Correlation-function approach to the momentum diffusion of atoms moving in standing waves

G. S. Agarwal* and K. Mølmer†
Max-Planck-Institut für Quantenoptik, W-8046 Garching, Germany
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We consider the momentum diffusion of atoms moving in a standing-wave laser field. We show how the correlation-function approach as originally applied to atoms at rest can be generalized to derive the velocity dependence of the momentum diffusion coefficient in standing waves, and that it gives results in agreement with the transport-equation approach to laser cooling. As an example we apply our calculations to determine the achievements of laser cooling in intense fields where cooling may occur around a nonvanishing velocity. Here we obtain temperatures which are 30% lower than the corresponding minimum obtained around zero velocity. Our explicit calculations involve usage of the optical Bloch equations, the quantum regression theorem, and the matrix continued-fraction method.

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I. INTRODUCTION

In the so-called semiclassical theories of laser cooling one attempts to derive and to solve an equation of motion involving only the phase-space distribution of atoms in position and momentum. Dealing, on one hand, with a classical description of the center-of-mass atomic motion and, on the other hand, with a quantum description of the atomic excitation and decay processes, the identification of the effects of one on the other presents both a paradigm for quantum physics and a problem of great practical interest. For these reasons it is satisfactory that by taking different approaches to the problem one arrives at identical expressions for the quantities of importance, the mean force and the momentum diffusion coefficient.

Two different approaches to the problem have been used: (a) Minogin [1] and more recently Javanainen [2] were able, under suitable approximations, to transform the transport equation for the Wigner function into a Fokker-Planck equation involving only the atomic center-of-mass phase-space distribution. The force and diffusion coefficient are easily identifiable quantities in this equation. Straightforward methods aiming directly at obtaining the velocity-dependent force and diffusion coefficient have been presented by Berg-Sørensen *et al.* [3]. (b) A correlation-function approach to momentum diffusion used by Cook [4] and by Gordon and Ashkin [5] has been proven by Dalibard and Cohen-Tannoudji [6] to be equivalent to the transport-equation treatment for atoms *at rest.*

In this paper we extend the work on the correlation-function approach and show that this represents an alternative derivation of the diffusion coefficient for moving atoms in a standing-wave field.

In a first approximation one has often considered only the linear part of the force, $F = -\alpha v$, and only the *zero-velocity* diffusion coefficient. These quantities lead to a Gaussian velocity distribution characterized by a temperature

$$k_B T = M\langle v^2\rangle = D/\alpha. \tag{1}$$

For a velocity distribution that probes the full velocity dependence of F and D this is far from sufficient. This was discussed by Berg-Sørensen *et al.* [3], and comparisons between the solutions to the Fokker-Planck equation, keeping $F(v)$ and $D(v)$, and exact quantum calculations were made. In particular for lighter atoms the velocity dependence of the diffusion coefficient and the force is important. Other situations where the velocity dependence of F and D is important arise even for heavier atoms when there is a nonzero stable velocity $F(v_0) = 0$ and where atoms may be cooled to velocities around v_0. To determine the temperature, one must calculate the slope of the force and the diffusion coefficient at this particular velocity $k_B T = D(v_0)/\alpha(v_0)$. This is, for example, the case for the studies of cooling around the so-called doppleron (velocity-tuned) resonances [7].

In Sec. II we present the relation between the diffusion coefficient and the two-time correlation functions associated with the fluctuations of the atomic dipole in a standing-wave laser field. In Sec. III we develop the framework for the evaluation of such correlation functions, and in Sec. IV we apply the formulation of Sec. III to examine cooling around a doppleron velocity in intense fields.

II. DIFFUSION COEFFICIENT AS A TWO-TIME CORRELATION FUNCTION

Consider a two-level atom with transition frequency ω_0, interacting with a standing-wave laser field of frequency ω. In addition the atom interacts with the vacuum field $\mathbf{E}_V(\mathbf{r})$. This accounts for the effects of spon-

taneous emission. The total Hamiltonian can be written as

$$H = \frac{P^2}{2M} + \frac{\hbar\omega_0}{2}(|e\rangle\langle e| - |g\rangle\langle g|) - \mathbf{d}\cdot\mathbf{E_0}\cos(kz)\cos(\omega t)(|e\rangle\langle g| + |g\rangle\langle e|) - \mathbf{d}\cdot\mathbf{E}_V(|e\rangle\langle g| + |g\rangle\langle e|) + H_{0R}, \quad (2)$$

where $\mathbf{d}$ is the dipole matrix element, $|e\rangle$ and $|g\rangle$ are the excited and ground states of the atom, $\frac{P^2}{2M}$ describes the center-of-mass motion, and H_{0R} gives the free Hamiltonian of the radiation field. We make the transformation to a frame rotating with the frequency ω and we drop all counter-rotating terms; then (2) reduces to

$$H = \frac{P^2}{2M} - \hbar\delta\left(\frac{|e\rangle\langle e| - |g\rangle\langle g|}{2}\right) - \frac{\hbar\Omega}{2}\cos(kz)(|e\rangle\langle g| + |g\rangle\langle e|) - (\mathbf{d}\cdot\mathbf{E}_V^{+}|e\rangle\langle g|e^{i\omega t} + \text{H.c.}) + H_{0R}, \quad (3)$$

where $\mathbf{E}_V^{+(-)}$ are the positive and the negative frequency components of the vacuum field. The detuning δ and the Rabi frequency Ω are defined by

$$\delta = \omega - \omega_0, \quad \Omega = \frac{\mathbf{d}\cdot\mathbf{E}_0}{\hbar}. \quad (4)$$

Let us now determine the force acting on the atom. Using (3) we obtain the Heisenberg equation for the momentum operator

$$\dot{P}_z = \frac{dP_z}{dt} = \frac{1}{i\hbar}[P_z, H] \equiv F_L + F_V, \quad (5)$$

$$F_L = -\frac{\hbar k\Omega}{2}\sin(kz)(|e\rangle\langle g| + |g\rangle\langle e|), \quad (6)$$

$$F_V = \frac{\partial}{\partial z}(\mathbf{d}\cdot\mathbf{E}_V^{+})e^{i\omega t}|e\rangle\langle g| + \text{H.c.} \quad (7)$$

At this stage we make the semiclassical approximation of taking the trajectory of the atoms as given by $z = vt$. Thus the mean value of the force is essentially determined by the dynamics of the internal variables like the dipole moment of the atom. Similarly the fluctuations in the force will be determined by the fluctuations in the dipole moment. The mean value of F_V vanishes. This is because the vacuum field involves a sum over all plane-wave modes, and by symmetry the average of $\mathbf{k}$ is zero.

From now on we will omit the suffix z from the force, etc. Thus

$$\langle F_z\rangle = \langle F\rangle = -\frac{\hbar k\Omega}{2}\sin kz(\rho_{ge} + \rho_{eg}), \quad (8)$$

where $\rho_{\alpha\beta}$ are the matrix elements of the density operator for the atomic system. Let us denote the quantum fluctuations in any quantity by

$$\Delta A = A - \langle A\rangle. \quad (9)$$

Then the diffusion coefficient is defined by

$$2Dt = \lim_{t\to\infty}\langle\Delta P^2(t)\rangle. \quad (10)$$

Using Eqs. (5) and (10) we therefore obtain

$$2D = \lim_{t\to\infty}\frac{\partial}{\partial t}\langle\Delta P^2(t)\rangle = \lim_{t\to\infty}\int_0^t d\tau\langle\Delta F(t)\Delta F(t-\tau) + \Delta F(t-\tau)\Delta F(t)\rangle \equiv 2D_L + 2D_V, \quad (11)$$

where D_V is the contribution from the vacuum modes [F_V term in (5)] and is known [4,6] to be given by

$$2D_V = \frac{2}{5}(\hbar k)^2\Gamma\rho_{ee}(t). \quad (12)$$

Here Γ is the rate of decay of the excited state to the ground state. Note that (12) has a simple interpretation in terms of the directional distribution of emitted photons.

The cross terms like $\langle F_L F_V\rangle$ do not contribute due to symmetry [8], and we are now left with the problem of evaluating the correlation function of F_L. Note that we are assuming the classical trajectory of the atom and hence the dynamics of the internal degrees of freedom can be described by the optical Bloch equations. On defining

$$u = \langle U\rangle = \left\langle\frac{1}{2}(|e\rangle\langle g| + |g\rangle\langle e|)\right\rangle = \frac{1}{2}(\rho_{ge} + \rho_{eg}),$$
$$v = \langle V\rangle = \left\langle\frac{1}{2i}(|e\rangle\langle g| - |g\rangle\langle e|)\right\rangle = \frac{1}{2i}(\rho_{ge} - \rho_{eg}), \quad (13)$$
$$w = \langle W\rangle = \left\langle\frac{1}{2}(|e\rangle\langle e| - |g\rangle\langle g|)\right\rangle = \frac{1}{2}(\rho_{ee} - \rho_{gg}),$$

the optical Bloch equations can be written as

$$\frac{du}{dt} = \delta v - \frac{\Gamma}{2}u,$$
$$\frac{dv}{dt} = -\delta u - \frac{\Gamma}{2}v + \Omega\cos(kz)w, \quad (14)$$
$$\frac{dw}{dt} = -\Gamma w - \frac{\Gamma}{2} - \Omega\cos(kz)v.$$

These are the equations for a static atom at the position z. For a moving atom with velocity v, the position z is replaced by vt.

With the notation of (13) the mean force (8) can be written as

$$\langle F_L\rangle = -\hbar k\Omega\ \sin(kvt)u(t). \quad (15)$$

The contribution D_L to diffusion using (6), (11), and (13) becomes

$$2D_L = \lim_{t\to\infty}\int_0^t d\tau(\hbar k\Omega)^2\sin(kvt)\sin[kv(t-\tau)] \times\langle\{\Delta U(t), \Delta U(t-\tau)\}\rangle. \quad (16)$$

Thus the mean force and the diffusion can be obtained from the knowledge of the solution of (14). This is because the two-time correlation function $\langle\{\Delta U(t),\Delta U(t-\tau)\}\rangle \equiv \langle\Delta U(t)\Delta U(t-\tau)+\Delta U(t-\tau)\Delta U(t)\rangle$ appearing in (16) can be evaluated from the solution of (14) and the quantum regression theorem [9]. Note that for a moving atom $2D_L$ in (16) does not converge to a constant value but to a periodic function as $t\to\infty$. In Sec. III we shall obtain the mean value of this function.

Most previous work has evaluated (16) for *atoms at rest*. We illustrate the procedure in order to indicate how it works. First, we rewrite (15) and (16) for atoms at rest,

$$\langle F_L\rangle = -\hbar k\Omega\sin(kz)u, \tag{17}$$

$$2D_L = (\hbar k\Omega)^2\sin^2 kz \times\lim_{t\to\infty}\int_0^t d\tau\langle\{\Delta U(t),\Delta U(t-\tau)\}\rangle. \tag{18}$$

If we write the optical Bloch equations as

$$\frac{d\psi}{dt} = M\psi + I,\ \psi = (u,v,w)^{\mathrm{tr}}, \tag{19}$$

where

$$M = \begin{pmatrix} -\frac{\Gamma}{2} & \delta & 0 \\ -\delta & -\frac{\Gamma}{2} & \Omega\cos(kz) \\ 0 & -\Omega\cos(kz) & -\Gamma \end{pmatrix}, \qquad I = \begin{pmatrix} 0 \\ 0 \\ -\frac{\Gamma}{2}, \end{pmatrix}, \tag{20}$$

then in the stationary state

$$M\psi + I = 0 \Rightarrow u = -(M^{-1}I)_1 = (M^{-1})_{13}\frac{\Gamma}{2}. \tag{21}$$

Note that with the definitions (13) the optical Bloch equations (14) and (19) can be viewed as equations for the mean values of the time-dependent (Heisenberg picture) operators (U,V,W). The fluctuations $(\Delta U,\Delta V,\Delta W)$ will therefore have mean values formally obeying (19) without the inhomogeneous term. Now, these quantities identically vanish for all times, but using the quantum regression theorem [9], we obtain that the correlation matrix [where the initial time $t-\tau$ is considered as fixed, i.e., we are considering the time evolution from $(t-\tau)$ to t]

$$\Psi(t,t-\tau) \equiv \begin{pmatrix} \langle\{\Delta U(t),\Delta U(t-\tau)\}\rangle \\ \langle\{\Delta V(t),\Delta U(t-\tau)\}\rangle \\ \langle\{\Delta W(t),\Delta U(t-\tau)\}\rangle \end{pmatrix} \tag{22}$$

satisfies the same set of homogeneous equations

$$\frac{d\Psi}{dt} = M\Psi. \tag{23}$$

The initial conditions on Ψ, i.e., values at $t\to(t-\tau)$ with $(t-\tau)$ fixed, can be obtained from the equal-time algebra of the atomic operators:

$$\Psi(t-\tau,t-\tau) = \begin{pmatrix} \langle\{U,U\}\rangle - 2\langle U\rangle\langle U\rangle \\ \langle\{V,U\}\rangle - 2\langle V\rangle\langle U\rangle \\ \langle\{W,U\}\rangle - 2\langle W\rangle\langle U\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2}-2u^2 \\ -2uv \\ -2uw \end{pmatrix}, \tag{24}$$

where we take the stationary values [from (21)] for (u,v,w). Equations (23) and (24) enable us to calculate the correlation function appearing in (18). However, according to (16) we do not need the explicit t dependence of the correlation matrix Ψ but only its integral over time. Using the value (24) for $\Psi(t-\tau,t-\tau)$ and the fact that the correlations (22) vanish in the limit of large time separation, we obtain from an explicit integration of (23), with respect to the later time t from $t-\tau$ to ∞ that D_L can be written as the first component of the algebraic solution

$$2D_L = -(\hbar k\Omega)^2\sin^2(kz)[M^{-1}\Psi(t-\tau,t-\tau)]_1. \tag{25}$$

It should be borne in mind that M, u, v, and w are functions of position through the $\cos kz$ term in (20). The above calculations (inversions of 3×3 matrices) are indeed not very complicated to carry out analytically [4,5] and the procedure easily generalizes to more complicated atomic transitions and laser configurations, e.g., in two and three dimensions [10].

III. DIFFUSION OF MOVING ATOMS

In this section we consider the application of the formalism of Sec. II to evaluate (15) and (16) for moving atoms. We note that the optical Bloch equations now involve terms which are periodic in time. The solutions of the optical Bloch equations with periodic terms have been studied extensively [11,12]. Also the fluctuations have been studied in order to obtain the spectrum of resonance fluorescence [13], for example, from atoms moving in standing-wave laser fields.

For moving atoms, $z=vt$, we write (19) in the form

$$\frac{d\psi}{dt} = M_0\psi + M_+e^{-ikvt}\psi + M_-e^{ikvt}\psi + I, \tag{26}$$

where

$$M_0 = \begin{pmatrix} -\frac{\Gamma}{2} & \delta & 0 \\ -\delta & -\frac{\Gamma}{2} & 0 \\ 0 & 0 & -\Gamma \end{pmatrix}, \qquad M_+ = M_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\Omega}{2} \\ 0 & -\frac{\Omega}{2} & 0 \end{pmatrix}. \tag{27}$$

The equation for the correlation matrix (22) is written in the similar form

$$\frac{d\Psi}{dt} = M_0\Psi + M_+e^{-ikvt}\Psi + M_-e^{ikvt}\Psi. \tag{28}$$

The periodic steady-state solution of (26) has the form

$$\psi(t) = \sum_{n=-\infty}^{\infty} \psi^{(n)}e^{inkvt}, \tag{29}$$

where the column $\psi^{(n)}$ (for each n value) has three components. All components of ψ are real and hence

$$(\psi^{(n)})^* = \psi^{(-n)}. \tag{30}$$

On using (29) and (30) in (15), and taking the time average we obtain

$$\langle F_L\rangle = \hbar k\Omega\,\mathrm{Im}(\psi_1^{(1)}). \tag{31}$$

The time average of D_V [defined by Eq. (12)] can also be written in terms of ψ:

$$2D_V = \tfrac{2}{5}(\hbar k)^2\Gamma \lim_{t\to\infty}(\tfrac{1}{2} + \langle w(t)\rangle) = \tfrac{2}{5}(\hbar k)^2\Gamma(\tfrac{1}{2} + \psi_3^{(0)}). \tag{32}$$

Thus the average force and diffusion D_V are known in terms of the Fourier components $\psi^{(n)}$.

To obtain the Fourier components $\psi^{(n)}$ we insert the decomposition (29) in (26) to obtain the recurrence relation, where δ_{n0} is the Kronecker delta,

$$0 = (M_0 - inkv)\psi^{(n)} + M_+\psi^{(n+1)} + M_-\psi^{(n-1)} + I\cdot\delta_{n0}. \tag{33}$$

The three term recursion relation (33) can be solved using matrix continued fractions [14]. By applying the ansatz

$$\psi^{(n)} = H_{n-1}\psi^{(n-1)} + r_n, \tag{34}$$

one obtains from Eq. (33) the equations for H_{n-1} and r_n:

$$H_{n-1} = -(M_0 - inkv + M_+H_n)^{-1}M_-,$$
$$r_n = -(M_0 - inkv + M_+H_n)^{-1}(M_+r_{n+1} + I\cdot\delta_{n0}). \tag{35}$$

By assuming that H_N and r_{N+1} vanish for sufficiently large N one can iterate these equations towards $n = 0$ where Eq. (30) allows us to terminate the resulting continued fraction and obtain an equation for $\psi^{(0)}$ and all higher Fourier components using (34). For details see the appendix of Ref. [3].

We now examine the solution of (28) for the two-time correlation functions $\Psi(t, t-\tau)$. We cannot, as in Sec. II, simply assume the stationary state at the initial time $t-\tau$ which would make Ψ a function of the time difference $\tau = t-(t-\tau)$ only. Instead we note that the system has a periodic steady state so that $\Psi(t, t-\tau)$ is a function which depends periodically on the location of the time interval $(t-\tau, t)$, and in a way that can be determined from the quantum regression theorem, on its duration τ. The two-time correlation matrix will therefore have a Fourier expansion, e.g., with respect to the later time,

$$\Psi(t, t-\tau) = \sum_{n=-\infty}^{\infty} \Psi^{(n)}(t-(t-\tau))e^{inkvt}, \tag{36}$$

where the expansion coefficients depend on the duration of the interval $(t-\tau, t)$. The time derivative in (28) is with respect to the later time t, assuming the initial time $t-\tau$ to be fixed, that is, it represents changes due to both a variation in t and a variation in the duration τ. Applying this to Eq. (36), we obtain terms from the Fourier coefficients, which can be written $(d\Psi^{(n)}/d\tau)\exp(inkvt)$, and terms from the exponentials $inkv\Psi^{(n)}\exp(inkvt)$. The right-hand side of (28) describes the evolution at time t, and rearranging terms we obtain the equations for $\Psi^{(n)}(\tau)$:

$$\frac{d\Psi^{(n)}}{d\tau} = (M_0 - inkv)\Psi^{(n)} + M_+\Psi^{(n+1)} + M_-\Psi^{(n-1)}. \tag{37}$$

As in the steps leading to (33) we have transformed a periodic time-dependent problem in three variables into a time-independent one in a higher number of variables (Fourier coefficients). By arrangement of all these variables in one single vector, (26) and (37) can be visualized as large sets of linear equations with all the M matrices put together in one single matrix with the M_0's $(-inkv)$ along the diagonal and the $M_\pm$'s just above or below the diagonal. The matrix continued-fraction method is convenient to invert a (block-) tridiagonal matrix problem such as (33) and (39) below.

Like for atoms at rest, we do not need the full τ dependence of the correlation matrix but only its integral, and an explicit integration of (37) yields for the integrated components

$$\hat{\Psi}^{(n)} \equiv \int_0^\infty \Psi^{(n)}(\tau)d\tau \tag{38}$$

the recurrence relations

$$-\Psi^{(n)}(0) = (M_0 - inkv)\hat{\Psi}^{(n)} + M_+\hat{\Psi}^{(n+1)} + M_-\hat{\Psi}^{(n-1)}, \tag{39}$$

where $\Psi^{(n)}(0)$ are determined from the initial values of the two-time correlation functions

$$\Psi(t-\tau, t-\tau) = \sum_{n=-\infty}^{\infty} \Psi^{(n)}(0)e^{inkv(t-\tau)}. \tag{40}$$

To obtain $\Psi^{(n)}(0)$ we use Eq. (24), substitute the solution of (33) for u, v and w and rearrange the resulting double series in the form of (40). The $\hat{\Psi}^{(n)}$'s are thus determined from the solution of the recurrence relations (39). These inhomogeneous equations can also be solved in terms of the matrix continued fractions Eqs. (34) and (35) where the $I\cdot\delta_{n0}$ term now is to be replaced by the set of coefficients obtained from Eq. (40).

In order to evaluate the diffusion coefficient (16) we have to consider the modified correlation function

$$\chi(t,t-\tau) \equiv \Psi(t,t-\tau)\sin[kv(t-\tau)]. \tag{41}$$

The correlations χ (for fixed t-τ) satisfy the same equation as Ψ, but when applying the analog of (39) we must use different initial conditions

$$\chi(t-\tau,t-\tau) = \sum_{n=-\infty}^{\infty} \chi^{(n)}(0)e^{inkv(t-\tau)}, \tag{42}$$

as $\chi(t-\tau,t-\tau)$ is given by the right-hand side of (24) multiplied by $\sin kv(t-\tau)$. Using the same notation for χ as for Ψ we obtain for the average D_L (16),

$$2D_L = (\hbar k\Omega)^2 \mathrm{Im}(\hat{\chi}_1^{(1)}). \tag{43}$$

To summarize, the recurrence relation (33) is solved to obtain the average force F_L and spontaneous diffusion coefficient D_V [Eqs. (31) and (32)]. The solution of (33) is also used, via (24), to obtain the coefficients $\chi^{(n)}(0)$ (41) and (42). These coefficients are then used in an equation similar to (39) to obtain the integrated $\hat{\chi}^{(n)}$, and finally one obtains the diffusion coefficient D_L. All quantities like the $\psi^{(n)}$'s and the $\chi^{(n)}$'s are functions of velocity due to the presence of v in the recurrence relations.

As an application of the procedures, just outlined, we present in Fig. 1 the velocity dependence of both the force F and the diffusion coefficient $D = D_L + D_V$. For the field parameters we assume $\Omega = 6\sqrt{2}\Gamma$ and $\delta = -3\Gamma$.

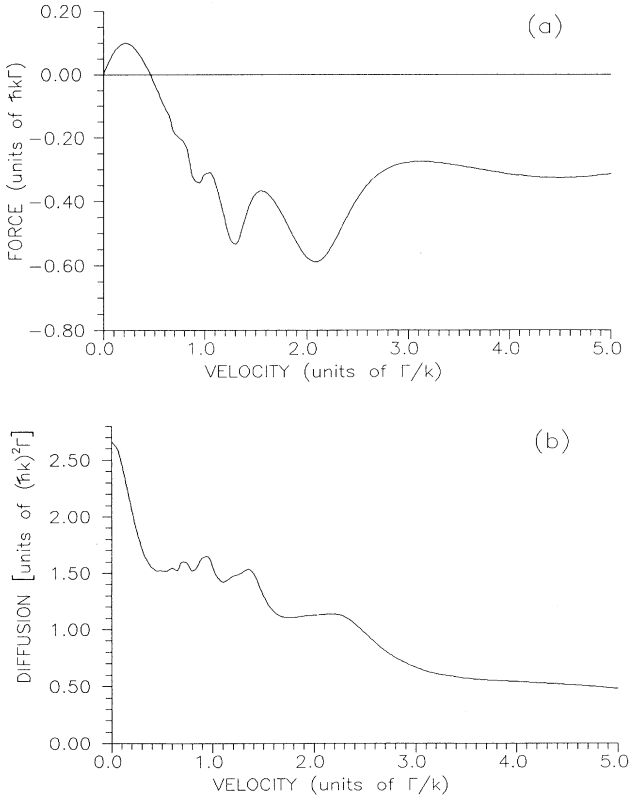

FIG. 1. (a) Force and (b) diffusion coefficient, shown as functions of the atomic velocity for the field parameters $\Omega = 6\sqrt{2}\Gamma$, $\delta = -3\Gamma$.

Our results coincide with the ones obtained by Berg-Sørensen *et al.* [3] based on a derivation from the transport equation approach. As a matter of fact, the equations applied by these authors for the zeroth moment (with respect to momentum) of the density matrix including also the atomic center-of-mass variables coincide with the equations for our (u,v,w) (14) and (19), and their equations for the Fourier coefficents of a suitably defined first moment are identical (except for a factor of $-\hbar k\Omega/2$) to the equations for our $\hat{\chi}^{(n)}$. The difference in spirit between the transport-theory approach and the correlation function approach is that one is performed in a Schrödinger-like picture in which the system evolves, the other in a Heisenberg-like picture where the evolution of operators is considered. With the transport equations one determines the evolution of the system described by its density matrix (represented in [3] by a Wigner matrix), and then one "measures" the corresponding changes in momentum and momentum spread. The approach presented here attributes such changes to the mean value and the fluctuations of a known force operator, which are then calculated directly.

It should be mentioned that the works of Minogin [1] and of Javanainen [2] do not only address the values of F and D, but attempt at deriving the equation of evolution for the atomic phase-space density. The Fokker-Planck equation, assumed in this work and in Ref. [3], is itself a result of their derivation.

IV. COOLING AROUND A DOPPLERON VELOCITY

Having obtained the velocity dependence of the diffusion coefficient and the force, we can apply these to determine the accomplishments of laser cooling in different situations. We shall here present an application to cooling around a nonzero velocity.

In Fig. 1(a), we observe the so-called *doppleron* structures in the laser cooling force. In a perturbative treatment these correspond to higher-order processes: $(n+1)$ absorptions from one and n stimulated emissions into the other traveling-wave component of the field is resonant at $(2n+1)kv = \delta$ [15]. The force is an odd function of the detuning δ, and over a small velocity range around zero one obtains cooling with a laser detuned to the *blue* side, which is in contrast to the usual scattering-force picture of laser cooling. This velocity range gets larger when the Rabi frequency and detuning increase, and in the limit of very intense fields the force and the diffusion are conveniently described in the dressed-atom picture [16]. Here the doppleron structures can be interpreted as Stueckelberg oscillations due to diabatic (Landau-Zener) transitions between the position-dependent dressed states [17,18].

Of our interest here is that for a range of Rabi frequencies and negative-frequency detunings the force is not a cooling force around zero velocity [11]. Instead, we observe a stable nonzero velocity ($v_0 \simeq 0.45\Gamma/k$ for the pa-

rameters studied in Fig. 1), and sufficiently heavy atoms may be cooled around this velocity. Since the force is an odd function of velocity, another fraction of the atoms may be cooled around $v = -v_0$. To apply Eq. (1) we need the slope of the force and the diffusion coefficient at $v = v_0$. Both quantities are available from the curves in Fig. 1, and as we see from Fig. 1(b), the relevant diffusion coefficient is smaller than the $v = 0$ result by nearly a factor of 2. This also suggests that cooling around such a doppleron velocity may be more efficient than around $v = 0$. To study this possibility we considered also the friction and diffusion coefficients at zero velocity. Experiments investigating the effects of the doppleron structure on laser cooling have been performed [7]. We present some results relevant to such experiments: with a fixed Rabi frequency we determine the stable velocity v_0 as a function of detuning δ [Fig. 2(a)], and from the corresponding slope $\alpha = -(\partial F/\partial v)|_{v_0}$ and the diffusion coefficient $D(v_0)$ we obtain the temperature shown as a solid curve in Fig. 2(b). The dashed curve shows the temperatures around zero velocity obtained with the same Rabi frequency but with positive detunings ($\delta \rightarrow -\delta$). The velocity spread corresponding to the temperature $k_B T$ should be sufficiently small that the atoms stay within the range of the linear force; since both $F(0)$ and $F(v_0)$ vanish this suggests that at least $\sqrt{\langle v^2 \rangle} < v_0$ be required for the applicability of the results of Fig. 2(b). Equation (1) makes this a requirement on the atomic mass.

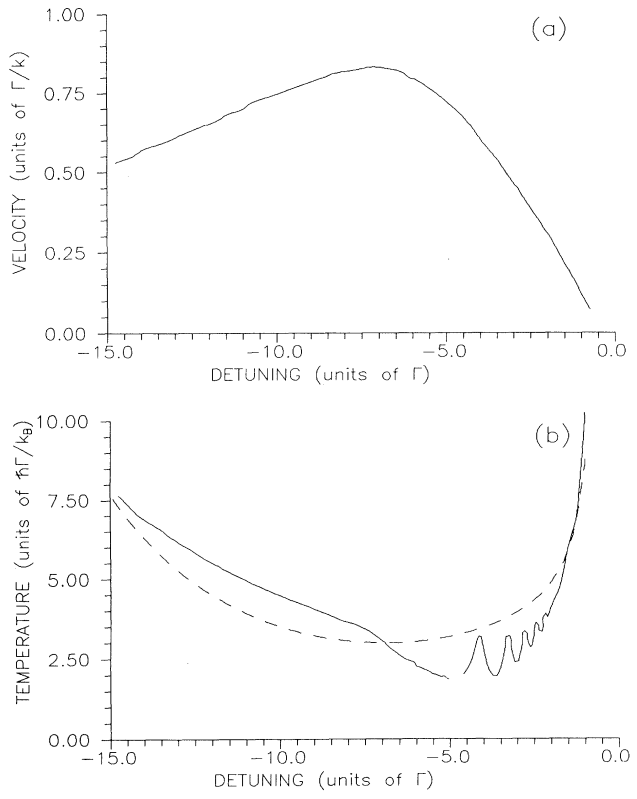

FIG. 2. (a) Stable velocity v_0 as a function of frequency detuning δ. (b) Temperature T around the velocity v_0 as a function of detuning δ (solid line). The dashed curve shows the temperatures around zero velocity obtained with a detuning of opposite sign (see text). The Rabi frequency Ω equals 10Γ.

V. DISCUSSION

We have presented a derivation of the momentum diffusion coefficient for atoms moving in a standing-wave laser field based on the two-time correlation functions of the force operator. Considering a Fourier expansion of the relevant equations we obtained a set of recurrence relations which lead directly to a matrix continued fraction solution for the quantities of interest, and this provides a very efficient method for the numerical treatment of the problem compared, e.g., with the straightforward numerical integration of the corresponding differential equations. This is largely so because the solutions (ψ and χ) are smooth and well-behaved functions and hence their Fourier series converge rapidly, whereas the differential equations have to be integrated over a period of the field, i.e., for a time $T = \lambda/v$, with a time step small compared to Ω^{-1}, δ^{-1}, and Γ^{-1}, which in particular for low velocities may be very time consuming.

The velocity dependence of the diffusion coefficient and of the force are of relevance to the achievements of laser cooling. In the general case, a Fokker-Planck equation must be solved to obtain the atomic velocity distribution, and as seen from Eq. (1), in particular lighter atoms will be exposed to the variation of F and D with velocity. One comment should be made at this point: the semiclassical analysis of laser cooling requires the atoms to be heavy, i.e., the dimensionless quantity $M\Gamma/\hbar k^2$ must be somewhat larger than unity. This enters the approximations in two different ways. First, via the assumption of a constant velocity, the force (of order $\hbar k\Gamma$) must not accelerate the atom substantially (of order Γ/k) during the time ($\simeq \Gamma^{-1}$) required by the mean values and correlation functions to reach their steady state. Second, we want simultaneously to be able to define the atomic position well within a wavelength ($2\pi/k$) and the velocity on the scale of Γ/k, without violating Heisenberg's uncertainty principle. Note that the quantity $M\Gamma/\hbar k^2$ is a constant specific to the atom and the cooling transition; in Ref. [3], with a value of 19 for this constant, corresponding to a transition in the Li^+-ion, the Fokker-Planck equation with F and D of Fig. 1 gave results in excellent agreement with an exact quantum treatment. Since the force is heating around zero velocity, Eq. (1) is meaningless in this case.

Like in most discussions of laser cooling, only the *position-averaged* force and diffusion coefficients are presented. In case one wishes to retain the full position dependence, e.g., in studies of localization effects in laser cooling [19], it is available from the higher-order coefficients in the Fourier expansions of Sec. III.

Finally, it should also be noted that the semiclassical treatment of laser cooling encounters fundamental difficulties in a number of situations where the time-scale separation between internal and external evolution (mentioned briefly above in connection with the atomic mass

criterion) breaks down [20]. Even with the recent advances in the full quantum treatments of laser cooling [21,22], one may still, however, anticipate an important role for the force and the diffusion coefficients providing both reasonable quantitative approximations to the exact results and means for interpretation of characteristic features in laser cooling.

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* Permanent address: School of Physics, University of Hyderabad, Hyderabad 500 134, India.
† Permanent address: Institute of Physics and Astronomy, Århus University, DK-8000 Århus C, Denmark.

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