Algebraic scattering theory and the geometric phase

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A nonstandard realization of the su(1,1) algebra is used to extract a two-parameter class of scattering potentials as well as to calculate the reflection coefficient of the associated one-dimensional scattering problem in the spirit of the algebraic scattering theory. The nontrivial geometric content of such realizations is discussed, and an interesting connection with geometric phases is pointed out. It is argued that using larger noncompact groups, realizations related to non-Abelian geometric phases may be useful for obtaining analytical expressions for interaction terms corresponding to higher-dimensional scattering problems.

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I. INTRODUCTION

The connection between solutions of partial differential equations and the theory of Lie groups is a well-known and fruitful field of physical applications. In fact we have learned that many of the differential equations of mathematical physics are simply expressions of the Casimir invariant of some Lie group in a particular representation. Since the appearance of algebraic scattering theory (AST) [1–3], such techniques based on the aforementioned idea have proved to be useful not only in describing bound-state but also scattering problems. The main goal of AST is to determine, purely algebraically, the S matrix of a scattering system described by a Hamiltonian expressed in terms of the Casimir operator of some noncompact group. Although this task can indeed be achieved merely by algebraic manipulations, without any recourse to a particlar coordinate realization, some authors [4] stressed the physical relevance of obtaining analytical expressions for scattering potentials by using explicit coordinate realizations of the group-theoretical Hamiltonian.

The aim of the present paper is to show that there is a geometrically nontrivial realization of particular interest of the Lie algebra governing the scattering process. Using such a realization we can derive a class of solvable potentials containing some of the known examples as a special case. In this paper we explicitly work out the simplest case where the symmetry group in question is $SU(1,1)\sim SO(2,1)/Z_2$. Since we mostly use the local properties of these groups we can write $SU(1,1)\approx SO(2,1)$ (cf. Ref. [3]), hence we can refer to both of them interchangeably. This group with the Lie algebra $su(1,1)\sim so(2,1)$, being also the archetypical example of AST, can be used to obtain similar realizations for larger noncompact groups.

The SU(1,1) Casimir operator C in this new realization can be regarded as a generalization of the one in Ref. [2] in two respects. (i) The eigenvalue problem for C yields a Schrödinger equation for a particle in the *difference of two* one-dimensional generalized Pöschl-Teller potentials indexed by *two* parameters giving a generalization of the *one*-parameter class of Ref. [2]. (ii) C commutes with an *effective* Hamiltonian arising from a *total* Hamiltonian with SU(1,1) dynamical symmetry describing *two* interacting subsystems, usually designated by slow (collective) and fast (internal) degrees of freedom, after adiabatically decoupling the fast ones from the slow ones using the standard Born-Oppenheimer (BO) method. Although we can get rid in this treatment of the fast variables by "integrating them out" of the original Hamiltonian, and forming thus the *effective* Hamiltonian for the slow subsystem, terms of geometric origin will still modify the algebra of conserved quantities [5], giving rise to a nonstandard realization of the $su(1,1)\sim so(2,1)$ algebra. Such ideas are well known from studies dealing with geometric phases, a topic which has raised considerable interest during the past few years [6].

The organization of this paper is as follows. In Sec. II we present our realization of the $su(1,1)\sim so(2,1)$ algebra used subsequently. After calculating the Casimir invariant we obtain the Schrödinger equation mentioned above with a two-parameter class of potentials. We can easily trace back this equation to one of the familiar equations of mathematical physics. The S matrix is then derived, as usual, from the asymptotic form of the eigenfunctions.

In Sec. III the same S matrix is calculated, purely algebraically, in the framework of AST. However, in order to do so we have to properly identify the domain of the generators. Indeed they act on the group manifold SU(1,1) rather than on the double-sheeted hyperboloid. As a consequence we can identify two commuting sets of SU(1,1) generators corresponding to the right and left actions on SU(1,1) (or, equivalently, to the two parameters needed to parametrize the potentials in Sec. II).

In Sec. IV we demonstrate that we can construct an effective Hamiltonian that commutes with our Casimir invariant. This is done by employing a simple SU(1,1) invariant model Hamiltonian containing two types of dynamical variables, and using the Born-Oppenheimer approximation. The conclusions and some comments are left for Sec. V.

II. NONSTANDARD REALIZATION OF THE so(2,1) ~ su(1,1) ALGEBRA

Let us consider the following set of operators:

$$J_1=-i(X_2\partial_3+X_3\partial_2)-n\frac{X_1}{r+X_3}\ ,$$
$$J_2=+i(X_3\partial_1+X_1\partial_3)-n\frac{X_2}{r+X_3}\ , \tag{2.1}$$
$$J_3=-i(X_1\partial_2-X_2\partial_1)-n\ ,$$

where X_i $(i=1,2,3)$ are Cartesian coordinates in $\mathbb{R}^3$, $\partial_i\equiv\partial/\partial X_i$, $r^2=-X_1^2-X_2^2+X_3^2>0$, and n is an integer or half integer. (Actually we can remove this last restriction on n by allowing its value to be an arbitrary real number; we will return to this point after we have clarified the meaning of n in Sec. III.) The surface characterized by $r^2=1$ is the double-sheeted hyperboloid parametrized, e.g., by the polar coordinates (ρ,φ) as

$$X_1=r\sinh\rho\cos\varphi,\quad X_2=r\sinh\rho\sin\varphi\ ,$$
$$X_3=r\cosh\rho\ , \tag{2.2}$$

where $-\infty\le\rho\le\infty$, $0\le\varphi<2\pi$. The double-sheeted hyperboloid can also be considered as the coset space O(2,1)/SO(2). This representation, expressing the fact that our hyperboloid consists of two disconnected pieces characterized by $X_3<-1$ and $X_3>1$, will be used in Sec. IV.

It is straightforward to check that

$$[J_1,J_2]=-iJ_3,\quad [J_2,J_3]=iJ_1,\quad [J_3,J_1]=iJ_2\ , \tag{2.3}$$

i.e., the operators give a realization of the so(2,1) ~ su(1,1) algebra. Notice that this set of operators generalizes the well-known one obtained from Eq. (2.1) by setting $n=0$.

Introducing the usual operators $J_\pm\equiv J_1\pm iJ_2$ and using Eq. (2.2), after a similarity transformation by $\sinh^{1/2}\rho$, we get

$$J_\pm=e^{\pm i\varphi}\left[\mp\partial_\rho+\coth\rho(\pm\tfrac{1}{2}-i\partial_\varphi-n)+\frac{n}{\sinh\rho}\right]\ , \tag{2.4a}$$

$$J_3=-i\partial_\varphi-n\ . \tag{2.4b}$$

[Compare with Eq. (8.3) of Ref. [2].]

We define the Casimir invariant by

$$C=-J_1^2-J_2^2+J_3^2=J_3^2+J_3-J_-J_+\ , \tag{2.5}$$

and the usual basis states

$$C|j,m,n\rangle=j(j+1)|j,m,n\rangle\ , \tag{2.6a}$$
$$J_3|j,m,n\rangle=m|j,m,n\rangle\ . \tag{2.6b}$$

Using Eqs. (2.4a) and (2.4b), the Casimir invariant can be expressed as

$$C=\partial_\rho^2+\frac{\partial_\varphi^2+2ni(\cosh\rho-1)\partial_\varphi+2n^2(\cosh\rho-1)+\frac{1}{4}}{\sinh^2\rho}-\frac{1}{4}\ . \tag{2.7}$$

The basis states satisfying Eqs. (2.6) have the form

$$|j,m,n\rangle=u_{j,m,n}(\rho)e^{i(m+n)\varphi}\ , \tag{2.8}$$

yielding the equation for $u_{j,m,n}(\rho)$

$$\left[-\frac{d^2}{d\rho^2}+\frac{m^2+n^2+2mn\cosh\rho-\frac{1}{4}}{\sinh^2\rho}\right]u_{j,m,n}(\rho)=-\left[j+\tfrac{1}{2}\right]^2u_{j,m,n}(\rho)\ , \tag{2.9}$$

which generalizes Eq. (8.4) of Ref. [2]. Using a continuous series of unitary representations of SU(1,1) corresponding to scattering states characterized [2] by

$$j=-\tfrac{1}{2}+ik\ , \tag{2.10}$$

where $k>0$ is a real number, Eq. (2.9) can be rewritten as a Schrödinger equation

$$\left[-\frac{d^2}{d\rho^2}+V_{m,n}(\rho)\right]u_{k,m,n}(\rho)=k^2u_{k,m,n}(\rho) \tag{2.11}$$

with the scattering potential

$$V_{m,n}(\rho)=\frac{(n+m)^2-\frac{1}{4}}{4\sinh^2\frac{\rho}{2}}-\frac{(n-m)^2-\frac{1}{4}}{4\cosh^2\frac{\rho}{2}}\ . \tag{2.12}$$

Our next task is to solve Eq. (2.11) and determine the S matrix from the asymptotic form of the solution. After transforming Eq. (2.9) back by the similarity transformation $\sinh^{-1/2}\rho$ and employing the variable $z=\cosh\rho$ instead of ρ, we obtain an equation familiar in mathematical physics in connection with representations of SU(1,1) studied by Vilenkin [7]. The solutions are the functions $B_{mn}^{-1/2+k}(z)$ that can be expressed in terms of the hypergeometric function (see Ref. [7]).

However, in order to present the solution of Eq. (2.11), we choose a different route that is simple [8], but the group-theoretical content is not explicit. After a change of variable $y=-\sinh^2\rho/2$ we obtain an equation for $u_{k,m,n}(y)$ from Eq. (2.11). The search for the solution of this equation in the form [8]

$$u_{k,m,n}(y)=y^\alpha(1-y)^\beta F_{k,m,n}(y)\ , \tag{2.13}$$

where

$$2\alpha=m+n+\tfrac{1}{2}\ , \tag{2.14a}$$
$$2\beta=m-n+\tfrac{1}{2}\ , \tag{2.14b}$$

yields directly the hypergeometric differential equation for F. Choosing only the physically acceptable solution that is regular at the origin $y=0$ [8], the final result is

$$u_{k,m,n}(y)=y^\alpha(1-y)^\beta\times F(m+\tfrac{1}{2}+ik,m+\tfrac{1}{2}-ik,m+n+1;y)\ . \tag{2.15}$$

Exploiting the asymptotic behavior of F,

$$\lim_{|Z|\to\infty} F(A,B,C;Z)=\frac{\Gamma(C)\Gamma(B-A)}{\Gamma(B)\Gamma(C-A)}(-Z)^{-A}+\frac{\Gamma(C)\Gamma(A-B)}{\Gamma(A)\Gamma(C-B)}(-Z)^{-B},$$

we obtain the reflection coefficient as

$$R(k)=\frac{\Gamma(\frac{1}{2}+m-ik)\Gamma(\frac{1}{2}+n-ik)}{\Gamma(\frac{1}{2}+m+ik)\Gamma(\frac{1}{2}+n+ik)}\frac{\Gamma(2ik)}{\Gamma(-2ik)}e^{-4ik\ln 2}. \tag{2.16}$$

In closing this section we note that the class of potentials in Eq. (2.12) is precisely the PI class of exactly solvable potentials of Ref. [9] discussed in connection with one-dimensional bound-state problems.

III. ALGEBRAIC SCATTERING THEORY

In order to investigate the scattering process in the framework of algebraic scattering theory (AST), we have to characterize also algebraically the number n entering our Casimir operator Eq. (2.7). Indeed in Sec. II, n was merely a number which appeared in our nonstandard realization of the su(1,1) algebra. This number is present in the characterization of the su(1,1) basis states $|j,m,n\rangle$ too. However, in AST, the asymptotic states $|j,m,n\rangle^\infty\equiv\lim_{\rho\to\infty}|j,m,n\rangle$ are expanded in terms of the incoming $|-k,m,n\rangle^\infty$ and outgoing $|+k,m,n\rangle^\infty$ waves [algebraically characterized [2] by the Euclidean algebra $E(2)$] in the form

$$|j,m,n\rangle^\infty=A_{m,n}|-k,m,n\rangle^\infty+B_{m,n}|+k,m,n\rangle^\infty. \tag{3.1}$$

For the calculation of the reflection coefficient $R_{m,n}=B_{m,n}/A_{m,n}$ by employing the concept of the Euclidean connection [2], we have to establish recursion relations between the functions $A_{m,n}(k)$ and $A_{m+1,n}(k)$, $A_{m,n+1}(k)$, etc. But so far we have not gotten any step operator raising or lowering the value of n in $|j,m,n\rangle^\infty$ and $|\pm k,m,n\rangle^\infty$. We can remedy this deficiency by introducing a new variable ψ and the associated differential operator defined by

$$K_3\equiv i\frac{\partial}{\partial\psi}, \tag{3.2}$$

and modifying Eq. (2.8) according to

$$|j,m,n\rangle=u_{j,m,n}(\rho)e^{i(m\varphi+n\psi)}. \tag{3.3}$$

The operators of Eq. (2.4) now have the form

$$J_\pm=e^{\pm i\varphi}\left[\mp\partial_\rho+\coth\rho(\pm\tfrac{1}{2}-i\partial_\varphi)-\frac{i}{\sinh\rho}\partial_\psi\right], \tag{3.4a}$$

$$J_3=-i\partial_\varphi. \tag{3.4b}$$

We next observe that the set of operators

$$K_\pm=e^{\mp i\psi}\left[\pm\partial_\rho+\coth\rho(\mp\tfrac{1}{2}-i\partial_\psi)-\frac{i}{\sinh\rho}\partial_\varphi\right], \tag{3.5a}$$

$$K_3=i\partial_\psi \tag{3.5b}$$

commutes with the set of operators in Eqs (3.4) and also satisfies the commutation relations of the su(1,1) algebra. Moreover, we have

$$C=J_3^2+J_3-J_-J_+=K_3^2+K_3-K_-K_+. \tag{3.6}$$

Using $-(C+\frac{1}{4})$ as our Hamiltonian acting on the states of Eq. (3.3), satisfying Eqs. (2.6), and taking into account Eq. (2.10), one obtained Eq. (2.11), which is our Schrödinger equation.

The set of operators defined by Eqs. (3.4) and (3.5) represents the left and right actions of the group SU(1,1) on itself [7]. The net result is that in order to describe the scattering process characterized by the Hamiltonian $-(C+\frac{1}{4})$ algebraically, we have to enlarge our group SU(1,1) to SU(1,1)⊗SU(1,1). As a consequence the suitable asymptotic description of the states $|\pm k,m,n\rangle^\infty$ is provided by the algebra $E(2)\otimes E(2)$. Note however that such an extension of our group is needed merely to tackle *one*-dimensional scattering with a *two*-parameter class of potentials and not in order to investigate scattering in *two* dimensions as in Ref. [3]. Actually, two operators (J_3,K_3) are needed to label the strength of the interaction.

It is straightforward to check that the operators

$$\mathcal{P}_\pm=-ie^{\pm i\varphi}\left[\partial_\rho\mp\frac{1}{\rho}(\pm\tfrac{1}{2}-i\partial_\varphi)\right], \tag{3.7a}$$

$$\mathcal{L}=-i\partial_\varphi \tag{3.7b}$$

(with $J_3=\mathcal{L}$), and the ones

$$\mathcal{Q}_\pm=-ie^{\mp i\psi}\left[\partial_\rho\pm\frac{1}{\rho}(\mp\tfrac{1}{2}-i\partial_\psi)\right], \tag{3.8a}$$

$$\mathcal{M}=i\partial_\psi \tag{3.8b}$$

(with $K_3=\mathcal{M}$) satisfy the commutation relations of the Euclidean algebra [2], and the two sets of operators commute with each other. Taking the limit $\rho\to\infty$ in Eqs. (3.4) and (3.5) and Eqs. (3.7) and (3.8), we can construct the asymptotic algebras of SU(1,1)⊗SU(1,1) and $E(2)\otimes E(2)$, respectively, with the corresponding eigenstates $|j,m,n\rangle^\infty$ and $|\pm k,m,n\rangle^\infty$, related to each other by Eq. (3.1). We can easily prove that

$$|j,n,n\rangle^\infty=A_{m,n}(k)e^{-ik\rho}e^{i(m\varphi+n\psi)}+B_{m,n}(k)e^{ik\rho}e^{i(m\varphi+n\psi)}. \tag{3.9}$$

The action of the operators needed to establish the necessary recursion relations are the following:

$$\mathcal{P}_\mp^\infty|\pm k,m,n\rangle^\infty=\pm k|\pm k,m+1,n\rangle^\infty, \tag{3.10a}$$

$$\mathcal{Q}_\mp^\infty|\pm k,m,n\rangle^\infty=\pm k|\pm k,m,n+1\rangle^\infty, \tag{3.10b}$$

$$J_\mp^\infty|\pm k,m,n\rangle^\infty=(\tfrac{1}{2}+m\mp ik)|\pm k,m+1,n\rangle^\infty, \tag{3.10c}$$

$$K_\mp^\infty|\pm k,m,n\rangle^\infty=(\tfrac{1}{2}+n\mp ik)|\pm k,m,n+1\rangle^\infty. \tag{3.10d}$$

Hence the connection formulas are as follows:

$$J_{\mp}^{\infty}=\pm\frac{1}{k}[(-\tfrac{1}{2}\mp ik)\mathcal{P}_{\mp}^{\infty}+\mathcal{L}^{\infty}\mathcal{P}_{\mp}^{\infty}]\ , \tag{3.11a}$$

$$K_{-}^{\infty}=\pm\frac{1}{k}[(-\tfrac{1}{2}\mp ik)\mathcal{Q}_{-}^{\infty}-\mathcal{M}^{\infty}\mathcal{Q}_{-}^{\infty}]\ , \tag{3.11b}$$

where $\pm$ corresponds to choosing the basis set $|\pm k,m,n\rangle^{\infty}$. Since the operators K_{-}^{∞} and $J_{\mp}^{\infty}$ commute, we can establish recursion relations by acting with $J_{\mp}^{\infty}K_{-}^{\infty}$ on the left and using Eqs. (3.10c) and (3.10d) on the right-hand side of Eq. (3.9). Reproducing the steps presented in Ref. [2] yields

$$R_{m+1,n+1}=\frac{(\frac{1}{2}+m-ik)(\frac{1}{2}+n-ik)}{(\frac{1}{2}+m+ik)(\frac{1}{2}+n+ik)}R_{m,n}\ . \tag{3.12}$$

Moreover, in the spirit of Ref. [2] we can allow m and n to be any real numbers corresponding to the projective representations of SU(1,1). The final result is

$$R(k)=\frac{\Gamma(\frac{1}{2}+m-ik)\Gamma(\frac{1}{2}+n-ik)}{\Gamma(\frac{1}{2}+m+ik)\Gamma(\frac{1}{2}+n+ik)}\Delta(k)\ , \tag{3.13}$$

where $\Delta(k)$ is an entire function of k [compare with Eq. (2.16)].

IV. NONSTANDARD REALIZATIONS AND THE GEOMETRIC PHASE

In this chapter we employ a simple model with SU(1,1) symmetry realized in a nonstandard way, giving rise to the appearance of the Casimir invariant [see Eq. (2.7)], our main concern in this paper. The idea underlying such a construction is based on recent reformulations [10,11] of the molecular Born-Oppenheimer approximation exhibiting an explicit U(1) gauge invariance due to the presence of nonintegrable phases. Such phases are just Berry's celebrated phase factors [12] reflecting the nontrivial topological properties of the configuration space of the slow (e.g., nuclear) degrees of freedom.

As our starting point we define an SO(2,1) invariant model Hamiltonian containing two types of dynamical variables $(\mathbf{P},\mathbf{X})$ and (p,q) called the slow (collective) and fast (internal) variables, respectively. The simplest choice for such a Hamiltonian H_{tot} is the following ($\hbar=1$):

$$H_{\text{tot}}(\mathbf{P},\mathbf{X};p,q)=\frac{1}{2M}\mathbf{P}\cdot\mathbf{P}+H(\mathbf{X};p,q)\ . \tag{4.1}$$

Here $\mathbf{X}$ and $\mathbf{P}$ are Cartesian coordinates confined to the double-sheeted hyperboloid and the conjugated momenta, respectively. The dot $(\cdot)$ refers to the scalar product $\mathbf{a}\cdot\mathbf{b}\equiv -a_1b_1-a_2b_2+a_3b_3$. In order to ensure the validity of the BO approximation we chose the parameter M to be sufficiently large. Note that Eq. (4.1) with $H(\mathbf{X};p,q)=0$ and $r^2=1$ corresponds to motion on the double-sheeted hyperboloid, which can also be represented as the coset space O(2,1)/SO(2).

In order to define $H(\mathbf{X};p,q)$, we form the following set of quadratic combinations from p and q:

$$S_1=\tfrac{1}{4}(p^2-q^2)\ , \tag{4.2a}$$

$$S_2=-\tfrac{1}{4}(qp+pq)\ , \tag{4.2b}$$

$$S_3=-\tfrac{1}{4}(p^2+q^2)\ , \tag{4.2c}$$

satisfying the commutation relations, Eqs. (2.3), of the su(1,1)$\sim$so(2,1) algebra ($[q,p]=i,\hbar=1$). Now we define

$$H(\mathbf{X};p,q)\equiv\mathbf{S}\cdot\mathbf{X}\ . \tag{4.3}$$

Note that by defining the quantities $2x\equiv X_1-X_3$, $2y\equiv X_2$, $2z\equiv -X_1-X_3$, Eq. (4.3) takes the following form:

$$H(x,y,z;p,q)=\tfrac{1}{2}[xq^2+y(qp+pq)+zp^2]\ , \tag{4.4}$$

which is the generalized harmonic oscillator, a well-known example of studies concerning the geometric phase [13–15]. Equation (4.3) can be regarded as a hyperbolic generalization of the Hamiltonian for a spin $(\mathbf{S})$ in a magnetic field $(\mathbf{X})$ with the important difference that now $\mathbf{X}$ also plays the role of a dynamic variable, not merely an external parameter.

Let us denote the set of operators in Eq. (2.1) for $n=0$ by $\mathbf{L}\equiv(L_1,L_2,L_3)$. Then one can verify the commutation rules

$$[\mathbf{L}+\mathbf{S},H_{\text{tot}}]=\mathbf{0}\ . \tag{4.5}$$

In fact $\mathbf{L}$ and $\mathbf{S}$ generate infinitesimal SO(2,1) rotations of the slow and fast subsystem, respectively.

We next adiabatically decouple the slow degrees of freedom from the fast ones using the BO approximation [10,11,16]. We write the usual BO expansion of the total wave function $\Psi(\mathbf{X},q)$, satisfying

$$(H_{\text{tot}}-E)\Psi=0 \tag{4.6}$$

in the form

$$\Psi(\mathbf{X},q)=\sum_{s=0}^{\infty}\chi_s(\mathbf{X})\psi_s(\mathbf{X},q)\ , \tag{4.7}$$

where $\psi_s(\mathbf{X},q)$ are the solutions of the equation

$$H(\mathbf{X})\psi_s(\mathbf{X},q)=E_s(\mathbf{X})\psi_s(\mathbf{X},q) \tag{4.8}$$

for every fixed value of $\mathbf{X}$ [with $H(\mathbf{X})$ having the form of Eq. (4.3)]. After adopting the convenient notation

$$|s(\mathbf{X})\rangle\equiv\psi_s(\mathbf{X},q)\ , \tag{4.9}$$

we multiply Eq. (4.6) from the left with $\langle t(\mathbf{X})|$, meaning integration with respect to q. Using Eq. (4.7) we can write down the projected form of the Schrödinger equation (4.6) as

$$\sum_{s=0}^{\infty}\left[\frac{1}{2M}(\mathbf{P}-\mathcal{A})\cdot(\mathbf{P}-\mathcal{A})+E_s(\mathbf{X})\cdot\mathbf{I}\right]_{ts}\chi_s(\mathbf{X})=E\chi_t(\mathbf{X})\ , \tag{4.10}$$

where $(\mathcal{A})_{ts}\equiv i\langle t(\mathbf{X})|\nabla|s(\mathbf{X})\rangle$, $\mathbf{P}=-i\nabla$, $\nabla\equiv(\partial/\partial X,\partial/\partial X_2,\partial/\partial X_3)$. In the following we assume that the motion described by the variables $\mathbf{X}$ is sufficiently slow [16], hence they will not induce transitions between the adjacent levels $E_t(\mathbf{X})$ and $E_s(\mathbf{X})$ $(t\neq s)$. Consequently, we can restrict our attention to the (one-dimensional) subspaces with eigenvalue $E_s(\mathbf{X})$. In other words, in this approximation we have to deal merely with the diagonal terms of the infinite dimensional matrix-valued operator appearing in the square brackets of Eq. (4.10). Isolating one eigensubspace with energy $E_s(\mathbf{X})$ for study, we obtain the *effective Hamiltonian* for this level [10,11,16] as

$$H^{(s)}_{\text{eff}}=-\frac{1}{2M}[\nabla-i\,\mathbf{A}^{(s)}(\mathbf{X})]\cdot[\nabla-i\,\mathbf{A}^{(s)}(\mathbf{X})]+\mathcal{U}^{(s)}(\mathbf{X})+\mathcal{V}^{(s)}(\mathbf{X})\ , \tag{4.11}$$

to be used in the equation

$$H^{(s)}_{\text{eff}}\chi_s(\mathbf{X})=E\chi_s(\mathbf{X})\ , \tag{4.12}$$

where

$$\mathcal{V}^{(s)}(\mathbf{X})\equiv E_s(\mathbf{X})\ , \tag{4.13a}$$

$$\mathbf{A}^{(s)}(\mathbf{X})\equiv i\langle s(\mathbf{X})|\nabla|s(\mathbf{X})\rangle\ , \tag{4.13b}$$

$$\mathcal{U}^{(s)}(\mathbf{X})\equiv\frac{1}{2M}\sum_{t(\neq s)}\langle\nabla s(\mathbf{X})|t(\mathbf{X})\rangle\cdot\langle t(\mathbf{X})|\nabla s(\mathbf{X})\rangle\ . \tag{4.13c}$$

Notice that Eq. (4.12) and the BO expansion Eq. (4.7) are invariant with respect to the local U(1) gauge transformation of the functions

$$|s(\mathbf{X})\rangle\rightarrow e^{i\alpha(\mathbf{X})}|s(\mathbf{X})\rangle\ , \tag{4.14a}$$

$$\chi_s(\mathbf{X})\rightarrow e^{-i\alpha(\mathbf{X})}\chi_s(\mathbf{X})\ , \tag{4.14b}$$

due to the presence of the *vector potential-like* term or U(1) gauge potential in Eq. (4.11).

Now we calculate the effective Hamiltonian for our model defined by Eqs. (4.1)–(4.3). We shall show that it can be expressed in terms of our Casimir invariant [see Eq. (2.7)], hence

$$[H^{(s)}_{\text{eff}},\mathbf{J}]=0 \tag{4.15}$$

with $\mathbf{J}\equiv(J_1,J_2,J_3)$ defined in Eq. (2.1), where the number n is fixed by the quantum numbers of the fast subsystem. As a first step we exploit the SO(2,1) symmetry of the Hamiltonian of Eq. (4.3) in order to solve its eigenvalue problem yielding the eigenvectors $|s(\mathbf{X})\rangle$ needed in Eqs. (4.13a)–(4.13c). [Although we are interested in the motion of the slow subsystem coupled to the fast one on the double-sheeted hyperboloid ($r^2=-X_1^2-X_2^2+X_3^2=1$), it is more illustrative to restore the dependence on r by demanding that $r^2>0$. In this way we obtain a deeper insight into the meaning of the terms in Eqs. (4.13a)–(4.13c). Of course we restrict our attention in the final result to the case with $r^2=1$.] Let us define the unitary operator

$$U(\mathbf{X})=U(\rho,\varphi)\equiv e^{-i\rho(\sin\varphi S_1-\cos\varphi S_2)}\ , \tag{4.16}$$

with S_1 and S_2 defined by Eqs. (4.2a) and (4.2b). Then one can show that

$$H(\mathbf{X})=\mathbf{S}\cdot\mathbf{X}=U(\rho,\varphi)\mathbf{S}\cdot\mathbf{X}_0U^{-1}(\rho,\varphi)\equiv U(\rho,\varphi)rS_3U^{-1}(\rho,\varphi)\ , \tag{4.17}$$

where $\mathbf{X}_0\equiv(0,0,r)$. The eigenfunctions of $H(\mathbf{X})$ are

$$|s(\mathbf{X})\rangle=U(\mathbf{X})|s(\mathbf{X}_0)\rangle\ , \tag{4.18}$$

$$S_3|s(\mathbf{X}_0)\rangle=-\tfrac{1}{2}(s+\tfrac{1}{2})|s(\mathbf{X}_0)\rangle\ , \tag{4.19}$$

where Eq. (4.19) follows from the fact that S_3 is just $-\frac{1}{2}$ times the usual harmonic-oscillator Hamiltonian with eigenvectors indexed by $s=0,1,2,\ldots$, a quantum number used also in Eq. (4.7). The first consequence of Eqs. (4.17)–(4.19) is the formula

$$\mathcal{V}^{(s)}(r)=-\tfrac{1}{2}r(s+\tfrac{1}{2})\ . \tag{4.20}$$

[Choosing the *negative* sign of the square root, i.e., $r=-(-X_1^2-X_2^2+X_3^2)^{1/2}$ ensures the positivity of the energy in Eq. (4.8). Hence the set of parameters $\mathbf{X}$ for r=const parametrizes the points on *one sheet* of the *two-sheeted* hyperboloid [15].]

Having determined the instantaneous eigenvectors, we are able to calculate the vector potential in Eq. (4.13b). First notice that since the matrix $\mathrm{U}^{-1}(\mathbf{X})\nabla\mathrm{U}(\mathbf{X})$ is anti-Hermitian and $\mathrm{U}\in\mathrm{SU}(1,1)$, $\mathrm{U}^{-1}\nabla\mathrm{U}$ is an element of the Lie algebra su(1,1). Consequently the coefficients ω_i^k of the one-form

$$\omega\equiv iU^{-1}dU\equiv iU^{-1}\nabla U d\mathbf{X}=\omega_i^kS_kdX^i,\quad i,k=1,2,3\ , \tag{4.21}$$

being the so-called Maurer-Cartan form [15] on the group manifold, are independent of the particular choice of representation used for the su(1,1) algebra. We can thus employ the simplest representation $(S_1,S_2,S_3)\equiv(\sigma_1/2i,\sigma_2/2i,\sigma_3/2)$ to calculate ω_i^k, yielding

$$A^{(s)}\equiv i\langle s(\mathbf{X}_0)|U^{-1}(\mathbf{X})dU(\mathbf{X})|s(\mathbf{X}_0)\rangle=\tfrac{1}{2}(s+\tfrac{1}{2})(1-\cosh\rho)d\varphi\ , \tag{4.22}$$

or in Cartesian components,

$$\mathbf{A}^{(s)}(\mathbf{X})=\frac{(s+\frac{1}{2})}{2r(r+X_3)}\begin{bmatrix}+X_2\\-X_1\\0\end{bmatrix}\ . \tag{4.23}$$

For the calculation of $\mathcal{U}^{(s)}(\mathbf{X})$ in Eq. (4.13c), we use the formula

$$\mathcal{U}^{(s)}(\mathbf{X})=\frac{1}{2M}\sum_{t(\neq s)}\frac{\langle s(\mathbf{X})|\nabla H(\mathbf{X})|t(\mathbf{X})\rangle\cdot\langle t(\mathbf{X})|\nabla H(\mathbf{X})|s(\mathbf{X})\rangle}{[E_t(\mathbf{X})-E_s(\mathbf{X})]^2}\ , \tag{4.24}$$

obtained after differentiating Eq. (4.8) and using the result in Eq. (4.13c). It can be proved [17] that it is sufficient to perform the calculation only at one point of the hyperboloid as a consequence of the SO(2,1) group action [15] on one sheet of the double-sheeted hyperboloid [i.e., the coset SO(2,1)/SO(2)$\sim$SU(1,1)/U(1)]. Choosing this point to be $\mathbf{X}_0$, we get

$$\mathcal{U}^{(s)}(r)=\frac{1}{2Mr^2}[\langle s(\mathbf{X}_0)|S^2|s(\mathbf{X}_0)\rangle-\langle s(\mathbf{X}_0)|\mathbf{S}|s(\mathbf{X}_0)\rangle\cdot\langle s(\mathbf{X}_0)|\mathbf{S}|s(\mathbf{X}_0)\rangle]\ . \tag{4.25}$$

Let us now represent our operators **S** in terms of creation ($a^\dagger$) and annihilation (a) operators ($[a,a^\dagger]=1$) as follows:

$$S_1=-\tfrac{1}{4}(aa+a^\dagger a^\dagger)\ , \qquad (4.26a)$$

$$S_2=+\frac{i}{4}(aa-a^\dagger a^\dagger)\ , \qquad (4.26b)$$

$$S_3=-\tfrac{1}{2}(a^\dagger a+\tfrac{1}{2})\ , \qquad (4.26c)$$

with basis states $|s\rangle\equiv|s(\mathbf{X}_0)\rangle$ ($a^\dagger a|s\rangle=s|s\rangle$, $a^\dagger|s\rangle=\sqrt{s+1}|s+1\rangle$, $a|s\rangle=\sqrt{s}\,|s-1\rangle$). After some algebra, one obtains

$$S^2=-S_1^2-S_2^2+S_3^2=-\frac{3}{16}\mathcal{I}\ . \qquad (4.27)$$

Using Eqs. (4.19) and (4.25), the final result is

$$\mathcal{U}^{(s)}(r)=-\frac{1}{8Mr^2}[s(s+1)+1]\ . \qquad (4.28)$$

As a next step we try to transform the first term in Eq. (4.11) into a more familiar form. Using Eq. (4.23), we have

$$r^2(\nabla-i\,\mathbf{A}^{(s)})\cdot(\nabla-i\,\mathbf{A}^{(s)})=r^2\nabla\cdot\nabla-2in\frac{1}{1+\cosh\rho}\partial_\varphi +n^2\frac{1-\cosh\rho}{1+\cosh\rho}\ , \qquad (4.29)$$

where

$$n=\tfrac{1}{2}(s+\tfrac{1}{2}),\quad s=0,1,2,\ldots \qquad (4.30)$$

Using the expression similar to the usual Laplacian

$$-\partial_1^2-\partial_2^2+\partial_3^2=\partial_r^2+\frac{2}{r}\partial_r-\frac{1}{r^2}L^2 =\frac{1}{r^2}\partial_r(r^2\partial_r)-\frac{1}{r^2}\frac{1}{\sinh\rho}\partial_\rho(\sinh\rho\partial_\rho) -\frac{1}{\sinh^2\rho}\partial_\varphi^2\ , \qquad (4.31)$$

with $\mathbf{L}=(L_1,L_2,L_3)$ defined in connection with Eq. (4.5) being the usual SO(2,1) "angular momentum" operators, we get finally

$$(\nabla-i\,\mathbf{A}^{(n)})^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\right]-\frac{1}{r^2}(J^2-n^2)\ , \qquad (4.32)$$

where J^2 is the Casimir invariant C corresponding to the set of operators J_i in Eq. (2.1), which has been our starting point. Moreover, within the framework of this model, we can interpret the number n appearing in the explicit form of the operators J_i as a quantum number of the *slow* system inherited from the *fast* ones. The allowed values of n are depending on the *particular representation* of the so(2,1)∼su(1,1) generators used to represent the symmetry transformations for the *fast* subsystem (**S**). In our case $n=\frac{1}{4},\frac{3}{4},\frac{5}{4},\frac{7}{4},\ldots$.

Inserting the results of Eqs. (4.20), (4.28), and (4.32) into Eq. (4.11) and employing the usual similarity transformation of $\sinh^{1/2}\rho$, we can represent the effective Hamiltonian in the form

$$H_{\rm eff}^{(s)}=\frac{1}{2M}\{C-\tfrac{1}{2}[s(s+1)+\tfrac{5}{8}]\}+\tfrac{1}{2}(s+\tfrac{1}{2})\ , \qquad (4.33)$$

where we have restricted our attention to the motion on one sheet of the hyperboloid $r^2=1$. Since the Casimir invariant defined by Eq. (2.7) is accompanied in Eq. (4.33) with terms multiplied by the identity operator, the relation expressed by Eq. (4.15) is established.

Notice also that the presence of C in Eq. (4.33) can be traced back to the appearance of the vector potential-like term [see Eq. (4.13b)] in Eq. (4.11), which is needed to restore the U(1) gauge invariance. Hence the Schrödinger equation (2.11) can be represented in the form [use Eq. (4.32) with $r^2=1$]

$$\{[-(\nabla-i\,\mathbf{A}^{(n)})^2+n^2]+\tfrac{1}{4}\}u_{k,m,n}(\rho)e^{im\varphi} =k^2u_{k,m,n}(\rho)e^{im\varphi},r^2=1 \qquad (4.34)$$

where the value of n is fixed by Eq. (4.30). For $n=0$ we obtain in Eq. (4.34) instead of the "covariant derivative" the ordinary one which, together with the constraint $r^2=1$, yields the usual Hamiltonian of Eq. (8.4) of Ref. [2].

V. CONCLUSIONS

In this paper we have shown how a nonstandard realization of the noncompact group SU(1,1) can be used to obtain a more general class of scattering potentials than the one known from Ref. [2]. The geometrical ideas underlying such a construction are embodied in Eq. (4.34). According to the algebraic scattering theory, the Hamiltonian governing the scattering process is constructed from the Casimir invariant C of SU(1,1) as $H=-(C+\frac{1}{4})$ or more generally [2,3] from an arbitrary function h of C as $H=h[-(C+\frac{1}{4})]$. In our case we managed to represent C as the square of a covariant derivative plus a constant. Our realization was achieved by the trick of representing the SU(1,1) symmetry on *two* types of dynamical variables corresponding to two subsystems with wildly different energy scales, making it possible to apply the Born-Oppenheimer approximation. Since the two types of variables make their presence in the appearance of two sets of quantum numbers (n,m), the reflection coefficient Eq. (2.16) exhibits explicit dependence on these numbers. This recognition can support the idea that such realizations, by making an imprint on the concrete form of the S matrix, may shed some light not only on the symmetry properties of the scattering process but also on the dynamics of the interacting subsystems involved.

The representation of the SU(1,1) Casimir invariant in a form which is manifestly U(1) gauge invariant (due to the presence of the covariant derivative) emphasizes the obvious connection with geometrical phases [see Eq. (4.14)]. Hence we expect that our generators in Eq. (2.1) can also be expressed in terms of covariant derivatives. This form is easily shown to be [use $n=(s+\frac{1}{2})/2$ in Eq. (4.23)]

$$J_1=-i[X_2(\partial_3-iA_3^{(n)})+X_3(\partial_2-iA_2^{(n)})]-n\frac{X_1}{r} ,$$
$$J_2=+i[X_3(\partial_1-iA_1^{(n)})+X_1(\partial_3-iA_3^{(n)})]-n\frac{X_2}{r} , \quad (5.1)$$
$$J_3=-i[X_1(\partial_2-iA_2^{(n)})-X_2(\partial_1-iA_1^{(n)})]-n\frac{X_3}{r} .$$

Notice that these generators are obtained not only by merely replacing the ordinary derivatives by the covariant one in **L**, but also by forming the mean value of the operator **S**, i.e.,

$$-\tfrac{1}{2}(s+\tfrac{1}{2})\frac{\mathbf{X}}{r}=\langle s(\mathbf{X})|\mathbf{S}|s(\mathbf{X})\rangle . \quad (5.2)$$

Heuristically we may argue that the set of operators **J** satisfying Eq. (4.15) can be obtained from the operator **L**+**S** of Eq. (4.5) by using covariant derivatives in **L** and taking the "adiabatic average" of **S** as shown by Eq. (5.2). In principle, the calculation of the modified set of generators J_i for other groups when nonintegrable phases or its non-Abelian generalizations [6] are present can be carried out by using the result of Ref. [5]. We remark that the set of operators of Eq. (5.1) is the hyperbolic generalization of the well-known set with SU(2) symmetry describing the modification of the angular momentum algebra when magnetic monopoles are present [5,6].

It is now obvious that the heart of our construction is the nonintegrable U(1) geometric phase. As a by-product we can explain why we have chosen the double-sheeted hyperboloid instead of the single-sheeted one with $r^2=X_1^2+X_2^2-X_3^2$, on which our operators **J** act. First recall that due to the symmetry properties of $H(\mathbf{X})$, the unitary operator needed to diagonalize it can be regarded as an element of SU(1,1). Since any eigenvector of $H(\mathbf{X})$ can be obtained from a fixed one by applying an SU(1,1) transformation according to Eq. (4.18), we expect that the set of eigenvectors can be indexed by the parameters of that particular SU(1,1) element. Of course this is not the case. Two vectors $|s(\mathbf{X})\rangle$ and $|s'(\mathbf{X})\rangle$ related by the *local* U(1) gauge transformation of Eq. (4.14a) are eigenvectors of $H(\mathbf{X})$ corresponding to the *same* eigenvalue. Hence the set of eigenvectors can be characterized by points of the coset SU(1,1)/U(1)$\sim$SO(2,1)/SO(2), which is one sheet of the *double-sheeted* hyperboloid parametrized by the pair (ρ,φ) in accordance with Eq. (4.16). For the single-sheeted hyperboloid such reasoning cannot be applied because in this case the relevant coset space is SU(1,1)/$\mathbb{R}\sim$SO(2,1)/SO(1,1), i.e., the subgroup is noncompact so that it cannot be interpreted as a geometric phase.

Observe also that the allowed values of n in Eq. (4.34) are depending on the particular representation of the su(1,1)$\sim$so(2,1) generators **S**, coupled to the usual ones **L** via the BO approximation. Choose for instance

$$S_1=-\frac{i}{2}(a^\dagger b^\dagger-ab) , \quad (5.3a)$$
$$S_2=-\tfrac{1}{2}(a^\dagger b^\dagger+ab) , \quad (5.3b)$$
$$S_3=+\tfrac{1}{2}(a^\dagger a+b^\dagger b+1) , \quad (5.3c)$$

acting on the states of the form

$$|s_a,s_b\rangle=(s_a!s_b!)^{-1/2}(a^\dagger)^{s_a}(b^\dagger)^{s_b}|0,0\rangle , \quad (5.4)$$

with

$$s_a-s_b\equiv s_0=\text{const} , \quad (5.5)$$

and the Casimir invariant

$$S^2=\tfrac{1}{4}(a^\dagger a-b^\dagger b)^2-\tfrac{1}{4}=\tfrac{1}{4}(s_0^2-1) . \quad (5.6)$$

It can be shown [18] that the states in Eq. (5.4) with the constraint given by Eq. (5.5) represent a basis for the discrete series unitary irreducible representation of SU(1,1) labeled by the number $|s_0|$. Hence the allowed values for n are

$$n=s_b+\tfrac{1}{2}(s_0+1) . \quad (5.7)$$

This "coupling of representations" can be used to gain some information from the range of the parameter n in the formula for the reflexion coefficient Eq. (2.16) on the concrete form of SU(1,1) representations characterizing the dynamics of the internal degrees of freedom.

The generalization of the present nonstandard realization to larger noncompact groups, e.g., for the group SO(3,2) used in Ref. [4], might deserve some attention in physical applications. It may be used, for example, to investigate models related to the modified Coulomb problems, which, according to Ref. [3], can be useful for the study of heavy-ion reactions. Such generalized realizations we shall present in future works.

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