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Generalized multimode squeezed states

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The notion of squeezing is generalized to the case of multimodes. The multimode generalized squeezing operator has similar algebraic properties to those of the single-mode case and reduces to the usual two-mode squeezing operator in the case of two modes. It is also shown that the generalized multimode squeezed state is a multimode minimum-uncertainty state if the squeezing parameters are real.

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In recent years squeezed states of the electromagnetic field have been widely studied, both theoretically and experimentally, because of their prospective applications in technology [1,2]. These are pure quantum-mechanical states of light that have reduced fluctuations in one field quadrature, when compared with coherent states. So far, most studies have been concentrated on single-mode and two-mode squeezing. Multimode squeezing, however, has not been studied as much. In this Brief Report we shall be interested mostly in the generalization of single-mode and two-mode squeezing to the multimode case. The generalized multimode squeezed states have been proved to be important in problems on electronic systems coupled to boson fields [3–6].

Single-mode squeezed states $|\beta,\alpha\rangle$ can be generated by means of a unitary squeezing operator defined by [7]

$$S(\beta) = \exp\{\tfrac{1}{2}(\beta a^{\dagger 2} - \beta^* a^2)\}, \tag{1}$$

from the vacuum state $|0\rangle$ as follows:

$$|\beta,\alpha\rangle = D(\alpha)S(\beta)\,|0\rangle, \tag{2}$$

where

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \tag{3}$$

is the unitary Weyl displacement operator, and a and $a^\dagger$ are the usual single-mode boson annihilation and creation operators, respectively. In configuration space the squeezed states correspond to Gaussian wave packets with widths distorted from that of the vacuum state. Also, the squeezed states are the eigenstates of the so-called squeezed annihilation operator

$$b = S^\dagger(\beta)\,a\,S(\beta) = \cosh(|\beta|)\,a + \frac{\beta}{|\beta|}\sinh(|\beta|)\,a^\dagger, \tag{4}$$

that is,

$$b\,|\beta,\alpha\rangle = \alpha\,|\beta,\alpha\rangle. \tag{5}$$

In the two-mode case a two-mode squeezed state is defined by [8,9]

$$|\beta,\alpha_1,\alpha_2\rangle = D_1(\alpha_1)D_2(\alpha_2)S_{12}(\beta)\,|0\rangle, \tag{6}$$

where $D_i(\alpha_i)$ is the displacement operator for the ith mode ($i=1,2$),

$$S_{12}(\beta) = \exp(\beta a_1^\dagger a_2^\dagger - \beta^* a_1 a_2) \tag{7}$$

is the unitary two-mode squeezing operator, and $|0\rangle$ is the two-mode vacuum state. This two-mode squeezed state is a highly correlated state of the two field modes that exhibits reduced quadrature noise in linear combinations of variables of both modes; however, squeezing is not observed in the fluctuations of individual modes. The two-mode squeezing operator transforms the annihilation operators according to

$$S_{12}^\dagger(\beta)\,a_1\,S_{12}(\beta) = \cosh(|\beta|)\,a_1 + \frac{\beta}{|\beta|}\sinh(|\beta|)\,a_2^\dagger, \tag{8}$$

$$S_{12}^\dagger(\beta)\,a_2\,S_{12}(\beta) = \cosh(|\beta|)\,a_2 + \frac{\beta}{|\beta|}\sinh(|\beta|)\,a_1^\dagger. \tag{9}$$

Thus, a typical process leading to two-mode squeezing is a process during which a mode is mixed with the conjugated field of another mode according to a Bogoliubov transformation. It should be noted that a two-mode squeezed state is not simply a direct product of two single-mode squeezed states $|\beta_1,\alpha_1\rangle\,|\beta_2,\alpha_2\rangle$, where

$$|\beta_i,\alpha_i\rangle = D_i(\alpha_i)S_i(\beta_i)\,|0\rangle; \tag{10}$$

in other words, the two-mode squeezing operator $S_{12}(\beta) \equiv \exp(\beta a_1^\dagger a_2^\dagger - \beta^* a_1 a_2)$ is not a simple product of the two single-mode squeezing operators: $S_1(\beta_1) \equiv \exp(\beta_1 a^{\dagger 2} - \beta_1^* a^2)$ and $S_2(\beta_2) \equiv \exp(\beta_2 a^{\dagger 2} - \beta_2^* a^2)$. Nevertheless, by a simple transformation the two-mode squeezing operator $S_{12}(\beta)$ can be expressed as a product of the following two new single-mode squeezing operators:

$$S_{12}(\beta) = \exp[\tfrac{1}{2}(\beta b_+^{\dagger 2} - \beta^* b_+^2)] \times \exp[-\tfrac{1}{2}(\beta b_-^{\dagger 2} - \beta^* b_-^2)], \tag{11}$$

where $b_\pm = (a_1 \pm a_2)/\sqrt{2}$.

To generalize the notion of squeezing to the multimode case, a possible definition of the generalized multimode squeezed state could be

$$|\,\beta_{12},...,\beta_{N-1,N},\alpha_1,...,\alpha_N\rangle$$

$$= \prod_{i=1}^{N} D_i(\alpha_i) \prod_{j<k=1}^{N} S_{jk}(\beta_{jk})\,|\,0\rangle, \quad (12)$$

where $D_i(\alpha_i)$ is the ith-mode displacement operator, $S_{jk}(\beta_{jk})$ is the two-mode squeezing operator for the jth and kth modes, and $|\,0\rangle$ is the multimode vacuum state. This definition of the generalized multimode squeezed state is, however, not very convenient because the operator product in Eq. (12) is often difficult to manipulate when N is large. The properties of these operator products are often difficult to use as well. For instance, general simple formulae for the unitary transformation of the annihilation and creation operators by these operator products are hard to find because of the complications caused by repeated applications of these two-mode operators. As a result, an alternative definition for the generalized multimode squeezed state is needed.

To begin with, let us introduce the generalized multimode squeezing operator in the following form:

$$\tilde{S}(\{\beta_{ij}\}) = \exp\left\{ \sum_{i<j=1}^{N} (\beta_{ij} a_i^\dagger a_j^\dagger - \beta_{ij}^* a_i a_j) \right\} \quad (13)$$

$$= \exp\left\{ \tfrac{1}{2} \sum_{\substack{i,j=1\\ i\neq j}}^{N} (\beta_{ij} a_i^\dagger a_j^\dagger - \beta_{ij}^* a_i a_j) \right\}, \quad (14)$$

where $\beta_{ij} = \beta_{ji}$. It is clear that for $N = 2$ the generalized multimode squeeze operator reduces to the usual two-mode squeeze operator given in Eq. (7). Then the generalized multimode squeezed state may be naturally defined as

$$|\,\{\beta_{ij}\},\{\alpha_i\}\rangle \;=\; \tilde{D}(\{\alpha_i\})\,\tilde{S}(\{\beta_{ij}\})\,|\,0\rangle, \quad (15)$$

where

$$\tilde{D}(\{\alpha_i\}) = \prod_{i=1}^{N} D_i(\alpha_i) = \exp\{ \sum_{i=1}^{N} (\alpha_i a_i^\dagger - \alpha_i^* a_i) \}. \quad (16)$$

It is not difficult to show that the generalized multimode squeezing operator transforms the annihilation and creation operators as follows:

$$\tilde{S}^\dagger\, a_i\, \tilde{S} = \sum_{j=1}^{N} (\mu_{ij} a_j + \nu_{ij} a_j^\dagger), \quad (17)$$

$$\tilde{S}^\dagger\, a_i^\dagger\, \tilde{S} = \sum_{j=1}^{N} (\mu_{ij}^* a_j^\dagger + \nu_{ij}^* a_j), \quad (18)$$

where $i = 1, 2, ..., N$, and

$$\mu_{ij} = \delta_{ij} + \frac{1}{2!}\sum_{k=1}^{N} \beta_{ik}\beta_{kj}^* + \frac{1}{4!}\sum_{k,l,m=1}^{N} \beta_{ik}\beta_{kl}^*\beta_{lm}\beta_{mj}^* + \cdots \quad (19)$$

$$\nu_{ij} = \frac{1}{1!}\beta_{ij} + \frac{1}{3!}\sum_{k,l=1}^{N} \beta_{ik}\beta_{kl}^*\beta_{lj} + \frac{1}{5!}\sum_{k,l,m,n=1}^{N} \beta_{ik}\beta_{kl}^*\beta_{lm}\beta_{mn}^*\beta_{nj} + \cdots. \quad (20)$$

Here we have set $\beta_{ii} = 0$ ($i = 1, 2, ..., N$) so that the restriction on the double sum in Eq. (14) can be lifted. We now define matrices $|\boldsymbol{\beta}|$ and $|\boldsymbol{\beta}|^{-1}$ in the following way:

$$(|\boldsymbol{\beta}|^2)_{ij} = \sum_{k=1}^{N} (|\boldsymbol{\beta}|)_{ik}(|\boldsymbol{\beta}|)_{kj} = \sum_{k=1}^{N} \beta_{ik}\beta_{kj}^*, \quad (21)$$

$$\delta_{ij} = \sum_{k=1}^{N} (|\boldsymbol{\beta}|^{-1})_{ik}(|\boldsymbol{\beta}|)_{kj}. \quad (22)$$

Then μ_{ij} and ν_{ij} can be rewritten as

$$\mu_{ij} = [\cosh(|\boldsymbol{\beta}|)]_{ij}, \quad (23)$$

$$\nu_{ij} = \sum_{k,l=1}^{N} [\sinh(|\boldsymbol{\beta}|)]_{ik}(|\boldsymbol{\beta}|^{-1})_{kl}(\boldsymbol{\beta})_{lj} = [\sinh(|\boldsymbol{\beta}|)\,|\boldsymbol{\beta}|^{-1}\,\boldsymbol{\beta}]_{ij}. \quad (24)$$

As a consequence, in the matrix notation one can express the unitary transformations in a form which closely resembles those of the single-mode case:

$$\tilde{S}^\dagger\,\mathbf{a}\,\tilde{S} = \cosh(|\boldsymbol{\beta}|)\,\mathbf{a} + \sinh(|\boldsymbol{\beta}|)\,|\boldsymbol{\beta}|^{-1}\,\boldsymbol{\beta}\,\mathbf{a}^\dagger, \quad (25)$$

$$\tilde{S}^\dagger\,\mathbf{a}^\dagger\,\tilde{S} = \cosh(|\boldsymbol{\beta}|^T)\,\mathbf{a}^\dagger + \sinh(|\boldsymbol{\beta}|^T)\,(|\boldsymbol{\beta}|^T)^{-1}\,\boldsymbol{\beta}^*\,\mathbf{a}, \quad (26)$$

where $|\boldsymbol{\beta}|^T$ is the transpose of $|\boldsymbol{\beta}|$, i.e. $(|\boldsymbol{\beta}|^T)_{ij} = |\boldsymbol{\beta}|_{ji}$, and $\mathbf{a}$ is the column vector consisting of annihilation operators a_i ($i = 1, 2, ..., N$), and $\mathbf{a}^\dagger$ is the vector of creation operators.

Next, to see the multimode squeezing property explicitly, we shall compute the variances for the quadrature amplitudes

$$\mathbf{a}_+ \;=\; \frac{\mathbf{a}+\mathbf{a}^\dagger}{2}, \qquad \mathbf{a}_- \;=\; \frac{\mathbf{a}-\mathbf{a}^\dagger}{2i}\;. \quad (27)$$

Using the transformations in Eqs. (25) and (26), it is straightforward to show that in a generalized multimode squeezed state the variances are given by

$$(\boldsymbol{\Delta}\mathbf{a}_+)^2 = \langle \boldsymbol{\Delta}\mathbf{a}_+ \boldsymbol{\Delta}\mathbf{a}_+^T \rangle = \tfrac{1}{8}\{\cosh(2|\boldsymbol{\beta}|) + \sinh(2|\boldsymbol{\beta}|)\,|\boldsymbol{\beta}|^{-1}\,\boldsymbol{\beta} + \cosh(2|\boldsymbol{\beta}|^T) + \sinh(2|\boldsymbol{\beta}|^T)\,(|\boldsymbol{\beta}|^T)^{-1}\,\boldsymbol{\beta}^*\}, \quad (28)$$

$$(\boldsymbol{\Delta}\mathbf{a}_-)^2 = \langle \boldsymbol{\Delta}\mathbf{a}_- \boldsymbol{\Delta}\mathbf{a}_-^T \rangle = \tfrac{1}{8}\{\cosh(2|\boldsymbol{\beta}|) - \sinh(2|\boldsymbol{\beta}|)\,|\boldsymbol{\beta}|^{-1}\,\boldsymbol{\beta} + \cosh(2|\boldsymbol{\beta}|^T) - \sinh(2|\boldsymbol{\beta}|^T)\,(|\boldsymbol{\beta}|^T)^{-1}\,\boldsymbol{\beta}^*\}, \quad (29)$$

$$
\begin{aligned}
(\Delta \mathbf{a}_{+-})^2 &= \tfrac{1}{2}[\langle \Delta \mathbf{a}_+ \Delta \mathbf{a}_- \rangle + (\langle \Delta \mathbf{a}_+ \Delta \mathbf{a}_- \rangle)^T] \\
&= \frac{1}{8i}\{-\cosh(2|\boldsymbol{\beta}|) + \sinh(2|\boldsymbol{\beta}|)\,|\boldsymbol{\beta}|^{-1}\boldsymbol{\beta} \\
&\qquad + \cosh(2|\boldsymbol{\beta}|^T) - \sinh(2|\boldsymbol{\beta}|^T)\,(|\boldsymbol{\beta}|^T)^{-1}\boldsymbol{\beta}^*\},
\end{aligned} \tag{30}
$$

where $\Delta \mathbf{a}_\pm = \mathbf{a}_\pm - \langle \mathbf{a}_\pm \rangle$ and

$$
(\Delta \mathbf{a}_+)^2 (\Delta \mathbf{a}_-)^2 = \tfrac{1}{16}\mathbf{I} + (\Delta \mathbf{a}_{+-})^2. \tag{31}
$$

For $\boldsymbol{\beta}$ being real, we have

$$
\Delta \mathbf{a}_+ = \tfrac{1}{2}\exp(\boldsymbol{\beta}), \qquad \Delta \mathbf{a}_- = \tfrac{1}{2}\exp(-\boldsymbol{\beta}), \tag{32}
$$

as well as

$$
\Delta \mathbf{a}_{+-} = \mathbf{0}, \tag{33}
$$

$$
\Delta \mathbf{a}_+ \,\Delta \mathbf{a}_- = \tfrac{1}{4}\mathbf{I}. \tag{34}
$$

Hence, provided $\boldsymbol{\beta}$ is real, the generalized multimode squeezed state is a multimode minimum-uncertainty state, which exhibits squeezing in the fluctuation of one field quadrature at the expense of an increase in the fluctuation of the other quadrature.

In summary, we have defined the generalized multimode squeezed state generated by the multimode squeezing operator. The generalized multimode squeezing operator has very similar algebraic properties to those of the single-mode case, and it reduces to the usual two-mode squeezing operator in the case of two modes. It has also been shown that the generalized multimode squeezed state is a multimode minimum-uncertainty state if the squeezing parameters are real. Our generalized multimode-squeezing-operator formulation has been proved to be very useful to problems in electronic systems coupled to boson fields (in particular, phonons)[3–6].

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