Stochastic quantum dynamics of a continuously monitored laser

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The stochastic dynamics of the quantum state of a laser conditioned on continuous intensity and phase-sensitive measurements on its output are examined. We first develop the essential model for a Poissonian laser, and then generalize it for regularly pumped lasers. We show that the rate of phase diffusion (which gives the laser linewidth) is not affected by the regularity of the pump. In both cases, heterodyne detection causes the phase variance in the conditioned laser state to become very small, yet remain significantly above (at least three times) that of a coherent state. The phase diffusion is manifested by a random walk undertaken by the mean phase, with the stochasticity arising from the local-oscillator shot noise. In contrast, intensity measurements have no effect on a Poissonian laser. For a perfectly regularly pumped laser, coarse-grained intensity measurements (for which an approximate theory is developed here) collapse the state to one with an arbitrarily small photon-number variance-to-mean ratio.

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I. INTRODUCTION

The emergence of quantum optics as an important field of theoretical physics was precipitated by two important technological developments: the laser and the efficient photodetector [1]. Laser light is unlike other optical light sources in that it has a large coherent amplitude. To treat this quantum mechanically, it was found extremely useful to use coherent-state representations, such as the Glauber-Sudarshan P function [2,3]. Also, it was necessary to develop new techniques for dealing with open quantum systems, such as the master equation [4]. Both of these methods are related to photodetection. Coherent states have been found to be precisely those states which are not affected by conventional detection techniques. The relationship between the openness of quantum systems and their continuous monitoring is still being explored.

In this paper we examine many of the early themes in quantum optics using very recently developed theories of quantum measurement [5,6]. While not leading directly to new experimental predictions (although see the final paragraph of Sec. VIII), these techniques provide new insights into the stochastic dynamics of individual quantum systems. In particular, we clarify the distinction between coherent states and the quantum state of the light inside a single mode laser. To do this, we first develop a simple master equation for an ideal laser producing light with Poisson photon statistics, as in a coherent state. An immediate consequence of this master equation is that the phase of the laser diffuses, with a linewidth inversely proportional to the mean photon number. An initially coherent state inside the cavity becomes a mixture of coherent states over all phases. The picture of a laser state with a well-defined phase is restored if we incorporate heterodyne detection. Using a stochastic master equation recently derived for such measurements [6], we show that the laser state is collapsed by a measurement to a near-coherent state with a phase which wanders randomly. The source of the stochasticity is ultimately the shot noise in the heterodyne photocurrent which conditions the laser state. The magnitude of the conditioned phase variance (three times that of a coherent state under ideal conditions) is a new and unanticipated result.

More recently in quantum optics, great interest has arisen in so-called squeezed states [7]. These are states in which the quantum uncertainty in one variable (such as one quadrature or the photon number) is less than that of a coherent state, and that in the conjugate variable (such as the other quadrature or the phase) is greater than that of a coherent state, as necessitated by Heisenberg's principle. There has been considerable theoretical work on laser systems which have photon-number variances below the classical (coherent) limit, and so produce sub-Poissonian photon statistics [8–16]. Here we consider ideal lasers with a sub-Poissonian pump, and derive a simple master equation for such lasers. This yields a number of results. Firstly we show that the linewidth, and the random phase walk under heterodyne detection, is the same as in Poissonian lasers. Now, however, the conditioned state is a (close to minimum uncertainty) squeezed state, rather than a mixture of coherent states as in a Poissonian laser.

Unlike the Poissonian laser, regularly pumped lasers are affected by intensity measurements. Direct photodetection is shown to collapse the conditioned state of the cavity mode into one with a photon-number variance below that of its steady-state unconditioned operation. The extra variance in the unconditioned (without measurement) state comes from classical random variation in the conditioned mean intensity. As in the random-phase walk, the source of this randomness is the shot noise in the measured photocurrent. To derive this result, it was necessary to develop an approximate theory of direct photodetection in which the state is conditioned by the photocurrent. Strictly, direct photodetection should be treated in terms of individual photodetections, but by coarse graining in time we heurestically derive a plausi-

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ble stochastic master equation which is valid for a suitable class of systems. The theory shows that the conditioned photon-number variance to mean ratio in a regularly pumped laser can be arbitrarily small in principle.

The organization of this paper is as follows. Firstly, in Sec. II we develop the essential model of a Poissonian pumped laser, and derive some simple results such as the laser linewidth. In Sec. III we review the quantum theory of heterodyne detection which we derived elsewhere [6], and motivate the theory from simple considerations. This theory is applied to the Poissonian laser in Sec. IV, and a physical picture of a laser with a randomly wandering phase is rigorously defined in terms of conditioning the state on the heterodyne photocurrent. The quantum theory of the ideal laser is generalized in Sec. V to lasers with a non-Poissonian pump. In Sec. VI we present an approximate quantum theory of direct photodetection, which is valid for systems such as a laser at steady state. The effect of such detection on the generalized laser model is examined in Sec. VII, as well as the effect of heterodyne detection. Section VIII is a discussion.

II. ESSENTIAL MODEL OF A LASER

In this section we present the simplest possible physical model for a laser. It reproduces the essential features of an ideal laser. The steady-state photon number distribution is Poissonian, as in a coherent state, but the phase diffuses, giving a finite linewidth. The laser consists of a single-mode optical cavity containing N three-level atoms. The atomic-level structure is shown in Fig. 1. The lowering operator from an upper to a lower level is denoted σ_{lu}. The cavity mode has the annihilation operator a, freely rotating at frequency ω. This is resonant with an atomic transition between levels $|1\rangle$ and $|2\rangle$. The dipole coupling constant is χ, which we take to be real for simplicity. The lower level $|1\rangle$ spontaneously decays to the atomic ground state $|0\rangle$ at rate γ. The atoms are incoherently pumped, causing the atom to be excited from $|0\rangle$ to $|2\rangle$ at rate β. Finally, the cavity mode is damped to the external continuum of electromagnetic modes at rate κ. Now eventually we will adiabatically eliminate the atoms, so we need consider only the interaction between the field and one atom, and subsequently scale the effect by N. The density operator W for the field plus one atom obeys the following master equation:

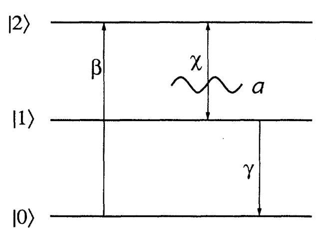

FIG. 1. Schematic diagram of atomic transitions in the ideal laser model. The dipole coupling constant (one-photon Rabi frequency) for the $|1\rangle$–$|2\rangle$ lasing transition is χ. Level $|0\rangle$ is incoherently excited to level $|2\rangle$ at rate β. The spontaneous-emission rate of $|1\rangle$ to $|0\rangle$ is γ, and that of level $|2\rangle$ is assumed negligible.

$$\dot{W} = -i\chi[a^\dagger\sigma_{12} + a\sigma_{12}^\dagger, W] + \gamma\,\mathcal{D}[\sigma_{01}]W + \beta\,\mathcal{D}[\sigma_{02}^\dagger]W + \kappa\,\mathcal{D}[a]W. \tag{2.1}$$

Here, $\mathcal{D}[c]$ is a superoperator taking the arbitrary operator c as an argument, defined by

$$\mathcal{D}[c] = \mathcal{J}[c] - \mathcal{A}[c], \tag{2.2}$$

where we are using the following notation:

$$\mathcal{J}[c]\rho \equiv c\rho c^\dagger, \tag{2.3}$$

$$\mathcal{A}[c]\rho \equiv \tfrac{1}{2}(c^\dagger c\rho + \rho c^\dagger c), \tag{2.4}$$

where ρ is an arbitrary density operator.

To achieve amplification by stimulated emission, it is necessary for level $|1\rangle$ to be rapidly depleted so that a population inversion on the lasing transition occurs. This requires that the spontaneous-emission rate γ be large, in the following sense:

$$\epsilon \equiv \frac{\chi^2\mu}{\gamma^2} \ll 1, \tag{2.5}$$

where μ is the mean photon number for which an expression will be given later. However, this condition is not sufficient to produce a laser. We also require that the stimulated emission events be Poissonian distributed in time in order to produce a Poissonian photon-number distribution. This will occur if the following condition is satisfied:

$$\zeta \equiv \frac{\beta}{\gamma\epsilon} \ll 1. \tag{2.6}$$

Finally, if we wish to adiabatically eliminate the atoms, then the cavity field must relax much more slowly than the atomic populations. This will be satisfied if

$$\delta \equiv \frac{\kappa}{\gamma\epsilon} \ll 1. \tag{2.7}$$

These three conditions could be satisfied (with $\epsilon \sim \delta \sim 0.1$, $\zeta \sim 10^{-8}$) by a set of realistic parameters such as $\kappa \sim 10^7\ \mathrm{s}^{-1}$, $\beta \sim 1\ \mathrm{s}^{-1}$, $\gamma \sim 10^9\ \mathrm{s}^{-1}$, $\chi \sim 10^4\ \mathrm{s}^{-1}$, and $\mu \sim 10^9$.

Now the above master equation (2.1) permits a solution of the form

$$W(t) = |0\rangle\langle 0| \otimes \rho^{00}(t) + |2\rangle\langle 2| \otimes \rho^{22}(t) + \left[|2\rangle\langle 1| \otimes \rho^{21}(t) + \mathrm{H.c.}\right] + |1\rangle\langle 1| \otimes \rho^{11}(t). \tag{2.8}$$

Here the ρ's are operators in the Hilbert space of the cavity mode. The leading term ρ^{00} is of order 1, while the others are much smaller, due to the above scalings. Specifically,

$$\rho^{22} = O(\zeta)\ , \quad \rho^{21} = O(\epsilon^{1/2})O(\zeta)\ , \quad \rho^{11} = O(\epsilon)O(\zeta). \tag{2.9}$$

These scalings will be shown to be self-consistent. The four cavity-mode ρ operators obey the following coupled equations:

$$\dot{\rho}^{00} = \gamma\rho^{11} - \beta\rho^{00} + \kappa\mathcal{D}[a]\rho^{00}, \tag{2.10a}$$

$$\dot{\rho}^{22} = -i\chi(a\rho^{12} - \rho^{21}a^\dagger) + \beta\rho^{00} + \kappa\mathcal{D}[a]\rho^{22}, \tag{2.10b}$$

$$\dot{\rho}^{21} = -i\chi(a\rho^{11} - \rho^{22}a) - \frac{\gamma}{2}\rho^{21} + \kappa\mathcal{D}[a]\rho^{21}, \tag{2.10c}$$

$$\dot{\rho}^{11} = -i\chi(a^\dagger\rho^{21} - \rho^{12}a) - \gamma\rho^{11} + \kappa\mathcal{D}[a]\rho^{11}. \tag{2.10d}$$

Consider first Eq. (2.10c). Using the above scalings (2.5)–(2.9), this can be rewritten as

$$[1 + O(\epsilon) + O(\delta)O(\epsilon)]\dot{\rho}^{21} = i\chi\rho^{22}a - \frac{\gamma}{2}\rho^{21}, \tag{2.11}$$

where the terms on the right-hand side are of equal order. Now this shows that ρ^{21} relaxes at a rate of order γ, whereas ρ^{22} will be shown to relax at a rate of order ϵ slower than this. Thus it is valid to say that ρ^{21} will be slaved to ρ^{22}, and replace the former by the steady-state value. Ignoring lower-order terms in Eq. (2.11) gives

$$\rho^{21} = \frac{2i\chi}{\gamma}\rho^{22}a = O(\epsilon^{1/2})O(\rho^{22}). \tag{2.12}$$

Substituting this into Eq. (2.10d) gives the following:

$$[1 + O(\delta)]\dot{\rho}^{11} = \frac{4\chi^2}{\gamma}a^\dagger\rho^{22}a - \gamma\rho^{11}. \tag{2.13}$$

As above, ρ^{22} is constant on the time scale over which ρ^{11} relaxes to its steady-state value of

$$\rho^{11} = \frac{4\chi^2}{\gamma^2}\mathcal{J}[a^\dagger]\rho^{22} = O(\epsilon)O(\rho^{22}), \tag{2.14}$$

where $\mathcal{J}$ is as defined in Eq. (2.3). Now we substitute Eq. (2.12) into Eq. (2.10b) to get

$$[1 + O(\delta)]\dot{\rho}^{22} = -\frac{4\chi^2}{\gamma}\mathcal{A}[a^\dagger]\rho^{22} + \beta\rho^{00}, \tag{2.15}$$

where $\mathcal{A}$ is as defined in Eq. (2.4). The relaxation time of ρ^{22} is of order $1/\delta$ times that of ρ^{00} so once again it is possible to replace the former by its slaved value,

$$\rho^{22} = \frac{\beta\gamma}{4\chi^2}\left\{\mathcal{A}[a^\dagger]\right\}^{-1}\rho^{00} = O(\zeta). \tag{2.16}$$

Substituting this into Eq. (2.14) gives

$$\rho^{11} = \frac{\beta}{\gamma}\mathcal{J}[a^\dagger](\left\{\mathcal{A}[a^\dagger]\right\}^{-1}\rho^{00}). \tag{2.17}$$

Now the reduced density operator for the cavity mode is given by

$$\rho = \mathrm{Tr}_{\mathrm{atom}}W = \rho^{00} + \rho^{11} + \rho^{22}. \tag{2.18}$$

Since the latter two terms are much smaller than the first (by a factor of order $\zeta \sim 10^{-8}$ with the parameters suggested above), we can ignore these and approximate ρ by ρ^{00}. Substituting Eq. (2.17) into Eq. (2.10a) gives the following master equation for ρ:

$$\dot{\rho} = \beta[\mathcal{J}[a^\dagger](\{\mathcal{A}[a^\dagger]\}^{-1}\rho) - \rho] + \kappa\mathcal{D}[a]\rho. \tag{2.19}$$

Multiplying the pumping term by the number of atoms N, we find the following elegant form for the master equation of an ideal Poissonian laser:

$$\kappa^{-1}\dot{\rho} = \left(\mu\mathcal{E}[a^\dagger] + \mathcal{D}[a]\right)\rho, \tag{2.20}$$

where $\mathcal{E}$ is defined by

$$\mathcal{E}[c] = \mathcal{J}[c]\mathcal{A}[c]^{-1} - 1. \tag{2.21}$$

This is to be compared with the definition of $\mathcal{D}$ (2.2). These simple definitions show clearly the distinction between the Poissonian damping process $\mathcal{D}[a]$ with rate κ times the mean photon number, and the Poissonian excitation process $\mathcal{E}[a^\dagger]$ with fixed rate κ times μ where

$$\mu \equiv N\beta/\kappa. \tag{2.22}$$

Using the Fock basis for the cavity mode gives the following:

$$\dot{\rho}_{n,m} = \mu\left(\frac{2\sqrt{nm}}{n+m}\rho_{n-1,m-1} - \rho_{n,m}\right) + \sqrt{(n+1)(m+1)}\,\rho_{n+1,m+1} - \tfrac{1}{2}(n+m)\rho_{n,m}. \tag{2.23}$$

Here, as in the remainder of this paper, we are measuring time in inverse units of the cavity linewidth κ. It is easy to see that μ is the steady-state mean photon number by verifying that the steady-state density matrix satisfying Eq. (2.23) is

$$\rho_{n,m}(\infty) = \delta_{n,m}e^{-\mu}\frac{\mu^n}{n!}. \tag{2.24}$$

This photon-number distribution is Poissonian, as in a coherent state of amplitude $\sqrt{\mu}$. Unlike a coherent state, it has a completely undefined phase as the off-diagonal elements are zero. This can be seen from an alternate representation of the steady-state density operator using the Glauber-Sudarshan P function,

$$\rho(\infty) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi|\sqrt{\mu}e^{i\varphi}\rangle\langle\sqrt{\mu}e^{i\varphi}|. \tag{2.25}$$

Thus, the stationary density operator can be expressed either as a Poissonian mixture of number states (with maximally determined intensity and completely undefined phase), or as a mixture of fixed amplitude coherent states (with well-defined intensity and phase). Why then is a laser often treated as being in a coherent state of unknown phase, but never as being in an imprecisely known Fock state? (Here we are talking of classical knowledge, in that the imprecision is merely due to the experimenter's ignorance.)

The basic answer to the above question is differential lifetimes, as pointed out by Gea-Banacloche [17]. If a laser were to be put into a Fock state, then under the master equation (2.23), the probability of it remaining in that state will decay at a rate of order $\kappa\mu$. Using the parameters suggested above, the lifetime of a number state is thus of the order of 10^{-16} s. On the other hand, the survival probability of a coherent state of amplitude $\sqrt{\mu}$ decays at a rate of order κ/μ, giving a lifetime of order minutes. This can be seen by calculating the infinitesimally evolved density matrix which is initially in the coherent state $|\alpha\rangle$ with $|\alpha|^2 = \mu$. From (2.23), the result is

$$\rho_{n,m}(dt) = e^{-|\alpha|^2} \frac{\alpha^n \alpha^{*m}}{\sqrt{n!m!}} \left(1 - \frac{dt(n-m)^2}{2(n+m)}\right). \quad (2.26)$$

For large μ, it is an excellent approximation to replace $n+m$ in the above denominator by 2μ. Then, the above short-time result (2.26) could have been obtained from the following master equation:

$$\dot{\rho} = \Gamma \mathcal{D}[a^\dagger a]\rho, \quad (2.27)$$

where we have defined (in units of κ)

$$\Gamma = 1/2\mu. \quad (2.28)$$

This short-time master equation is valid for an initial coherent state of amplitude $\sqrt{\mu}$. By linearity, it will also be valid for all initial states which are mixtures of such coherent states. But it is easy to verify that the class C_μ of such mixtures is invariant under the master equation (2.27). Thus, this approximate master equation is in fact valid for all times.

From (2.27), we can derive the following result:

$$\frac{d}{dt}\langle a^\dagger(t)\rangle = -\frac{\Gamma}{2}\langle a^\dagger(t)\rangle. \quad (2.29)$$

Then, by the quantum regression theorem and the fact that $a\rho$ is a member of C_μ if ρ is a member, we have

$$\langle a^\dagger(t)a(0)\rangle = e^{-\Gamma t/2}\mu. \quad (2.30)$$

Thus the laser has a Lorentzian spectrum with full width at half maximum equal to Γ. This is as expected since a master equation containing a double commutator with $a^\dagger a$ will cause diffusion in the conjugate variable phase. This is perhaps more easily seen from the Fokker-Planck equation for the P function. Since this is nonzero only for complex amplitudes α of modulus $\sqrt{\mu}$, the P function has only one real argument, $\varphi = \arg(\alpha)$. It satisfies a pure diffusion equation

$$\dot{P}(\varphi) = \frac{\Gamma}{2}\frac{\partial^2}{\partial\varphi^2}P(\varphi). \quad (2.31)$$

The Green's function for this equation [4]

$$P(\varphi, t|\varphi_0, 0) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{in(\varphi-\varphi_0)}e^{-\Gamma t n^2/2} \quad (2.32)$$

is an alternate route to the linewidth stated above, which agrees with that of Louisell in the same limits [4].

III. QUADRATURE MEASUREMENTS

The fact that the inverse of the laser linewidth is of order minutes is often used as a justification for approximating the laser by a coherent state. On the time scale of many experiments, a laser initially in a coherent state will remain in that state to a very good approximation. However, this appears to beg the question of how the laser got into a coherent state in the first place. An obvious answer is that measuring the phase of the laser collapses it into a coherent state (or alternatively, determines which coherent state it really was in). In practice, phase measurements are made using homodyne or heterodyne detection. In this section we review briefly the quantum theory of such phase measurements which we have presented in detail elsewhere [5,6].

To investigate the collapse of the laser state due to measurement we need a theory which describes the *selective* evolution of the system. That is to say, we need to know what the state of the laser is given the result of the measurement. The nonselective evolution of the laser is simply given by the master equation (2.27). Different measurement schemes on the output light of the laser will give different selective evolution equations. If the result of the measurement is averaged over, these selective evolutions necessarily reproduce the nonselective evolution for the density operator.

The simplest kind of measurement on the output light is direct photodetection. The selective evolution of the system is easy to describe. In the infinitesimal time interval $(t, t+dt)$, a photodetector of quantum efficiency η placed at the output of a cavity will register a count with probability $\eta P_c(t)dt$, where $P_c(t) = \mathrm{Tr}\{\mathcal{J}[a]\rho_c(t)\}$. Recall that we are measuring time in inverse units of the cavity linewidth. The subscript c on $\rho_c(t)$ indicates that the state of the system at time t is in general conditioned on previous photocounts. If a photodetection occurs, the new state of the system is $\rho_c(t+dt) = \mathcal{J}[a]\rho_c(t)/P_c(t)$. If no photodetection occurs, the state of the system evolves such that the average evolution of the system is given by its master equation. This is shown explicitly in Ref. [6].

It is possible to generalize this theory of photodetection by adding a local oscillator to the amplitude of the output field before it is detected. This is achieved by using a beam splitter, and it is advantageous to use two detectors, one for each of the beams leaving the beam splitter. If the local oscillator is tuned to cavity resonance, then we have a balanced homodyne measurement of the quadrature of the system in phase with the local oscillator. For monitoring both quadratures (or equivalently, the amplitude and phase) of the system, one can use so-called eight-port homodyne detection, where the output light from the cavity is split into two independent balanced homodyne detectors. The same result can be achieved by using heterodyne detection, in which the local oscillator is significantly detuned from the cavity. The two Fourier amplitudes of the resulting photocurrent at the detuning frequency give the two quadrature measurements.

Since we are interested in monitoring the phase of the laser, this last scheme is most appropriate. We measure the phase of the laser relative to that of the local oscillator, which is defined as a coherent state with amplitude β. If the local-oscillator amplitude is much greater than the system amplitude, then the rate of photocounts is much higher than it would be without the local oscillator, but the change in the conditioned system state due to each photodetection is much less. In the limit where the local-oscillator amplitude goes to infinity, it is possible to convert from discrete photocounts to a continuous photocurrent. Denote the photocurrent measuring the quadrature in phase with the local oscillator by $I_c^{\cos}(t)$, and that out of phase by $I_c^{\sin}(t)$. Then it is useful to define a "complex photocurrent" $I_c^{\exp}(t) = I_c^{\cos}(t)+iI_c^{\sin}(t)$, which is given simply by

$$I_c^{\exp}(t) = \beta[\eta\langle a^\dagger\rangle_c(t) + \sqrt{\eta}\xi(t)], \tag{3.1}$$

where $\xi(t)$ represents complex Gaussian white noise satisfying

$$E(\xi(t)\xi^*(t')) = \delta(t-t'), \tag{3.2}$$

with all other first- and second-order expectation values vanishing. Such ensemble averages denoted by E are not to be confused with conditioned quantum averages such as $\langle a^\dagger\rangle_c(t)$ which is given by $\mathrm{Tr}[a^\dagger\rho_c(t)]$. The conditioning of the system density operator on the photocurrent is given explicitly via the following equation (which is to be interpreted in the Ito formalism of stochastic calculus [18]):

$$\dot\rho_c(t) = \mathcal{L}\rho_c(t) + \sqrt{\eta}\{\xi(t)[a - \langle a\rangle_c(t)] + \mathrm{H.c.}\}\rho_c(t). \tag{3.3}$$

Although Eq. (3.3) is generally unfamiliar in content and in form, we can motivate it from three simple principles. Firstly, the conditioned system evolution must be completely determined from the conditioned complex photocurrent (3.1). This is true by the definition of a conditioned system state. Secondly, if we ignore the result of the measurement by averaging over all possible results, we must return to the original master equation for the unconditioned state $\dot\rho = \mathcal{L}\rho$. In this case, this amounts to setting the noise terms in (3.1) equal to their average value of zero, or to setting the efficiency of the detector to zero. Thus, the only additional terms in the master equation must be linear in the photocurrent noise. Since the stochastic evolution equation must preserve the Hermiticity of $\rho_c(t)$, the obvious solution suggested by these first two principles is

$$\dot\rho_c(t) = \mathcal{L}\rho_c(t) + \sqrt{\eta}[\xi(t)\mathcal{F}\rho_c(t) + \mathrm{H.c.}], \tag{3.4}$$

where $\mathcal{F}$ is an as-yet undetermined superoperator which may be nonlinear in its action on $\rho_c(t)$.

The third principle is that the stochastic equation (3.4) must yield the correct correlation functions for the complex photocurrent (3.1), as found by more standard analyses. For example, the two-time correlation function for the heterodyne photocurrent is defined as the ensemble average

$$R(t,t+\tau) = E(I_c^{\exp}(t+\tau), I^{\exp}(t)^*), \tag{3.5}$$

where we are using the notation $E[A,B] = E[AB] - E[A]E[B]$. From standard quantum regression techniques [19], this is given by

$$R(t,t+\tau) = \eta^2\mathrm{Tr}\{a^\dagger e^{\mathcal{L}\tau}[a\rho(t)]\} \\ -\eta^2\mathrm{Tr}[a^\dagger e^{\mathcal{L}\tau}\rho(t)]\mathrm{Tr}[a\rho(t)] + \eta\delta(\tau), \tag{3.6}$$

where $\rho(t)$ is the density operator at time t. That this is assumed given is the reason that the current at time t in Eq. (3.5) is not conditioned; whether or not the system state at time t is conditioned on previous measurement results is irrelevant. The current at time $t+\tau$ is, however, conditioned on the results of the homodyne measurements in the interval $[t, t+\tau)$. In particular, the state of the system at time $t+\tau$ is conditioned on the noise in the photocurrent at time t.

It is easy to see [5] that, upon substituting Eq. (3.1) into Eq. (3.5) and using Eq. (3.2), the terms remaining after cancellations are

$$R(t,t+\tau) = E(\mathrm{Tr}[\eta a^\dagger\rho_c(t+\tau)]\sqrt{\eta}\xi^*(t)) + \eta\delta(\tau). \tag{3.7}$$

For this to agree with Eq. (3.6), we simply require that

$$E(\rho_c(t+\tau)\xi^*(t)) = \sqrt{\eta}e^{\mathcal{L}\tau}\{[a\rho(t)] - \mathrm{Tr}[a\rho(t)]\rho(t)\}. \tag{3.8}$$

The left-hand side of this equation is not zero because the conditioned density operator at time $t+\tau$ is influenced by the photocurrent noise at time t. Specifically, we see from our postulated Eq. (3.4) that

$$\rho_c(t+dt)\xi^*(t) = \sqrt{\eta}\mathcal{F}\rho_c(t), \tag{3.9}$$

where we have used Eq. (3.2) and discarded terms of higher order in dt. Between $t+dt$ and $t+\tau$, the ensemble average evolution of $\rho_c(t)$ is simply given by the Liouville superoperator $\mathcal{L}$. That is, we get to lowest order in dt

$$E(\rho_c(t+\tau)\xi^*(t)) = \sqrt{\eta}e^{\mathcal{L}\tau}[\mathcal{F}\rho(t)]. \tag{3.10}$$

A more rigorous derivation of this result is given in Ref. [5]. Comparison with Eq. (3.8) shows that we must define the superoperator $\mathcal{F}$ by

$$\mathcal{F}\rho = \{a - \mathrm{Tr}[a\rho]\}\rho. \tag{3.11}$$

This is precisely the definition needed to reproduce the correct equation (3.3).

By use of this equation, we will show that the conditioned laser state does have a well-defined phase at all times, relative to the fixed phase of the local oscillator. However, this appears to beg the question as to how the fixed phase of the local oscillator is produced. In fact, it is not necessary for the local oscillator to have a fixed phase; it can come from a completely phase-diffused laser. Provided that the local-oscillator linewidth is much less than that of the system, the dynamics of the relative phase will be dominated by the system dynamics, as given in Eq. (3.3). The absolute phase is as undetermined as that of the local oscillator. This is usually unimportant, because in practice only relative phase is measurable at optical frequencies.

IV. MONITORING THE POISSONIAN LASER

First we determine the conditioned state of the laser under direct photodetection. We showed in Sec. II that the laser at steady state is in a mixture of coherent states of amplitude of modulus $\sqrt{\mu}$. From the theory in Sec. III, it is easy to see that direct photodetection has no effect on the laser state. This is because the coherent states $|\alpha\rangle$ are eigenstates of the jump superoperator $\mathcal{J}[a]$, and the eigenvalues are $|\alpha|^2$, which is independent of phase. If the uncertainty in the laser intensity was greater (or less) than that of a coherent state, then direct photodetection would have an effect on the conditioned system state. Moreover, the effect is distinctly different in the two cases, as will be seen in Sec. VII.

Heterodyne detection, on the other hand, has a dramatic effect on the laser at steady state. The stochastic master equation for the conditioned laser state at steady state is

$$\dot{\rho}_c(t) = \Gamma\mathcal{D}[a^\dagger a]\rho + \sqrt{\eta}\{\xi(t)[a\rho_c(t) - \langle a\rangle_c(t)\rho_c(t)] + \text{H.c.}\}. \tag{4.1}$$

As noted in Sec. II, the P function for the laser at steady state is a function of the phase φ only. It obeys

$$\dot{P}_c(\varphi,t) = \left(\frac{\Gamma}{2}\frac{\partial^2}{\partial\varphi^2} + \sqrt{\eta\mu}\left\{\xi(t)\left[e^{i\varphi} - \int d\varphi' e^{i\varphi'}P_c(\varphi',t)\right] + \text{c.c.}\right\}\right)P_c(\varphi,t). \tag{4.2}$$

Now we will show shortly that the long-time solutions to this equation have a variance in φ of order $1/\mu \ll 1$. Keeping terms of lowest order in $1/\mu$ in Eq. (4.2) gives

$$\dot{P}_c(\varphi,t) = \left\{\frac{\Gamma}{2}\frac{\partial^2}{\partial\varphi^2} + \sqrt{2\eta\mu}\,\xi_c^\phi(t)[\varphi - \phi_c(t)]\right\}P_c(\varphi,t). \tag{4.3}$$

Here, $\xi_c^\phi(t)$ represents real normalized Gaussian white noise, defined by

$$\xi_c^\phi(t) = -\text{Im}[\sqrt{2}e^{i\phi_c(t)}\xi(t)], \tag{4.4}$$

and $\phi_c(t)$ is the central angle of the distribution

$$\phi_c(t) = \int d\varphi\, P_c(\varphi,t)\varphi. \tag{4.5}$$

Since the variance in the phase is very small, we can ignore the periodicity requirement on $P_c(\varphi,t)$ and instead use the following ansatz for the solution of Eq. (4.3):

$$P_c(\varphi,t) = \frac{1}{\sqrt{2\pi U_c(t)}}\exp\{-[\varphi - \phi_c(t)]^2/2U_c(t)\}. \tag{4.6}$$

If $P_c(t)$ is initially in such a Gaussian state, it will remain so under the evolution of Eq. (4.3), with the conditioned mean and variance evolving via

$$\dot{\phi}_c(t) = \sqrt{2\eta\mu}\,\xi_c^\phi(t)U_c(t), \tag{4.7}$$

$$\dot{U}_c(t) = \Gamma - 2\eta\mu U_c^2(t). \tag{4.8}$$

These equations are derived in Appendix A. Note that the effect of the measurement on the variance is deterministic, and causes it to reduce as expected. If the laser state is taken to be initially in a coherent state, the normally ordered variance is $U_c(0) = 0$. From Eq. (4.8) we then see that

$$U_c(t) = U_\infty \tanh(2\eta\Gamma\mu t), \tag{4.9}$$

where the steady-state P-function phase variance is

$$U_\infty = \sqrt{\Gamma/2\eta\mu}. \tag{4.10}$$

If there is no excess phase diffusion in the laser, then $\Gamma = 1/2\mu$, as derived in Sec. II. Assuming in addition that $\eta = 1$, this gives the minimum steady-state conditioned P phase variance of $U_\infty = 1/2\mu$. As the mean photon number μ goes to infinity, this goes to zero. Nevertheless, on a quantum scale, it is always significantly higher than the value of zero for a coherent state. In fact, it is equal to the Q-function phase variance of a coherent state. Thus it is not true that monitoring the phase of a laser collapses it to a coherent state.

Equation (4.9) shows that the phase variance relaxes to its steady-state value at a rate of order the cavity linewidth, which is of order μ times the laser linewidth. Thus, we may replace $U_c(t)$ in Eq. (4.7) by its steady-state value. This gives

$$\dot{\phi}_c(t) = \sqrt{\Gamma}\xi_c^\phi(t). \tag{4.11}$$

This equation is precisely of the form of the stochastic differential equation for the laser phase which would be derived from the original Fokker-Planck equation for the P function (2.31). That is not to say, however, that all we have achieved is a very lengthy and obscure derivation of a standard result. The phase $\phi_c(t)$ is not a mathematical artifact. It has a physical interpretation as the mean phase of the laser, conditioned on the results of the continuous heterodyne measurement. The conditioning occurs via the noise in Eq. (4.11) which is physically derived from the photocurrent shot noise, not merely a formal device producing a stochastic equation equivalent to a Fokker-Planck equation. We thus have the following picture for the dynamics of a continuously monitored standard laser. The laser state is a Gaussian mixture of equal-amplitude coherent states with a constant phase variance inversely proportional to the mean photon number. The mean phase of the mixture undergoes a random walk on a time scale inversely proportional to the laser linewidth.

V. REGULARLY PUMPED LASER MODEL

The laser model developed in Sec. II was based on the assumption that the excitation process was Poissonian. Specifically, this was achieved by assuming that the rate of decay from the lower lasing level to the ground state was much greater than the rate of excitation from the ground state to the upper lasing level. This assumption has the benefit that the Poissonian laser master equation has a simple form, and that the laser steady state is a mixture of equal-amplitude coherent states, which gives Poissonian photon statistics in the output light. Other assumptions regarding the decay and excitation rates in a three-level atomic system can give rise to sub-Poissonian statistics [12]. Furthermore, the more atomic levels involved, the better the quantum noise reduction in the laser output [13,14]. As the number of levels increases indefinitely with matched transition rates equal to half the pump rate, it is possible to get arbitrarily low photocount variance-to-mean ratios for long times [14].

The explanation for this quantum noise reduction is that the reexcitation of the upper laser level following a stimulated emission is regularized due to the many intermediate steps. It is important that the transition rates are matched; if one is much slower than the others, then this becomes the rate-determining step, and the excitation process becomes Poissonian, as in Sec. II. A regular electronic excitation can also be achieved by externally imposing a regular pump process on the laser [15,16], such as by a regular injection of atoms. Either method of sub-Poissonian excitation can be treated simply using the superoperators derived in Sec. II. The following argument is similar to that of Ref. [16].

Consider a short time dt in which the number of stimulated emissions $n(dt)$ is expected to be much greater than 1. If the mean rate of excitations is μ, as in Sec. II, then we can write

$$n(dt) = \mu dt + \sqrt{\mu(q+1)}dW, \tag{5.1}$$

where dW is an infinitesimal Weiner increment and q is the Mandel Q parameter [20] for the excitation process, equal to 0 for a Poisson process and -1 for a perfectly regular process. For an m level (including the two lasing levels) atomic system with matched transition rates, $q = -(m-2)/(m-1)$ [14]. Now the effect of one stimulated emission is, from Sec. II, given by the excitation superoperator $\mathcal{E}[a^\dagger]$. Thus, the infinitesimally evolved density operator is

$$\rho(t+dt) = (1+\mathcal{E}[a^\dagger])^{n(dt)}\rho(t). \tag{5.2}$$

Since $\mathcal{E}[a^\dagger]$ represents a small change to the state of the mode (adding one photon to μ photons), we can assume that it is much less than unity, and expand Eq. (5.2) to get

$$\begin{aligned}\rho(t+dt) = \{&1 + n(dt)\mathcal{E}[a^\dagger] \\ &+ \tfrac{1}{2}n(dt)[n(dt)-1]\mathcal{E}[a^\dagger]^2\}\rho(t).\end{aligned} \tag{5.3}$$

Substituting in the above expression for $n(dt)$ and averaging over the uncertainty in the number of stimulated emissions gives the following master equation:

$$\dot{\rho} = \mu\left\{\mathcal{E}[a^\dagger] + \frac{q}{2}\mathcal{E}[a^\dagger]^2\right\}\rho + \mathcal{D}[a]\rho. \tag{5.4}$$

Here we have restored the damping term with rate 1. For the Poissonian pump ($q=0$) this reduces to that derived in Sec. II (2.20), as expected.

To elucidate Eq. (5.4), we write it in the Fock basis,

$$\begin{aligned}\dot{\rho}_{n,m} = \mu&\left(\frac{2q\sqrt{nm(n-1)(m-1)}}{(n+m)(n+m-2)}\rho_{n-2,m-2} + \frac{2(1-q)\sqrt{nm}}{n+m}\rho_{n-1,m-1} - \frac{2-q}{2}\rho_{n,m}\right) \\ &+\sqrt{(n+1)(m+1)}\rho_{n+1,m+1} - \tfrac{1}{2}(n+m)\rho_{n,m}.\end{aligned} \tag{5.5}$$

Looking at photon-number populations $P_n = \rho_{n,n}$ gives

$$\begin{aligned}\dot{P}_n = \mu&\left[\frac{q}{2}P_{n-2} + (1-q)P_{n-1} - \frac{2-q}{2}P_n\right] \\ &+(n+1)P_{n+1} - nP_n.\end{aligned} \tag{5.6}$$

From this we can easily derive the following for the mean and variance of the photon number:

$$\dot{\bar{n}} = \mu - \bar{n}, \tag{5.7}$$

$$\dot{\sigma^2} = \bar{n} + (1+q)\mu - 2\sigma^2. \tag{5.8}$$

These equations show that the steady-state mean photon number is independent of q, as expected, and that the steady-state variance is given by

$$\sigma^2 = \mu(1+q/2). \tag{5.9}$$

This can obviously be sub-Poissonian, with a minimum of $\mu/2$ when $q=-1$. Furthermore, Eq. (5.7) allows the steady-state second-order correlation function to be calculated exactly as

$$\langle :\hat{n}(t+\tau)\hat{n}(t):\rangle = \mu^2 + \tfrac{1}{2}\mu q e^{-\tau}. \tag{5.10}$$

This gives the following normalized noise spectrum for the laser intensity output:

$$S(\omega) = \mu\left[1 + \frac{q}{1+\omega^2}\right]. \tag{5.11}$$

For perfect pump regularity, the noise is reduced to zero at low frequencies.

The above results for photon-number statistics are as obtained by previous workers. However, they make no mention of the effect of pump regularity on the laser phase. To investigate this, we convert the master equation (5.4) into a Fokker-Planck equation for the Wigner function $W(\alpha,\alpha^*)$. It is necessary to use the Wigner function rather than the P function as in Sec. II because now we have to deal with nonclassical states (see, for example, Ref. [19]). The result, which is derived in Appendix B, is

$$\dot{W} = \left\{ \frac{\partial}{\partial\alpha}\left[\frac{-\mu}{2\alpha^*} + \frac{\alpha}{2} + \left(1+\frac{q}{2}\right)\frac{\mu}{4|\alpha|^2\alpha^*}\right] + \text{c.c.}\right\} W + \frac{1}{2}\left\{ \frac{\partial^2}{\partial\alpha^2}\frac{q\mu}{4\alpha^{*2}} + \frac{\partial^2}{\partial\alpha^{*2}}\frac{q\mu}{4\alpha^2} + \frac{\partial^2}{\partial\alpha\partial\alpha^*}\left[1+\left(1+\frac{q}{2}\right)\frac{\mu}{|\alpha|^2}\right]\right\} W. \tag{5.12}$$

This equation can be considerably simplified by transforming to intensity and phase variables $n = |\alpha|^2 - \frac{1}{2}$ and $\varphi = \ln(\alpha/\alpha^*)/2i$. Then the Fokker-Planck equation for $W(n,\varphi)$ becomes

$$\dot{W} = \left\{ \frac{\partial}{\partial n}(n-\mu) + \frac{1}{2}\frac{\partial^2}{\partial n^2}[n + (1+q)/\mu] + \frac{1}{8}\frac{\partial^2}{\partial\varphi^2}\frac{n+\mu}{n^2}\right\} W. \tag{5.13}$$

From this, the results (5.7) and (5.8) may easily be verified.

By replacing n by its mean μ in the diffusion terms, we get the simple expression

$$\dot{W} = \left[\frac{\partial}{\partial n}(n-\mu) + \frac{1}{2}\mu(2+q)\frac{\partial^2}{\partial n^2} + \frac{1}{2}\frac{1}{2\mu}\frac{\partial^2}{\partial\varphi^2}\right] W. \tag{5.14}$$

This clearly shows that regular pumping does not affect the phase-diffusion term. That is, reducing the intensity fluctuations in this case does not increase the phase fluctuations. We thus expect the effect of heterodyne detection on phase diffusion to be much the same as in Sec. IV. However, the effect on the laser intensity variance needs to be investigated. Also, since we have a nonclassical state, direct photodetection will now have an effect on the conditioned state of the laser. To deal with this, we need the theory of coarse-grained intensity measurements developed in the next section.

VI. COARSE-GRAINED INTENSITY MEASUREMENTS

The quantum theory of direct photodetection has already been presented in Sec. III, in terms of the probability of a photodetection occurring in an infinitesimal time increment, and the effect on the conditioned system state. However, this theory is of limited use in practice, because with many systems the photon flux is so high that only a photocurrent is measurable, not a photocount. From a theoretical point of view, the noise associated with stochastic arrival times of photons is much less tractable than the Gaussian noise typically associated with steady photocurrents. For these reasons, we wish to develop a quantum theory of intensity measurements appropriate for systems with large, stable intensities. Such a theory could be called a coarse-grained theory of photodetection, in that we must average over many individual photocounts in order to get a photocurrent. However, this is not the approach we adopt here to derive a stochastic equation for the conditioned system state. Rather, we develop a heuristic model in the manner of the motivation for the stochastic equation for heterodyne detection given in Sec. III.

Consider a cavity with a stable mean photon number $\bar{n} \simeq \mu \gg 1$, and photon-number variance of order μ. Let dt be a small-time increment such that the average photocount in that time ($\sim \mu dt$) is very large, but the change in the system ($\sim dt$) is very small. Obviously this requires $\mu \gg 1$, which will be satisfied in a laser with $\mu \sim 10^9$. Now since the change in the system over the time dt may be ignored, the photocount dm will be a Poissonian random variable with mean and variance equal to $\eta\bar{n}dt$, where η is the efficiency of the photodetector. Since this is very large, it is well approximated by a Gaussian random variable with equal first- and second-order moments. That is, we approximate dm by

$$dm = \eta\bar{n}dt + \sqrt{\eta\bar{n}}dW, \tag{6.1}$$

where dW is an infinitesimal Weiner increment. Since the mean photon number is assumed stable, we can replace $\bar{n}$ by μ in the noise term of this photocurrent. This is necessary for the theory developed below. Then a photocurrent defined by $I(t) = dm(t)/dt$ will be given by

$$I_c(t) = \eta\langle a^\dagger a\rangle_c(t) + \sqrt{\eta\mu}\,\xi(t), \tag{6.2}$$

where $\xi(t) = dW(t)/dt$. Here we have added a conditioned subscript to the mean photon number $\langle a^\dagger a\rangle_c(t) = \mathrm{Tr}[a\rho_c(t)a^\dagger]$ in anticipation of the impending development of the theory.

Now we wish to know how the photocurrent (6.2) conditions the system state. Presumably, it would be possible to derive this from the summation of the individual effect of each photocount. However, this appears to be very difficult, so instead we derive an evolution equation for the conditioned state based on the three conditions specified in Sec. III. These were (i) the stochasticity must be determined by the measured photocurrent; (ii) averaging over the noise must restore the original master equation; and (iii) it must yield the correct autocorrelation function for the photocurrent. As in Sec. III, the first two conditions suggest a stochastic master equation of the form

$$\dot{\rho}_c(t) = [\mathcal{L} + \sqrt{\eta}\xi(t)\mathcal{I}]\rho_c(t), \tag{6.3}$$

where $\mathcal{L}$ is the generator of the original master equation, and $\mathcal{I}$ is the nonlinear superoperator for intensity measurements which is to be determined.

The photocurrent autocorrelation function which we require is [19]

$$\begin{aligned} E(I_c(t+\tau), I(t)) &= \eta^2 \mathrm{Tr}\{a^\dagger a e^{\mathcal{L}\tau}[a\rho(t)a^\dagger - \langle a^\dagger a\rangle(t)\rho(t)]\} \\ &\quad + \eta\langle a^\dagger a\rangle_c(t)\delta(\tau). \end{aligned} \tag{6.4}$$

Within the approximations mentioned above, it is permissible to replace the final shot noise term by $\eta\mu\delta(\tau)$. Using the method of Sec. III, we can show from Eq. (6.2) that the stochastic master equation Eq. (6.3) gives the result

$$E(I_c(t+\tau), I(t)) = \eta^2\sqrt{\mu}\,\mathrm{Tr}\{a^\dagger a e^{\mathcal{L}\tau}[\mathcal{I}\rho(t)]\} + \eta\mu\delta(\tau). \tag{6.5}$$

We see immediately that the required superoperator is defined by

$$\mathcal{I}\rho = \frac{1}{\sqrt{\mu}}\left[a\rho a^\dagger - \mathrm{Tr}(a\rho a^\dagger)\rho\right]. \tag{6.6}$$

That is to say, with the photocurrent given by Eq. (6.2), the stochastic evolution equation for the conditioned system state is

$$\dot{\rho}_c(t) = \left(\mathcal{L} + \sqrt{\frac{\eta}{\mu}}\,\xi(t)\left\{\mathcal{J}[a] - \langle a^\dagger a\rangle_c(t)\right\}\right)\rho_c(t). \tag{6.7}$$

Having "derived" this master equation, we must emphasize that it is not valid in general. Although it obviously preserves Hermiticity and trace, it does not in general preserve the positivity of density operators. For example, consider an initially pure state with mean photon number μ in a simply decaying cavity. It is easy to show that, to leading order in the infinitesimal time dt,

$$\mathrm{Tr}[\rho(dt)^2] = 1 + \frac{2dW}{\sqrt{\mu}}(\langle a^\dagger\rangle\langle a\rangle - \langle a^\dagger a\rangle), \tag{6.8}$$

where all of the quantum averages are evaluated at $t = 0$ and we have assumed $\eta = 1$. Unless the initial state is a coherent state, the infinitesimally evolved state has a 50% chance of being nonpositive, with $\mathrm{Tr}[\rho(dt)^2] > 1$. This shortcoming is a consequence of the fact that this theory is intended to apply to coarse-grained measurements. A stochastic master equation describing fine-grained measurements [such as (3.3) with $\eta = 1$] is equivalent to a stochastic Schrödinger equation for a state vector [6], providing that the nondamping evolution is unitary. The density matrix formed from a state vector is of course positive definite, so fine-grained measurements will always give valid master equations. Despite this problem, Eq. (6.7) is useful for those applications for which it was intended. That is, if the system has a well-defined stationary mean photon number (with variance of the same order as the mean), the stochastic master equation will generate valid density-matrix evolution. This will be shown to be the case for an ideal laser. The proof relies upon keeping only terms of leading order in the mean photon number.

VII. MONITORING THE REGULAR LASER

We first investigate the effect of direct photodetection on the regularly pumped laser state, using the theory of the previous section. Converting Eq. (6.7) into a Fokker-Planck equation for the Wigner function and adding the ideal laser terms gives

$$\begin{aligned} \dot{W}_c(t) = \Bigg[&\frac{\partial}{\partial n}(n-\mu) + \frac{1}{2}\mu(2+q)\frac{\partial^2}{\partial n^2} + \frac{1}{2}\frac{1}{2\mu}\frac{\partial^2}{\partial\varphi^2} \\ &+ \sqrt{\frac{\eta}{\mu}}\,\xi(t)\left(n - \bar{n}_c(t) + \frac{\partial}{\partial n}n + \frac{1}{2}\frac{\partial^2}{\partial n^2}n\right)\Bigg] \\ &\times W_c(t), \end{aligned} \tag{7.1}$$

where $\bar{n}_c(t) = \langle a^\dagger a\rangle_c(t)$. Now we linearize the measurement-induced terms as above, and discard terms of order $1/\sqrt{\mu}$. In doing this we make the crucial assumption that the photon-number variance of the conditioned state is of the same order as the mean. Since the phase dependence of the Wigner function is not influenced by the intensity measurement, we remove this degree of freedom to obtain

$$\begin{aligned} \dot{W}_c(n,t) = \Bigg[&\frac{\partial}{\partial n}(n-\mu) + \frac{1}{2}\mu(2+q)\frac{\partial^2}{\partial n^2} \\ &+ \sqrt{\frac{\eta}{\mu}}\,\xi(t)\left(n - \bar{n}_c(t) + \mu\frac{\partial}{\partial n}\right)\Bigg] W_c(n,t). \end{aligned} \tag{7.2}$$

Using the method of Appendix A, we find that this equation has a Gaussian solution. The mean $\bar{n}$ and variance $\sigma_c^2(t)$ of this distribution obey the following equations:

$$\dot{\bar{n}}_c(t) = -\bar{n}_c(t) + \mu + \sqrt{\frac{\eta}{\mu}}\left[\sigma_c^2(t) - \bar{n}_c(t)\right]\xi(t), \tag{7.3}$$

$$\dot{\sigma^2}_c(t) = -2\sigma_c^2(t) + \mu(2+q) - \frac{\eta}{\mu}\left[\sigma_c^2(t) - \bar{n}_c(t)\right]^2. \tag{7.4}$$

These equations are similar to those for the moments of the phase distribution in Sec. IV, in that the noise only appears in the equation for the mean. Note the different effect on the conditioned mean depending on whether the state is super- or sub-Poissonian. For a super-Poissonian state ($\sigma^2 > \bar{n}$), the conditioned mean photon number increases when a higher than average intensity is measured at the photodetector [$\xi(t) > 0$]. This is what is expected from a classical measurement; the higher photocurrent indicates that the true mean photon number of the system is larger than we had previously thought, and so our estimate ($\bar{n}_c$) is adjusted upwards. For a sub-Poissonian state, this intuition fails in that a higher photocurrent causes the conditioned mean to decrease. This is precisely the mechanism by which squeezed light gives rise to a subshot noise photocurrent, and highlights the difference between classical and nonclassical states. The explanation lies in the fact that an above average photocurrent is caused by an increase in the number of photons leaving the cavity. If the photon-number variance is sufficiently small (less than the mean), then the effect of this in dropping the mean photon number in the cavity overrides the "classical" effect described above.

The deterministic equation for the variance will cause the photon-number variance to collapse rapidly to its steady-state value of

$$\sigma_\infty^2 = \frac{\mu}{\eta}[-1+\eta+\sqrt{1+q\eta}]. \tag{7.5}$$

If $\eta = 1$ (perfect detection), we have simply

$$\sigma_\infty^2 = \mu\sqrt{1+q}, \tag{7.6}$$

which is to be compared with the nonconditioned variance of $\mu(1+q/2)$. From this it is obvious that the conditioned variance is less than or equal to the unconditioned variance for $\eta = 1$. In fact, this is true for any value of η, and as $\eta \to 0$ we recover the nonconditioned value. If $q = 0$, then the conditioned variance is equal to the nonconditioned variance μ. This is as we concluded earlier: for a Poissonian pumped laser, intensity measurements have no effect. For a sub-Poissonian pumped laser ($q < 0$), the nonclassical nature of the intracavity field is enhanced in the conditioned state. In particular, if $\eta \to 1$ and $q \to -1$, then we get $\sigma_\infty^2 \to 0$. In this regime, our approximations break down, as we have assumed that the variance is always of the same order as the mean. Nevertheless, it indicates that under ideal conditions, the ratio of the variance to the mean in the conditioned state will be arbitrarily small.

Substituting the steady-state value for the conditioned variance into the equation (7.3) for the conditioned mean gives

$$\dot{\bar{n}}_c = -\bar{n}_c(t) + \mu + \sqrt{\frac{\mu}{\eta}}\left[-1+\sqrt{1+q\eta}\right]\xi(t). \tag{7.7}$$

If $\eta = 0$, then there is no noise in the mean, as expected. The same applies if $q = 0$. In general, $\bar{n}_c$ undergoes a random walk with a linear restoring force. That is, it obeys an Ornstein-Uhlenbeck process [18]. Averaging over the ensemble of conditioned states, we have the following steady-state expectation values

$$E(\bar{n}_c) = \mu, \tag{7.8}$$

$$E(\bar{n}_c^2 - \mu^2) = \frac{\mu}{\eta}[1-\sqrt{1+q\eta}+\tfrac{1}{2}q\eta]. \tag{7.9}$$

The total uncertainty in the photon number in the ensemble of conditioned states is equal to the photon-number variance in each conditioned state (7.5) plus the ensemble variance in the mean photon number (7.9),

$$\sigma_\infty^2 + E(\bar{n}_c^2-\mu^2) = \mu\left(1+\frac{q}{2}\right). \tag{7.10}$$

This is precisely equal to the unconditioned variance in the photon number (5.9). Here, this variance has a new interpretation in terms of two components: the quantum variance σ_∞^2 around each conditioned mean $\bar{n}_c$, and the classical (ensemble) variance due to the random walk of $\bar{n}_c$ under photodetection. The relative proportion of these two contributions depends on the efficiency of the detection.

We now examine heterodyne detection on the regular laser. Converting the heterodyne stochastic master equation (3.3) into a linearized stochastic Fokker-Planck equation for the Wigner function gives

$$\begin{aligned}\dot{W}_c(t) = \Big\{&\frac{\partial}{\partial n}(n-\mu) + \frac{1}{2}\mu(2+q)\frac{\partial^2}{\partial n^2} + \frac{1}{2}\frac{1}{2\mu}\frac{\partial^2}{\partial\varphi^2}\\ &+\sqrt{\frac{\eta}{2\mu}}\,\xi_c^\mu(t)\left[n-\bar{n}_c(t)+\mu\frac{\partial}{\partial n}\right]\\ &+\sqrt{2\mu\eta}\,\xi_c^\phi(t)\left[\varphi-\phi_c(t)+\frac{1}{4\mu}\frac{\partial}{\partial\varphi}\right]\Big\}W_c(t),\end{aligned} \tag{7.11}$$

where $\phi_c(t)$ is the mean phase of the Wigner function, and we now have two real white-noise terms,

$$\xi_c^\mu(t) = \mathrm{Re}[\sqrt{2}e^{i\phi_c(t)}\xi(t)], \tag{7.12}$$

$$\xi_c^\phi(t) = \mathrm{Im}[\sqrt{2}e^{i\phi_c(t)}\xi(t)], \tag{7.13}$$

where $\xi(t)$ is the complex white noise in Eq. (3.3). It is evident that, in this linearized regime, the effect of heterodyne detection on the photon-number distribution is the same as the effect of direct photodetection, with the efficiency reduced by a factor of 2. This is not unexpected, as the heterodyne photocurrent contains equal information about the intensity and phase of the cavity mode. The above results for direct photodetection thus go over to heterodyne detection with the replacement of η by $\eta/2$.

The effect of heterodyne detection on the phase is the same as it was in the Poissonian laser. To see this we derive the coupled stochastic differential equations, again using the method of Appendix A

$$\dot{\phi}_c(t) = \sqrt{\eta\mu}\left[2V_c(t)-\frac{1}{2\mu}\right]\xi_c^\phi(t), \tag{7.14}$$

$$\dot{V}_c(t) = \frac{1}{2\mu} - \eta\mu\left[2V_c(t)-\frac{1}{2\mu}\right]^2, \tag{7.15}$$

where $V_c(t)$ is the conditioned Wigner phase variance. These are identical to the equations for the Poisson laser (4.7) when one remembers that $V_c(t) = U_c(t)+1/4\mu$. Assuming that $\eta = 1$, we have a conditional phase variance of

$$V_\infty = \frac{3}{4\mu}. \tag{7.16}$$

Meanwhile, putting $\eta = 1/2$ into Eq. (7.5) gives the heterodyne detection photon-number variance under the same conditions as

$$\sigma_\infty^2 = \mu(\sqrt{2}-1) \simeq 0.41\mu, \tag{7.17}$$

where we have put $q = -1$. Thus the conditioned state for a perfectly regular laser under heterodyne detection is highly nonclassical, and has a Wigner phase-space area of

$$\sqrt{V_\infty \sigma_\infty^2} = \tfrac{1}{2}\sqrt{3(\sqrt{2}-1)} \simeq 0.56. \tag{7.18}$$

This is only just above the minimum of 0.5 allowed by the Heisenberg uncertainty relations for number and phase.

VIII. DISCUSSION

The stochastic dynamics of a continuously monitored laser are essentially given in Sec. VII. There we considered a laser with arbitrary pump statistics under both direct photodetection and heterodyne detection. The picture which emerged is this: With no measurement, the unconditioned state of the laser has a well-defined intensity (with variance determined by the regularity of the pump), but completely undefined phase. Under heterodyne detection, the laser state is collapsed rapidly (on the time scale of the cavity linewidth) to a state with a well-defined phase. For an ideal laser with unit efficiency detection, the steady-state phase variance is three times that of a coherent state. The conditioned mean phase undergoes a random walk. The source of randomness is the shot noise in the heterodyne photocurrent on which the state of the system is conditioned.

For a Poissonian pumped laser, heterodyne detection has no effect on the conditioned photon-number variance, but in all other cases the conditioned variance is less than the unconditioned variance. In particular, the conditional number squeezing in a regularly pumped laser is enhanced. The conditioned mean photon number also has a random trajectory related to the noise in the photocurrent. Unlike heterodyne detection, direct photodetection has no effect on the phase of the laser. To compensate, its effect on the amplitude variance is enhanced by a factor of 2 over that of heterodyne detection. Unit efficiency direct detection of a perfectly regularly pumped laser causes the conditioned photon-number variance to become arbitrarily smaller than that of a coherent state.

An aspect of interest in the analysis of the conditioning of the laser state on external photodetection is that it reinforces the distinction between classical and nonclassical states. By "classical" states we mean those with a positive Glauber-Sudarshan P function [2,3]. The state of an open quantum system conditioned on external photodetection (whether it be direct detection, or with the addition of a local oscillator) will not be nonclassical unless the nonconditioned state is nonclassical. This can be viewed either as a statement about the nature of external photodetection (photodetection cannot produce nonclassical states) or about the definition of coherent states (coherent states are those states which are uninfluenced by photodetection). In this context, it is worth noting that the ideal model of a laser which we developed contains the definition of another quantum optical process which is inherently classical. This process of incoherent excitation, which we have denoted by the superoperator $\mathcal{E}[a^\dagger]$, can be added to the list of classical processes such as damping, detuning, and coherent driving.

Although the results we have derived for the conditioned state of a laser are helpful pedagogically, it is reasonable to ask what practical use they have. Usually, the conditioned state of a system is of little interest to experimenters and some would even doubt whether the concept is meaningful. In fact, the conditioned states such as we have investigated are meaningful experimentally, because they are precisely those states which can be stabilized by using the measured photocurrent (on which they are conditioned) in a feedback loop. The proof of this statement will appear in future work. This implies that the steady-state conditioned variances which we derived here represent the best achievable steady-state variances under feedback. In summary, the practical significance of this work is that conditioning is realized by feedback.

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APPENDIX A: GAUSSIAN SOLUTION TO STOCHASTIC FOKKER-PLANCK EQUATION

Consider a nonlinear Fokker-Planck equation in one dimension with an Ito stochastic term [18] of the form

$$\dot{P}(x,t) = \sqrt{\lambda}\xi(t)[x - \bar{x}(t)]P(x,t), \tag{A1}$$

where $\xi(t)$ represents real Gaussian white noise and

$$\bar{x}(t) = \int dx\, P(x,t)x. \tag{A2}$$

We call Eq. (A1) a Fokker-Planck equation because, if we assume a solution of the form

$$P(x,t) = \frac{1}{\sqrt{2\pi U(t)}} \exp\{-[x-\bar{x}(t)]^2/2U(t)\}, \tag{A3}$$

then we can rewrite Eq. (A1) as

$$\dot{P}(x,t) = -\sqrt{\lambda}\xi(t)U(t)\frac{\partial}{\partial x}P(x,t), \tag{A4}$$

which has the form of a nonlinear drift term. However, it will be seen to have an effect on both the variance and the mean of the distribution. To see this we note that the rules of Ito stochastic calculus give

$$1 + \sqrt{\lambda}dW(t)[x-\bar{x}(t)] = \exp\left\{\sqrt{\lambda}dW(t)[x-\bar{x}(t)] - \frac{\lambda}{2}dt[x-\bar{x}(t)]^2\right\}. \tag{A5}$$

Applying this to the ansatz (A3) and rearranging gives

$$P(x,t+dt) = \frac{1}{\sqrt{2\pi U(t)}} \exp\left\{-[x-\bar{x}(t)]^2\left[\frac{1}{2U(t)}+\frac{\lambda}{2}dt\right]+\sqrt{\lambda}dW[x-\bar{x}(t)]\right\} \tag{A6}$$

$$= \frac{1}{\sqrt{2\pi U(t+dt)}} \exp\{[x-\bar{x}(t+dt)]^2/2U(t+dt)\}, \tag{A7}$$

where we have defined

$$\bar{x}(t+dt) = \bar{x}(t) + \sqrt{\lambda}U(t)dW, \tag{A8}$$

$$U(t+dt) = U(t) - \lambda U(t)^2 dt. \tag{A9}$$

That is to say, the nonlinear stochastic equation (A1) does have a Gaussian solution. The mean obeys the stochastic equation expected from Eq. (A4), and the variance obeys a deterministic equation which causes it always to shrink.

APPENDIX B: WIGNER FUNCTION FOKKER-PLANCK EQUATION FOR REGULARLY PUMPED LASER

We wish to convert the regularly pumped laser master equation

$$\dot{\rho} = \mu\left(\mathcal{E}[a^\dagger] + \frac{q}{2}\mathcal{E}[a^\dagger]^2\right)\rho + \mathcal{D}[a]\rho \tag{B1}$$

into a Fokker-Planck equation for the Wigner function. Recall that we are using the notation

$$\mathcal{E}[a^\dagger] = \mathcal{J}[a^\dagger]\mathcal{A}[a^\dagger]^{-1} - 1\ ,\quad \mathcal{D}[a] = \mathcal{J}[a] - \mathcal{A}[a], \tag{B2}$$

where

$$\mathcal{A}[c]\rho = \tfrac{1}{2}(c^\dagger c\rho + \rho c^\dagger c)\ ,\quad \mathcal{J}[c]\rho = c\rho c^\dagger. \tag{B3}$$

First, consider the density operator σ defined by

$$\rho = \tfrac{1}{2}(aa^\dagger\sigma + \sigma aa^\dagger). \tag{B4}$$

If $W(\alpha,\alpha^*)$ is the Wigner function for ρ and $V(\alpha,\alpha^*)$ is that of σ, then standard operator correspondences [19] give

$$W = \left(|\alpha|^2 + \frac{1}{2} - \frac{1}{4}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)V. \tag{B5}$$

Assuming that the mean photon number is very large, we can invert this differential operator to second order in $|\alpha|^{-2}$ to find

$$V = \left(\frac{1}{|\alpha|^2+\frac{1}{2}} + \frac{1}{2|\alpha|^2}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\frac{1}{2|\alpha|^2}\right)W. \tag{B6}$$

That is to say, we have the following approximate superoperator correspondence:

$$\mathcal{A}[a^\dagger]^{-1}\rho \to \left(\frac{1}{|\alpha|^2+\frac{1}{2}} + \frac{1}{2|\alpha|^2}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\frac{1}{2|\alpha|^2}\right)W. \tag{B7}$$

Next, it is easy to show that

$$\mathcal{J}[a^\dagger]\rho \to \left(|\alpha|^2 + \frac{1}{2} - \frac{1}{2}\frac{\partial}{\partial\alpha}\alpha - \frac{1}{2}\frac{\partial}{\partial\alpha^*}\alpha^* + \frac{1}{4}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)W. \tag{B8}$$

Thus, to second order in the mean photon number we have

$$\mathcal{E}[a^\dagger]\rho \to \frac{1}{2}\left\{\left[\frac{\partial}{\partial\alpha}\left(-\frac{1}{\alpha^*}+\frac{1}{2|\alpha|^2\alpha^*}\right)+\text{c.c.}\right] + \frac{\partial^2}{\partial\alpha\partial\alpha^*}\frac{1}{|\alpha|^2}\right\}W, \tag{B9}$$

and

$$\mathcal{E}[a^\dagger]^2\rho \to \frac{1}{4}\left[\left(\frac{\partial}{\partial\alpha}\frac{1}{|\alpha|^2\alpha^*}+\text{c.c.}\right)+\left(\frac{\partial^2}{\partial\alpha^2}\frac{1}{\alpha^{*2}}+\text{c.c.}\right) + \frac{\partial^2}{\partial\alpha\partial\alpha^*}\frac{2}{|\alpha|^2}\right]W. \tag{B10}$$

Finally, it is a standard result that

$$\mathcal{D}[a]\rho \to \frac{1}{2}\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^* + \frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)W. \tag{B11}$$

Putting all of these results together gives the Fokker-Planck equation quoted in the text (5.12)

$$\dot{W} = \left\{\frac{\partial}{\partial\alpha}\left[\frac{-\mu}{2\alpha^*}+\frac{\alpha}{2}+\left(1+\frac{q}{2}\right)\frac{\mu}{4|\alpha|^2\alpha^*}\right]W + \text{c.c.}\right\} + \frac{1}{2}\left\{\frac{\partial^2}{\partial\alpha^2}\frac{q\mu}{4\alpha^{*2}} + \frac{\partial^2}{\partial\alpha^{*2}}\frac{q\mu}{4\alpha^2} + \frac{\partial^2}{\partial\alpha\partial\alpha^*}\left[1+\left(1+\frac{q}{2}\right)\frac{\mu}{|\alpha|^2}\right]\right\}W. \tag{B12}$$

[1] J.R. Klauder and E.C.G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968).
[2] R.J. Glauber, Phys. Rev. **130**, 2529 (1963); **131**, 2766 (1963).
[3] E.C.G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).
[4] W.H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).
[5] H.M. Wiseman and G.J. Milburn, Phys. Rev. A **47**, 642 (1993).
[6] H.M. Wiseman and G.J. Milburn, Phys. Rev. A **47**, 1652 (1993).
[7] Special issue on squeezed states of the electromagnetic

field of J. Opt. Soc. Am. B **4**, 1453 (1987).
[8] Y. Yamamoto, N. Imoto, and S. Machida, Phys. Rev. A **33**, 3243 (1986).
[9] D.F. Walls, M.J. Collett, and A.S. Lane, Phys. Rev. A **42**, 4366 (1990).
[10] H. Ritsch, Quantum Opt. **2**, 189 (1990).
[11] H.M. Wiseman and G.J. Milburn, Phys. Rev. A **44**, 7815 (1991).
[12] A.M. Khazanov and G.A. Koganov, Phys. Rev. A **42**, 3065 (1990).
[13] T.C. Ralph and C.M. Savage, Opt. Lett. **16**, 1113 (1991).
[14] H. Ritsch, P. Zoller, C.W. Gardiner, and D.F. Walls, Phys. Rev. A **44**, 3361 (1991).
[15] M.A.M. Marte and P. Zoller, Phys. Rev. **40**, 5774 (1989).
[16] F. Haake, S.M. Tan, and D.F. Walls, Phys. Rev. A **40**, 7121 (1989).
[17] J. Gea-Banacloche, in *New Frontiers in Quantum Electrodynamics and Quantum Optics*, edited by A.O. Barut (Plenum, New York, 1990).
[18] C.W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1985).
[19] C.W. Gardiner, *Quantum Noise* (Springer-Verlag, Berlin, 1991).
[20] L. Mandel, Opt. Lett. **4**, 205 (1979).