARD-TRANSIT-TIME OPERATOR

The development of the forward-transit-time operator parallels that of the transit-time operator as presented in Sec. II; however, straightforward modifications allow a separation of the transmitted and reflected contributions to transit time. Instead of treating a finite region around the barrier, we consider the dwell time in an infinite domain, $D'=(-\infty,b]$. In addition, only positive time integration is used in the expression corresponding to (4). With these adjustments the forward transit time of a reflected particle is infinity. The use of positive time integration prevents transmitted particles from also having infinite forward transit time. Otherwise, transmitted particles have infinite "total" transit time in D' due to contributions from the remote past [i.e., $t\in(-\infty,0)$] where the particle is incoming from $x=-\infty$.

In addition to letting the boundary of D' be one sided, we locate the right boundary, b, far to the right of the barrier, permitting the use of the $x\to\infty$ asymptotic form of the energy eigenstates in the neighborhood of the right boundary.

As in Sec. II, we are actually interested in transit across a finite interval, (a,b). In order to treat the finite transit of interest, it is necessary to localize the initial particle state, $\hat{\rho}$, to the neighborhood of a. (To make this choice of particle state precise we will, in some of the succeeding developments, let $\hat{\rho}$ be a coherent state centered about $x=a$, with momentum distribution centered about p, where $p>0$ and a is chosen far to the left of the barrier.) Also, the initial state must project almost entirely (that is, but for an exponentially small contribution) onto the space of left incoming energy eigenstates, the property required of the more general initial state considered in Sec. II.

We now proceed with the definition of forward transit time in the same manner used to define $\hat{\tau}$ in Sec. II. Thus, we define the "survival probability" as

$$C(t)=\mathrm{Tr}[\hat{d}\,'\hat{\rho}_t]=\mathrm{Tr}[\hat{d}\,'_{-t}\hat{\rho}] , \tag{15}$$

which is the probability that the particle is in D' at time t, subsequent to preparation in state $\hat{\rho}$ at time 0; $\hat{d}\,'$ projects onto D' just as $\hat{d}$ projects onto D. Associated with $C(t)$ is the "tunneled flux,"

$$P(t)=-\frac{dC(t)}{dt}=\mathrm{Tr}[\hat{j}\hat{\rho}_t] , \tag{16}$$

where

$$\begin{aligned}\hat{j} &\equiv \left.\frac{d}{dt}\hat{d}'_t\right|_{t=0} = -\frac{i}{\hbar}[\hat{H},\hat{d}\,'] \\ &= \frac{i}{\hbar}\left\{\left[\frac{\hat{p}^2}{2m},\hat{x}\right]\delta(\hat{x}-b)+\delta(\hat{x}-b)\left[\frac{\hat{p}^2}{2m},\hat{x}\right]\right\} \\ &= \frac{1}{2m}[\hat{p}\delta(\hat{x}-b)+\delta(\hat{x}-b)\hat{p}]\end{aligned} \tag{17}$$

is termed the "flux operator." Equation (17) makes use of the von Neuman equation for operator time evolution, the analog of Eq. (2) for $\hat{d}\,'$, and

$$\frac{d\theta_{D'}(x)}{dx} = -\delta(x-b) .$$

It is shown below that $P(t)$ can be viewed as the transit-time probability distribution for the interval (a,b).

An expression for the mean "residence time" in D' can, in principle, be obtained by integrating the survival probability, $C(t)$, over all positive t. However, $C(t)$ as defined in Eq. (15) does not decay to zero as $t\to\infty$ and the integral does not converge. The asymptotic value, $C(\infty)$, represents the total reflection probability since the reflected part of the evolving state stays in D' indefinitely [14]. The divergence of the mean residence time therefore should be identified as arising from the contribution of an infinite residence time associated with the reflected part of the wave packet. With this in mind, we identify the integral of the "subtracted" survival probability, $C(t)-C(\infty)$, as the contribution associated with the transmitted part of the wave packet. For example, $T\equiv C(0)-C(\infty)$ is the probability that a particle prepared in state $\hat{\rho}$ is initially in D' but eventually leaves, never to return; i.e., the "tunneling probability." From this perspective, $[C(t)-C(\infty)]/T$ can be viewed as the "tunneling-specific" survival probability; i.e., the probability that a tunneling particle is still in D' at t. Note this latter interpretation is appropriate only if $C(t)-C(\infty)\geq 0$ for all time. As is shown below, this condition is satisfied in the limit of $b\to\infty$ if $\hat{\rho}$ is specific to the channel "incoming from the left," precisely the case of interest.

Integration of the tunneling-specific survival probability determines a tunneling time

$$\tau^{+} = \int_0^{\infty} dt \frac{C(t)-C(\infty)}{T} , \tag{18}$$

which is the mean residence time in D' of a tunneling particle with initial position and momentum in the neighborhood of a and p, respectively. This tunneling time is related to the expectation of the "forward-transit-time operator." Using the definition of $C(t)$, we have

$$\begin{aligned}\tau^{+} &= \frac{1}{T}\int_0^{\infty} dt[\hat{d}_{-t}-\hat{d}_{-\infty}]\hat{\rho} \\ &= \frac{1}{T}\mathrm{Tr}(\hat{\tau}^{+}\hat{\rho}) ,\end{aligned} \tag{19}$$

where

$$\hat{\tau}^{+} = w\int_0^{\infty} dt[\hat{d}_{-t}-\hat{d}_{-\infty}] . \tag{20}$$

From the definition of the flux operator, in Eq. (17),

$$\hat{d}_{-t} = \hat{d}_{-\infty} + w\int_{-\infty}^{-t} dt'\,\hat{j}_{t'} . \tag{21}$$

Substituting this expression into Eq. (20) yields

$$\begin{aligned}\tau^{+} &= w\int_0^{\infty} dt \int_{-\infty}^{-t} dt'\,\hat{j}_{t'} \\ &= w\int_{-\infty}^{0} dt' \int_0^{-t'} dt\,\hat{j}_{t'} \\ &= w\int_0^{\infty} dt'\,t'\,\hat{j}_{t'} .\end{aligned} \tag{22}$$

This form of the forward-transit-time operator can be used in Eq. (19) to provide the desired interpretation of the tunneled flux. The substitution in question gives

$$\begin{aligned}\tau^{+} &= \frac{1}{T}\int_0^{\infty} dt\, t\, \mathrm{Tr}[\hat{j}_{-t}\hat{\rho}] \\ &= \frac{1}{T}\int_0^{\infty} dt\, tP(t) ,\end{aligned} \tag{23}$$

where the definition of $P(t)$ has been used. If $P(t)$ is a positive function of time, then it can be viewed as a component of the probability density for forward transit time associated with state $\hat{\rho}$. The piece missing from $P(t)$, which prevents it from being the full probability density, is a contribution at $t=\infty$ which has integrated weight equal to the reflection probability. Incorporation of this contribution into $P(t)$ would make the integral of Eq. (23) diverge. This corresponds to the inclusion of the infinite transit time of reflected particles. Since $P(t)$ excludes this contribution, it is viewed as the tunneling-specific component of the forward-transit-time distribution. In this context, $\hat{j}_{-t}$ represents the projector onto the forward-transit-time eigenspace associated with eigenvalue t, and we define a normalized tunneling-transit-time distribution, $\Pi(t)$, via

$$\Pi(t) = \frac{P(t)}{T} . \tag{24}$$

If $\Pi(t)$ is everywhere positive then it must indeed have the desired interpretation. This requirement is demonstrated below, both numerically and within a semiclassical limit framework. Note that had we been interested in reflection times we would have instead taken $D'=[c,\infty)$, with c far to the left of a. In this case $P(t)$ would be a distribution of reflection times—all tunneling contributions would be concentrated at $t=\infty$.

IV. SEMICLASSICAL LIMIT OF THE TUNNELED FLUX

The derivation of the forward-transit-time operator, and hence of the tunneling-time distribution, required an initial state localized in both position and momentum. The expression for $P(t)$ simplifies considerably in the case that $\hat{\rho}$ is taken to be a pure state, e.g., a localized wave packet. Setting

$$\hat{\rho} = |\psi\rangle\langle\psi|$$

and using (17) for the flux operator yields

$$P(t) = \langle\psi_t|\hat{j}|\psi_t\rangle = \frac{\hbar}{m}\mathrm{Im}[\psi_t^{*}(b)\psi_t'(b)] , \tag{25}$$

which provides a completely straightforward expression for the tunneling-time distribution in terms of the time-evolved initial wave packet. Note that this relation for the tunneling-time distribution is quantum mechanically exact (within our asymptotic scattering approximation) for any pure, localized, initial state.

A well known and extremely convenient example of such a localized wave packet is the coherent state, i.e., a minimum uncertainty Gaussian wave packet. The wave function of a coherent state localized in phase space at (q,p) is given (up to a choice of phase) by

$$\psi(x;q,p)=\left[\frac{1}{\pi\sigma^2}\right]^{1/4}\exp\left[-\frac{(x-q)^2}{2\sigma^2}+\frac{i}{\hbar}p(x-q)\right],$$

where the parameter σ has units of length and is, in general, arbitrary. However, in the following we wish to consider the semiclassical limit, $\hbar\to 0$, and want the coherent-state wave packet to correspond, in that limit, to a particle in a classical phase space. To this end we impart the "natural"—in the sense that the coherent state of a harmonic oscillator of frequency ω has $\sigma=\sqrt{\hbar/m\omega}$—dependence of σ on $\hbar$ and take $\sigma=\sqrt{\hbar}\alpha$. With this choice, the position space "width" of the coherent state is $\sqrt{\hbar}\alpha$ and as $\hbar\to 0$ the Gaussian wave packet becomes increasingly localized in position. Similarly, the momentum space width, easily shown to be given by $\sqrt{\hbar}/\alpha$, also approaches zero in the semiclassical limit. The (arbitrary) parameter, α, controls the asymmetry of the state in phase space.

With all $\hbar$ dependence thus accounted for, the initial wave packet in our prototypical tunneling process is given by

$$\psi_{t=0}(x)=\left[\frac{1}{\pi\hbar\alpha^2}\right]^{1/4}\exp\left[-\frac{(x-a)^2}{2\hbar\alpha^2}+i\frac{p(x-a)}{\hbar}\right],\tag{26}$$

which is localized, as desired, about point a, far to the left of the barrier, and with momentum $p>0$ (see Fig. 1).

Equation (25) expresses the reactive flux simply in terms of the time-evolved coherent state at $x=b$. This time-evolved state is simplified using incoming specific energy representation [15]

$$\psi_t(x)=\int_0^\infty dE\,e^{-iEt/\hbar}\langle E1+|\psi\rangle\psi_{E1+}(x)\ .\tag{27}$$

Here, $\psi_{E1+}(x)\equiv\langle x|E1+\rangle$ is the energy eigenstate in position representation. Use has been made of the fact that $|\psi\rangle$ projects only onto states incoming from the left (i.e., $l=1$).

Recall that $\psi_{t=0}(x)$ is localized in the neighborhood of $x=a$, a point far to the left of the barrier. As a result, the matrix element $\langle E1+|\psi\rangle$, which determines $\psi_t(x)$, is evaluated to desired accuracy (simply make $|a|$ sufficiently large) in terms of the $x\to-\infty$ asymptotic form of $\psi_{E1+}(x)$. Specifically,

$$\langle E1+|\psi_{t=0}\rangle=\left[\frac{m}{p(E)}\right]^{1/2}[\langle p(E)|\psi_{t=0}\rangle+R^*(E)\langle -p(E)|\psi_{t=0}\rangle]\tag{28}$$

$$=\left[\frac{m^2\alpha^2}{\pi\hbar p^2(E)}\right]^{1/4}\left[\exp\left[-\frac{\alpha^2[p-p(E)]^2}{2\hbar}-i\frac{p(E)a}{\hbar}\right]+R^*(E)\exp\left[-\frac{\alpha^2[p+p(E)]^2}{2\hbar}+i\frac{p(E)a}{\hbar}\right]\right].\tag{29}$$

Note the use of momentum eigenstates, $\langle x|\pm p(E)\rangle=\exp[\pm ip(E)x/\hbar]/\sqrt{2\pi\hbar}$, in Eq. (28). The final expression utilizes the simplicity of the coherent-state momentum representation.

In the semiclassical limit the second term in the square brackets of Eq. (29) is exponentially small compared with the first term, and it results in an $O[\exp(-\alpha^2p^2/2\hbar)]$ contribution to the integral of Eq. (27). [Note that $p(E)$ extends from 0 to ∞.] We therefore neglect this "reflection term" in favor of the "incoming wave term."

Since b is a point far to the right of the barrier, the $x\to\infty$ asymptotic form of $\psi_{E1+}(x)$ can be used to evaluate $\psi_t(x=b)$ with Eq. (27). Equation (27) now takes the form

$$\psi_t(b)=m\left[\frac{\alpha^2}{4\pi^3\hbar^3}\right]^{1/4}\int_0^\infty dE\,e^{-iEt/\hbar}\frac{T(E)}{p(E)}\exp\left[-\frac{\alpha^2[p-p(E)]^2}{2\hbar}+i\frac{p(E)(b-a)}{\hbar}\right].\tag{30}$$

The semiclassical transmission coefficient [16]

$$T(E)\sim\exp\left[\frac{i}{\hbar}[s(E)-p(E)(b-a)]\right]\tag{31}$$

can be used to evaluate the integral of (30). Here $s(E)$ is the classical action associated with passage from $x=a$ to $x=b$, including the underbarrier (imaginary) contribution [17]

$$s(E)=\int_a^b dx\sqrt{2m[E-V(x)]}\ .\tag{32}$$

After substituting the semiclassical tunneling formula (31) into (30), the energy integral can be evaluated using the method of steepest descent. The integrand has the form $\exp[F(E,t)/\hbar]/p(E)$, where

$$F(E,t)=-\frac{\alpha^2[p-p(E)]^2}{2}+i[s(E)-Et]$$

and the contour of integration is deformed to pass through the "stationary point" of $F(E,t)$, i.e., the solution to the nonlinear equation

$$\frac{\partial F(E,t)}{\partial E}=\frac{\partial}{\partial E}\left[-\frac{\alpha^2[p-p(E)]^2}{2}+i[s(E)-Et]\right]=0\ .\tag{33}$$

Using the definition of the energy at the saddle point this equation takes the form,

$$\frac{p-p(E)}{p(E)}+i\frac{\tau(E)-t}{\tau_0}=0\ ,\tag{34}$$

where

$$\tau(E)=\frac{\partial s(E)}{\partial E}=\int_a^b dx\sqrt{m/2[E-V(x)]} \tag{35}$$

is the classical time for passage from a to b at energy E, and $\tau_0=m\alpha^2$ is a characteristic unit of time. Note that $\tau(E)$ has an imaginary contribution associated with passage under the barrier.

Equation (34) introduces $\tau_0=m\alpha^2$ which, since $\alpha\sqrt{\hbar}$ is the width of the initial wave packet in configuration space, has the following fanciful interpretation: $m\alpha^2\hbar$ is an effective moment of inertia, and τ_0 corresponds to the rotation period of the initial wave packet undergoing virtual rotation about its center with angular momentum $2\pi\hbar$.

The formula for the semiclassical limit reactive flux is now easily obtained. First, solve (34) for the "saddle-point" energy $E^\dagger$ at which the phase is stationary. To make the notation somewhat clearer, we denote all quantities evaluated at the saddle point with a superscript dagger. Then employ the standard steepest-descent formula for the asymptotic form of $\psi_t(x)$ to get

$$\psi_t(b)\sim m\left[\frac{\alpha^2}{4\pi^3\hbar^3}\right]^{1/4}\left[\frac{2\pi\hbar}{\left|\frac{\partial^2F(E^\dagger,t)}{\partial E^2}\right|}\right]^{1/2}\times\frac{\exp[F(E^\dagger,t)/\hbar-i\theta/2]}{p(E^\dagger)}, \tag{36}$$

where θ is the phase of $\partial^2F(E^\dagger,t)/\partial E^2$. Once obtained, this expression is substituted into Eq. (25). Note that the semiclassical asymptotic form is given by neglecting the term involving the derivative, with respect to b, of the preexponential factor. The other term, which we include, is larger by $O(1/\hbar)$. The result is

$$P(t)\sim\left[\frac{\alpha^2}{4\pi^3\hbar^3}\right]^{1/2}\left[\frac{2\pi\hbar m}{\left|\frac{\partial^2F(E^\dagger,t)}{\partial E^2}\right|}\right]\mathrm{Im}\left[\frac{d}{db}[F(E^\dagger,t)]\right]\frac{\exp\{2\,\mathrm{Re}[F(E^\dagger,t)]/\hbar\}}{|p(E^\dagger)|^2} \tag{37}$$

$$=\left[\frac{\alpha^2}{\pi\hbar}\right]^{1/2}\left[\frac{m\,\mathrm{Re}p(E^\dagger)}{\left|\frac{\partial^2F(E^\dagger,t)}{\partial E^2}\right|}\right]\frac{\exp\{2\,\mathrm{Re}[F(E^\dagger,t)]/\hbar\}}{|p(E^\dagger)|^2}. \tag{38}$$

Equation (38) relies on $\partial s(E)/\partial b=p(E)$, which determines the b dependence of $F(E^\dagger,t)$. In principle, $F(E^\dagger,t)$ also depends on b implicitly via $E^\dagger$. However, this dependence on b drops out because of the stationarity of $F(E,t)$ with respect to E at the saddle point. To get the semiclassical tunneling-time probability distribution [see Eq. (24)], $P(t)$ is divided by the total probability T, which is just the integral of $P(t)$ over all time. The tunneling-time probability distribution is therefore the normalized tunneled flux.

The semiclassical tunneled flux as defined in (38) is clearly positive. Thus, at this level of approximation, it is legitimate to interpret $\Pi(t)=P(t)/T$ as the tunneling-time probability distribution. Note that in addition to the explicit time dependence of this quantity there is an implicit time dependence carried by the saddle-point energy $E^\dagger$. It is this latter time dependence which produces the richness of behavior described below.

V. NUMERICAL STUDIES

Equation (38) determines an $\hbar\to 0$ asymptotic tunneling-transit-time distribution. As a semiclassical expression it reflects characteristics of tunneling manifest at the "almost classical" level of approximation. Nevertheless, because tunneling is an intrinsically quantal phenomenon, (38) also contains a great deal of quantum mechanics. This section closely examines both the semiclassical and the exact quantal tunneling flux, and extracts a number of useful insights regarding the quantal tunneling transit time.

Much of the insight extracted here results from examination of relation (34) which determines the saddle-point energy. For example, note that if $t=\tau^\dagger\equiv\tau(E^\dagger)$, then $p^\dagger\equiv p(E^\dagger)=p$, and the saddle-point energy is just $p^2/2m$, the "central energy" of the incoming coherent state. In the case of classically allowed motion, this time is real and is the time at which the tunneled flux is a maximum (see below). In the presence of a barrier, $p^\dagger=p$ determines a complex time, the imaginary part of which is associated with the underbarrier portion of the classical trajectory directly connecting a and b. Since our tunneling study utilized the time domain, where only real times are of interest, we never encounter $p^\dagger=p$ in the case of tunneling.

A. Most probable tunneling transit time

We begin our examination of the semiclassical tunneled flux by extracting the "most probable tunneling transit time," i.e., the time of maximum flux, found by setting the derivative of $P(t)$ to zero. Note that in the semiclassical limit the derivative of $P(t)$ is determined strictly in terms of the derivative of the exponent, $\mathrm{Re}[F(E^\dagger,t)]$. That is,

$$\frac{d}{dt}P(t)=0$$

implies

$$\frac{d}{dt}\mathrm{Re}[F(E^\dagger,t)]$$
$$=\mathrm{Re}\left[\left.\frac{\partial F(E,t)}{\partial E}\right|_{E=E^\dagger}\frac{dE^\dagger}{dt}+\frac{\partial F(E^\dagger,t)}{\partial t}\right]\sim 0\ . \quad (39)$$

Since

$$\left.\frac{\partial F(E,t)}{\partial E}\right|_{E=E^\dagger}=0$$

by definition of $E^\dagger$, and

$$\frac{\partial F(E^\dagger,t)}{\partial t}=-iE^\dagger\ ,$$

(39) simplifies to

$$\mathrm{Im}[E^\dagger]=0\ . \quad (40)$$

Thus, the most probable tunneling time occurs when the saddle-point energy is real. Since $E^\dagger=(p^\dagger)^2/2m$, $p^\dagger$ must also be real. Referring to (34), we find that the most probable tunneling time, $\bar{t}$, satisfies

$$-i(\tau^\dagger-\bar{t})=\tau_0\frac{p-p^\dagger}{p^\dagger}\ , \quad (41)$$

with both p and $p^\dagger$ real.

The important feature of (41) is that its right side is real. Therefore, $\bar{t}$ must exactly cancel the real part of $\tau^\dagger$. As already discussed, in the case of classical allowed motion this corresponds to $p^\dagger=p$ and semiclassical mechanics reduces essentially to classical mechanics. However, in the case of tunneling $\tau^\dagger$ has an imaginary part and (41) is a nontrivial equation for $p^\dagger$. In particular, the most probable tunneling transit time $\bar{t}=\mathrm{Re}(\tau^\dagger)$, and $p^\dagger$ satisfies the real equation

$$\frac{p-p^\dagger}{p^\dagger}=\mathrm{Im}\left[\frac{\tau^\dagger}{\tau_0}\right]\ , \quad (42)$$

or

$$p=p^\dagger[1+\mathrm{Im}(\tau^\dagger/\tau_0)]\ . \quad (43)$$

Equation (43) provides a simple expression for p in terms of $p^\dagger$. The inverse of this relation determines the saddle-point momentum, $p^\dagger$, in terms of p, that is, in terms of the coherent-state central momentum. This inverse function is easily obtained graphically and examples are presented in Fig. 2(a) for the case the potential is a Gaussian barrier of unit height and width; i.e.,

$$V(x)=\exp(-x^2)\ . \quad (44)$$

Also, $m=\frac{1}{2}$ is chosen so that $p=1$ corresponds to the top of the barrier.

The essential features of the inverse of (43) are described as follows.

(1) Since $\mathrm{Im}(\tau^\dagger)<0$ for real $p^\dagger<1$, p is consistently smaller than $p^\dagger$. The degree to which $p^\dagger$ is greater than p is determined by the relative sizes of $\mathrm{Im}(\tau^\dagger)$ and $\tau_0=m\alpha^2$. Thus, for example, large α produces $p^\dagger$ close to p, whereas small α corresponds to a big difference.

(2) For $p^\dagger\geq 1$, $\mathrm{Im}(\tau^\dagger)=0$ and $p=p^\dagger$. However, as $p^\dagger$ approaches 1 from below p approaches a value less than 1. There exists, therefore, a range of p without any corresponding value of $p^\dagger$. For these p values the method of steepest descent fails. This is a consequence of the discontinuity in $\mathrm{Im}[\tau(E)]$ at $E=1$, which in turn results from the singularity in the primitive semiclassical transmission coefficient employed above [we specifically refer to the branch points in the action integral—see Eq. (32)]. This problem can be avoided through the use of uniform semiclassical eigenstates in the vicinity of $E=1$, which would result in a rapid but smooth connection between the curves in Fig. 2(a) and the $p^\dagger=p$ straight lines that are appropriate when $p>1$. This approach is not considered in the present article, however. Rather, we simply assume p small enough for the solution, $p^\dagger$, to exist in the sense described above [i.e., as a solution to Eq. (43)]. This restriction can be viewed as a restriction to cases of "true tunneling" (at least from a semiclassical standpoint). In cases where $p^\dagger=1$, the transmission probability is dominated by the momentum space tail of the initial wave packet with energy above the top of the potential barrier. From our current standpoint such transmission is not true tunneling.

Note that the restriction to cases of true tunneling constrains α as well as p. Specifically, α is restricted to sufficiently large values, for $0<p<1$, since there are no solutions to Eq. (43) when α is small (~ 2 or less). Moreover, the range of tunneling p values is very limited unless α is fairly large. These ranges are demonstrated in Fig. 2.

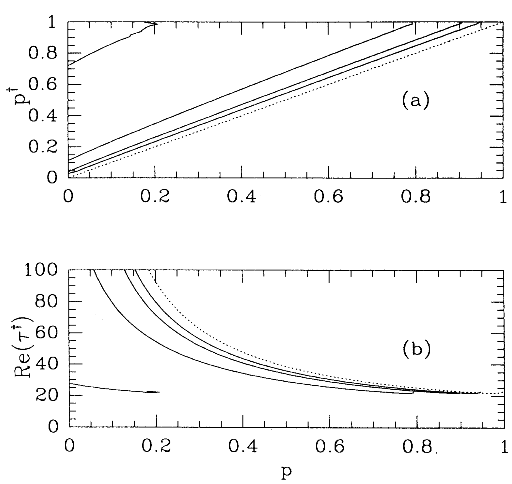

FIG. 2. Solutions to the nonlinear equation for the saddle-point momentum. Panel (a) shows saddle-point-momentum values, $p^\dagger$, as functions of p, for four different values of the asymmetry parameter, $\alpha=2$, 4, 6, and 8. Note that there is an upper bound on p for which "true tunneling" (i.e., with $p^\dagger<1$) occurs. This bound decreases markedly as α decreases below 4. The corresponding most probable tunneling times, $\bar{t}=\mathrm{Re}(\tau^\dagger)$, are shown in panel (b) as functions of p. The dashed lines in the two figures are for the case of $\alpha=\infty$, wherein $p^\dagger=p$. In particular, in panel (b) the dashed line gives the time for simple classical motion from a to b at momentum p, assuming instantaneous crossing between turning points. Note that the latter is almost always larger than the actual most probable tunneling time.

(3) The solution to Eq. (43), $p^{\dagger}$, is generally not close to p. Thus, the barrier generally cannot be viewed as a small perturbation on the energy of the particle. Since there is no $\hbar$ dependence in Eq. (43), even in the semiclassical limit, particle energy is strongly perturbed when the particle is "forced" to tunnel through the barrier (or go over it, as in cases of large p and/or small α). An important consequence of this observation is that the energy representation picture of tunneling time has little relevance to the time domain tunneling studied here. In the energy representation one considers the transit-time observable (or time delay) as described in Sec. II. The connection between energy and time representations is usually made in the limit of a narrow energy distribution where the "interaction process" (i.e., scatterer) causes only a small perturbation to the particle energy [18]. This permits the expression of the time delay in terms of energy derivatives of stationary (i.e., energy-specific) quantities. In contrast to other formulations of tunneling time [9(c)] where a narrow range of energies associated with the initial wave packet is sufficient to obtain energy-specific results, the current presentation requires a highly skewed phase space asymmetry ($\alpha \rightarrow \infty$) in order for $\bar{t}$ to be specific to energy E (i.e., the case of $p^{\dagger} \rightarrow p$). Thus, in the semiclassical limit, even though the absolute energy range $[O(\hbar/\alpha^2)]$ is small, $p^{\dagger}$ may be far from p due to a not-too-large α value. In this case $\bar{t}$ is not energy specific and our tunneling-time distribution is not related to a corresponding energy domain point of view.

We interpret $p^{\dagger}$ as the "effective initial momentum" with which the bulk of the particle probability tunnels through the barrier. Thus, our results show that the particle tunnels with larger than its initial central momentum value, p. Effectively, the particle "attempts" to go over the barrier, and as a result it travels "faster" and the most probable tunneling transit time, $\bar{t} = \text{Re}[\tau(E^{\dagger})]$, is generally smaller than the time $\text{Re}[\tau(E)]$ required along a simple "classical" path associated with the initial particle energy—on this path tunneling is treated as an instantaneous jump between turning points which occurs when the classical particle reaches the leftmost point. Most probable tunneling times are presented in Fig. 2(b). Note that for each value of α there is a restriction on the range of p as described above.

The range of p we consider consists of those momenta insufficiently large for the particle to surmount the barrier. As α approaches infinity the shift from p to $p^{\dagger}$ decreases, and this range increases to the full interval (0,1). The case of $\alpha = \infty$ corresponds to the dotted lines shown in Fig. 2. The circumstance of large α, which keeps $p^{\dagger}$ close to p, corresponds to an initial wave packet which is very narrow in momentum space. In this case, the particle momentum distribution falls off rapidly away from its central value p. The effective momentum $p^{\dagger}$ is drawn from the positive tail of the momentum distribution. Shrinking this tail by increasing α forces $p^{\dagger}$ to approach p, as observed.

If reflection had been the process of interest, a semiclassical reflection coefficient would replace $T(E)$ in (30) (other simple changes would also be necessary). In this case $\tau^{\dagger}$ would always be real and the most probable reflection time would reduce to the simple classical value at energy E.

B. Tunneling-time distribution

To emphasize the relevance of the semiclassical computations presented in this section we first consider a comparison of the semiclassical tunneled flux with that computed using essentially exact quantum mechanics. Figure 3 shows such a comparison for a sequence of $\hbar$ values. Here, $p = 0.2$ and $\alpha = 4$. Figure 4 shows similar results for the case of a Lorentzian potential (rather than a Gaussian one), also of unit height and width,

$$V(x) = \frac{1}{1+x^2} . \tag{45}$$

The semiclassical data were determined by solving Eq. (34) for $p(E^{\dagger}) = p^{\dagger}$, and then substituting into Eq. (38). The "exact" quantum-mechanics results were obtained using a converged fast-Fourier-transform algorithm [19] to compute $\psi_t(x)$, and Eq. (25). The principle conclusions of this and other comparisons are as follows: even when $\hbar$ is quite large the semiclassical formula provides a close approximation to the quantal results—qualitative characteristics are essentially reproduced. Thus, any insight into the tunneled flux which we draw from the semiclassical analysis is also insight into exact quantal tunneling. A specific feature of this comparison is the relatively rapid convergence, as $\hbar \rightarrow 0$, of the most probable tunneling time. The amplitude of the tunneled flux converges more slowly, and generally converges from below. Note that although both the Gaussian and Lorentzian potentials are of unit "width," the Lorentzian is wider at $p = 0.2$. Thus, the Lorentzian tunneled fluxes are narrower and of lower amplitude for a given value of $\hbar$.

The quantum-speed-up effect is illustrated in Figs. 3

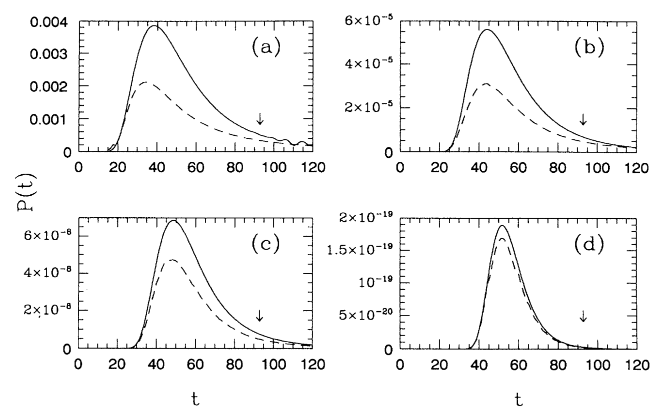

FIG. 3. Tunneled fluxes, or unnormalized tunneling-time distributions, for a Gaussian barrier potential. $\hbar = 1.8$, 0.6, 0.3, and 0.1, for panels (a), (b), (c), and (d), respectively. Otherwise, $p = 0.2$ and $\alpha = 4$. Dashed lines give semiclassical data, while solid lines are for exact quantum mechanics. Downward arrows indicate times for simple classical motion, $\text{Re}[\tau(E)]$. Note that the integral over all time of these curves determine the total tunneling probability at the p and α values in question.

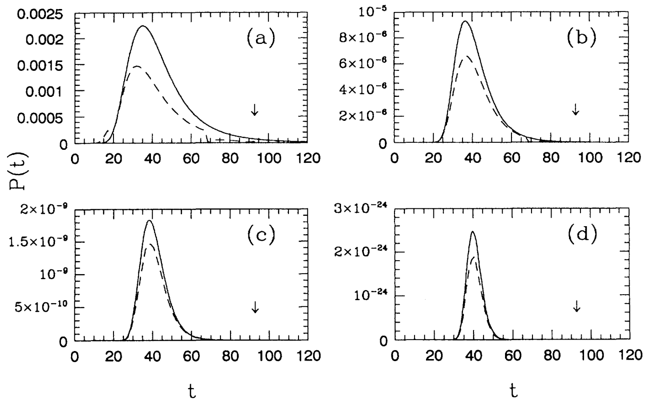

FIG. 4. As in Fig. 2 except that a Lorentzian barrier potential was used.

and 4 by the inclusion of arrows labeling the time, $\mathrm{Re}[\tau(E)]$, which is associated with simple classical motion from a to b at the initial particle momentum, p. In this idealized motion, the particle traverses the classically forbidden (underbarrier) portion of the interval instantaneously.

Figure 5 shows a set of semiclassical and exact quantal tunneled fluxes with different p and α values, all for the case of a Gaussian potential. Here we see broader distributions and enhanced quantum speed-up for smaller p values. The smaller α value is also associated with a greater relative difference between $\mathrm{Re}[\tau(E)]$ and $\mathrm{Re}[\tau(E^\dagger)]$, but it produces narrower distributions, as one might expect.

Apparent in Figs. 5(c), 5(e), 4(a), and 4(b), are discontinuities in the semiclassical tunneled flux at small and/or large times. These appear to associated with peculiarities of Eq. (34) at times well away from $\bar{t}$. For example, long times typically correspond to $E^\dagger$ with a small real part. In this case, the associated turning points can approach or extend beyond a and b (particularly if the potential is Lorentzian). The semiclassical approximation is not valid under such circumstances and spurious results appear. (In fact, it was necessary to cut the Lorentzian potential data off at not too large times since unphysical exponentially large terms were seen.) Short times are associated with $\mathrm{Re}(E^\dagger)$ approaching unity. In this case, the primitive semiclassical transmission coefficient becomes invalid (as discussed above), and the data can therefore be unreliable at short times as well.

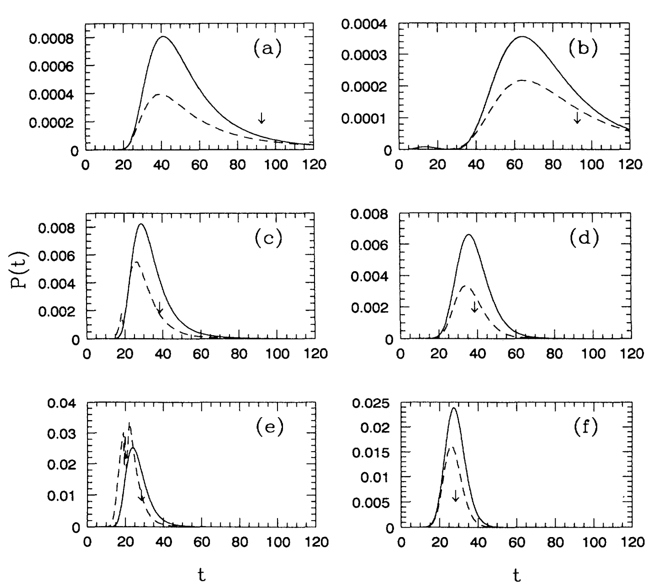

FIG. 5. As in Fig. 2 except that $\hbar=1$, while p and α are varied. Panels (a) and (b) are for $p=0.2$, panels (c) and (d) are for $p=0.5$, and panels (e) and (f) are for $p=0.7$, $\alpha=2$ in panels (a), (c), and (e), while $\alpha=4$ in panels (b), (d), and (f).

Aside from the "faster than classical" most probable tunneling times, the most notable feature of these data is the dependence of the spread of $P(t)$ on the phase space asymmetry of the initial wave packet, as measured by α. The spread in $P(t)$ scales roughly in proportion to $\tau_0=m\alpha^2$ which determines a qualitative measure of the tunneling-time-distribution time scale. This interpretation of τ_0 can be seen analytically via Eq. (34), which can be rewritten in the form

$$p^\dagger=\frac{p}{1+i(t-\tau^\dagger)/\tau_0} .$$

In terms of the saddle-point energy, $E^\dagger=(p^\dagger)^2/2m$, this expression takes the form

$$E^\dagger=\frac{E}{[1+i(t-\tau^\dagger)/\tau_0]^2} . \tag{46}$$

From this expression we get $\mathrm{Im}E^\dagger$, which determines the logarithmic derivative of $P(t)$ in the semiclassical limit [see Eqs. (38) and (40)]:

$$\begin{aligned}\frac{d\ln P(t)}{dt} &\sim \frac{2}{\hbar}\frac{\partial \mathrm{Re}[F(E^\dagger)]}{\partial t}\\ &=\frac{2}{\hbar}\mathrm{Im}[E^\dagger] .\end{aligned} \tag{47}$$

This demonstrates an extremely simple relationship between the tunneling-time distribution and the saddle-point energy.

It proves instructive to note that the tunneling-time distribution can be represented, through use of (46), in the form

$$\frac{d\ln P(t)}{dt}=-\frac{4}{\hbar}\frac{E\lambda\tilde{t}}{(\lambda^2+\tilde{t}^2)^2} , \tag{48}$$

where

$$\tilde{t}=\frac{t-\mathrm{Re}[\tau^\dagger]}{\tau_0} \tag{49}$$

is a scaled and shifted time variable and

$$\lambda=1+\mathrm{Im}\left[\frac{\tau^\dagger}{\tau_0}\right] \tag{50}$$

is an effective width parameter as suggested by Eq. (48).

If we assume that $\tau^{\dagger}$ is linear in time [20] then Eq. (48) can be integrated to determine a model tunneling-time distribution. In the resulting model $\ln P(\tilde{t})$ is Lorentzian with width essentially given by $\bar{\lambda} \equiv 1+\mathrm{Im}[\bar{\tau}^{\dagger}/\tau_0]$ (this translates to a width of $\tau_0+\mathrm{Im}[\bar{\tau}^{\dagger}]$ in terms of unscaled time, t), where $\bar{\tau}^{\dagger}$ is the saddle-point classical time at $t=\bar{t}$. This model is easy to derive in the case of constant λ; however, it is also valid in the more general case in which λ depends linearly on t. Skewing of the model Lorentzian, consistent with observed tunneling-time distributions, is obtained if an appropriate linear time dependence of λ is incorporated.

A few general conclusions can be drawn from the model of the semiclassical tunneling-time distribution constructed here. First, we see that the width of the distribution is characterized by $(\tau_0+\mathrm{Im}[\bar{\tau}^{\dagger}])/\hbar$. The $\hbar$ dependence results when we account for the overall $1/\hbar$ scaling of the model Lorentzian. Since $\mathrm{Im}[\bar{\tau}^{\dagger}]$ is negative, its contribution reduces the width of $P(t)$ which otherwise scales via τ_0 as the square of the relative coordinate space spread of the initial wave packet.

The skewing of $P(t)$ towards longer times is related to the variation of $\mathrm{Im}[\tau^{\dagger}]$ as a function of time (through its energy dependence). Typically, as time increases, $\mathrm{Im}[\tau^{\dagger}]$ increases making the effective width, λ, larger at long times. This time dependence of $\mathrm{Im}[\tau^{\dagger}]$ may be understood by reexamining Eq. (34). Specifically, Eq. (43) leads to

$$\frac{d}{dt}\mathrm{Im}\tau^{\dagger}=\tau_0\frac{d}{dt}\mathrm{Re}\frac{p}{p^{\dagger}}=-\frac{\tau_0}{E^{\dagger}}\frac{d}{dt}\mathrm{Re}p^{\dagger} . \tag{51}$$

Thus, $\mathrm{Im}[\tau^{\dagger}]$ increases with t since $\mathrm{Re}[p^{\dagger}]$ decreases. The behavior of $\mathrm{Re}[p^{\dagger}]$ is understood physically. Smaller saddle-point momenta determine the arrival of the particle at b at longer times. These longer-time contributions naturally correspond to slower components of the evolving wave packet.

The expression of $P(t)$ as the exponential of a Lorentzian is nonphysical at sufficiently long times. This follows because the model tunneled flux is not integrable, and, in fact, does not even decay to zero as $t\rightarrow\infty$. The true tunneled flux decays to zero at long times and integrates to give the finite total tunneling probability, T. Thus, either one or both of the models determined here, or the semiclassical approximation as presented above, must break down at sufficiently long times. (Note our earlier discussion in this regard.)

C. Husimi-transform evolution

To illustrate the quantum-speed-up effect pictorially, we have computed Husimi transforms of time-evolving wave packets [21]. The Husimi transform is accomplished by projecting the wave function at any given time onto a basis of reference coherent states

$$h_t(z)=|\langle z|\psi_t\rangle|^2 ,$$

where $|z\rangle=|(x/\alpha_h+i\alpha_h p)/\sqrt{\hbar}\rangle$ is the general coherent state associated with phase space asymmetry, α_h (which need not be the same as the asymmetry of the original wave packet). The Husimi transforms provide information about the phase space distribution of a wave packet as it evolves, tunneling through and reflecting off the potential barrier. Figure 6 shows contour plots for a time sequence of $h_t(z)$ with $p=0.2$, $\alpha=4$, and $\hbar=0.3$. Note that a logarithmic scale for the contour values was chosen. The solid horizontal lines specify the initial particle momentum and its reflected value, $-p$. The dashed line denotes the saddle-point momentum, $p^{\dagger}$, associated with the most probable tunneling time. The "tunneled-wave-packet lobe" (clearest in the bottom row) exhibits skewing typical of free particle motion and appears to be centered at $p^{\dagger}$. The latter result emphasizes the significance of Eq. (43) which produces this higher momentum. The reflected particle is not as clear in these plots, as it generally interferes with the incoming particle. Nevertheless, we see that the reflected particle, at least initially, appears to travel to the left faster than the incoming particle travels to the right. At long time, we expect $-p$ to give the reflected particle momenta (and the last panel is consistent with this expectation). The other noteworthy feature of these calculations is the very broad

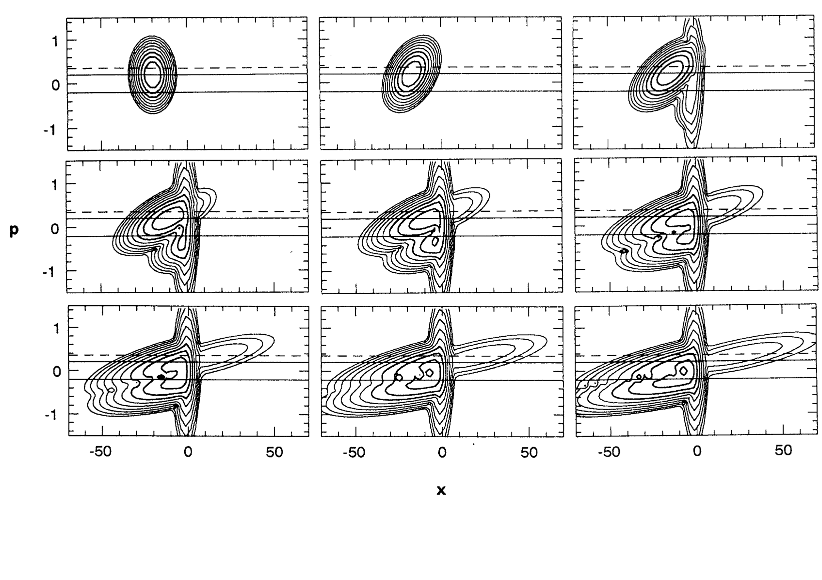

FIG. 6. Contour plots of Husimi-transform time evolution. The upper left panel gives the Husimi transform of the initial coherent state wave packet. Note that a logarithmic scale is used for contour values, which are 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1, 10, 10^2, and 10^3 (contours with value ≥ 1 are heavier for clarity). Time increases from left to right, and then from top to bottom. Time values shown are from 0 to 64, in increments of 8. The horizontal solid lines denote initial wave-packet momentum, $p=0.2$, and its reflected value. The dashed lines denote saddle-point-momentum values, $p^{\dagger}=0.35$, associated with the most probable tunneling time. The phase-space symmetry parameter associated with the Husimi transforms shown here is $\alpha_h=4$.

range of momenta exhibited "under" the barrier. It is interesting to observe that these large momentum components do not venture outside of the barrier region as time evolves.

VI. SUMMARY AND DISCUSSION

Localized wave-packet tunneling through a one-dimensional barrier has been investigated with regard to tunneling-time distribution. We have shown that, strictly speaking, a tunneling-time probability distribution does not exist. Nevertheless, in terms of the time evolution of an initially localized wave packet, we have constructed a quantity—the tunneled flux—with the essential properties of a tunneling-time distribution. This quantity was investigated numerically as well as analytically via a semiclassical steepest-descent calculation. The semiclassical calculation provides a good approximation to numerical tunneled fluxes. It also afforded means of extracting qualitative characteristics of tunneled flux (and hence the semiclassical tunneling-time distribution). The principal features of tunneling-time distribution thereby extracted are summarized as follows.

(1) The most probable tunneling time is determined by the classical motion associated with a certain "saddle-point" momentum given by the solution of a simple nonlinear equation.

(2) Tunneling is faster than simple classical motion from one side of the barrier to the other, even assuming instantaneous transit between the turning points. Effectively, the barrier acts as a filter for the large-momentum components of the initial wave packet. This effect was illustrated pictorially in phase space by numerical computations of the Husimi-transformed time-evolving wave packet.

(3) The semiclassical tunneling-time distribution has been modeled, in agreement with observation, as the exponential of a skewed Lorentzian function. The width of the distribution scales as $1/\hbar$ and is otherwise determined by two opposing terms. The first of these terms scales as the square of the relative coordinate space width of the initial wave packet; while the second term is just the imaginary time associated with the forbidden classical motion between the turning points at the saddle-point momentum.

The role of the imaginary part of the time in determining characteristics of the tunneling-time distribution represents one of the most interesting and controversial issues regarding the question of tunneling transit time. A number of previous studies in this area have emphasized this quantity [1(a),4,10]. Within our approach, the imaginary time is found to have a peripheral, but not insignificant, role. Specifically, the imaginary time controls the magnitude of a positive shift in the energy of the classical particle motion which determines the most probable tunneling transit time. That is, a relatively large imaginary time effectively "deflects" the real classical motion to higher velocities. This generally has the effect of reducing the tunneling transit time. In terms of exact quantum mechanics, the barrier can be viewed as a filter which selects out the large momentum components of the wave packet, and the imaginary time contributes to the degree of filtering. In addition, the imaginary time (in combination with the phase-space asymmetry of the initial wave packet) affects the spread of the tunneling-time distribution. Increasing the imaginary time decreases the spread of the transit-time distribution which would otherwise be determined by the width of the initial wave packet alone. A more subtle effect, resulting from the time dependence of the saddle-point imaginary time, is responsible for the skewing of observed distributions to longer times.

Although we have presented our results within the framework of simple one-dimensional scattering, many of the developments generalize straightforwardly to more complicated cases. For example, future investigations will consider the multidimensional scattering generalization of tunneling time, as well as the nature of tunneling time in cases other than "deep tunneling," i.e., when the classical turning points are near the top of the barrier [22]. Additionally, we believe it will soon be possible to compare our predictions with experimental results [23].

Tunneling from a metastable state is more problematic since it is not generally possible to take the initial wavepacket to be "far" from the barrier, making the validity of various asymptotic approximations questionable. Our hope is that by expressing the tunneling-time distribution in a more general language—specifically, the path-integral formulation of quantum mechanics—we will be able to unravel the complexities of tunneling from a metastable state. A future contribution will make the connection between the current semiclassical analysis and the stationary-phase approximation of the coherent-state path-integral representation of the tunneled flux [24].

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[1] For recent reviews of the subject see (a) E. H. Hauge and J. A. Støvneng, Rev. Mod. Phys. **61**, 917 (1989); (b) M. Büttiker, in *Electronic Properties of Multilayers and Low-Dimensional Semiconductor Structures,* edited by J. M. Chamberlain *et al.* (Plenum, New York, 1990); (c) M. Büttiker and R. Landauer, Phys. Rev. Lett. **49**, 1739

(1982), and references therein.

[2] A. Ranfagni, D. Mugnai, P. Fabeni, and G. P. Pazzi, Appl. Phys. Lett. **58**, 774 (1990), and references therein.

[3] Another treatment of tunneling time distributions is provided in H. A. Fertig, Phys. Rev. Lett. **65**, 2321 (1990).

[4] Imaginary tunneling times have been the focus of much interest in the past. See, for example, E. Pollok and W. H. Miller, Phys. Rev. Lett. **53**, 115 (1984); D. Sokolovski and L. M. Baskin, Phys. Rev. A **36**, 4604 (1987); and C. R. Leavens and G. C. Aers, Solid State Commun. **63**, 1101 (1987).

[5] (a) Phil Pechukas (private communication). (b) Reference [1] describes a quantum-speed-up effect for tunneling in terms of transmission-probability-weighted averages of energy-specific dwell times (see Fig. 5 of Ref. [1]). However, this speed-up differs from that described in the current article since we instead consider distributions of forward transit times (see Sec. III).

[6] A. Ranfagni and D. Mugnai (private communication).

[7] See Ref. [1(a)] above as well as A. Ranfagni and D. Mugnai, in *Lectures on Path Integration, Trieste 1991,* The Adriatico Research Conference, edited by H. Cerdeira *et al.* (World Scientific, Singapore, 1992).

[8] See, for example, Ref. [1(c)] and M. Büttiker, Phys. Rev. B **27**, 6178 (1983). It was this approach which precipitated the current interest in tunneling.

[9] (a) W. Jaworski and D. M. Wardlaw, Phys. Rev. A **37**, 2843 (1988); (b) E. H. Hauge, J. P. Falck, and T. A. Fjeldly, Phys. Rev. B **36**, 4203 (1987). For a comparison of these two related but different approaches see (c) W. Jaworski and D. M. Wardlaw, Phys. Rev. A **38**, 5404 (1988).

[10] E. Pollok, J. Chem. Phys. **83**, 1111 (1985).

[11] Note, however, that a and b cannot be the classical turning points as these are energy dependent and the current formalism is in the time domain.

[12] A derivation of transit time based on the density-matrix formalism is provided in A. Bohm, *Quantum Mechanics* (Springer-Verlag, New York, 1979), Sec. XVIII.2.

[13] The lack of a simple breakdown of mean transit time into transmitted and reflected components provides a central theme of Ref. [1(a)].

[14] The limit $C(\infty)$ exists, in general, because this is a scattering problem and the Hamiltonian therefore has a continuous spectrum.

[15] Equations (25) and (27) show that our tunneling-time distribution (or its mean) is represented as a double integral over energy. In particular, it is not a simple transmission-probability-weighted average over energy (i.e., a single energy integral). This is in contrast to the tunneling time defined in Ref. [9(a)] (see also [5(b)] above). This difference is due to our use of the forward-transit-time operator, which permitted the complete separation of tunneling and reflection times.

[16] L. D. Landau and Lifshitz, *Quantum Mechanics (Non-Relativistic Theory),* 3rd ed. (Pergamon, New York, 1977).

[17] The classical action, $s(E)$, which appears in Eq. (31) is defined as the branch with positive imaginary part. Otherwise, it contributes an unphysical exponentially large tunneling coefficient. The desired branch corresponds to a contour integral [see Eq. (32)] which follows a path passing below the left turning point, and above the right, in the complex plane.

[18] J. R. Taylor, *Scattering Theory* (Wiley, New York, 1972), Sec. 13-c.

[19] See, for example, M. D. Feit and J. A. Fleck, Jr., J. Chem. Phys. **78**, 301 (1983); **80**, 2578 (1984).

[20] This assumption is substantiated by computations not shown here. In any event, it is a reasonable assumption in the semiclassical limit where truncated expansion about the most probable tunneling time is sufficient to determine $P(t)$ in the time range where it is not negligibly small.

[21] K. Husimi, Proc. Phys. Math. Soc. Jpn. **22**, 264 (1940).

[22] T. L. Marchioro II and R. S. Dumont (unpublished).

[23] A. Ranfagni and D. Mugnai (private communication).

[24] T. L. Marchioro II, T. L. Beck, and R. S. Dumont (unpublished).