# Lie-algebra methods in quantum optics: The Liouville-space formulation

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(Received 14 January 1993)

Lie-algebra methods for investigating quantum optical systems are presented within the framework of the Liouville-space formulation. The generalized decomposition formulas for exponential functions of the generators of su(2) and su(1,1) Lie algebras are derived and their expectation values are calculated for typical states in quantum optics. The general procedure for using Lie algebras in the Liouville space to treat quantum optical processes is given in terms of generalized decomposition formulas and their use is demonstrated by calculating the absorption line shape and photon echo signal in a localized electron-phonon system. It is also shown that the photon-counting probability can be calculated by using the su(1,1) Lie algebra and that the electron-counting probability can be calculated by using the su(2) Lie algebra. The su(1,1) Lie algebra is also used to investigate a quantum-nondemolition measurement of photon number in the four-wave-mixing model.

PACS number(s): 42.50.Hz, 03.65.Fd, 42.50.Dv

## I. INTRODUCTION

Many kinds of phenomena in quantum optical systems can be studied in terms of the su(2) and su(1,1) Lie algebras. The quantum correlation, phase coherence, and squeezing of photons, for example, have been investigated intensively by using the su(1,1) and su(2) algebras and the generalized coherent states associated with these algebras [1]. The degenerate and nondegenerate parametric amplifiers which generate the quadrature squeezed state are typical systems described by the su(1,1) Lie algebra [2,3]. The time-evolution equations of various optical systems have been solved by using the su(1,1) algebra [4] following the Wei-Norman method [5,6] and the Magnus method [6,7]. The quantum coherence in the two-photon Jaynes-Cummings model is also investigated by using the su(1,1) Lie algebra [8]. The SU(1,1) generalized coherent states, such as the pair coherent state by Barut and Girardello [9] and the correlated two-photon coherent state by Perelomov [10,11], are important in this kind of analysis, and their phase properties have been studied by several authors [12].

The beam splitter [13], and the Mach-Zehnder and Fabry-Pérot interferometers [14] which are key components in interferometric experiments, have been described in terms of the su(2) Lie algebra. The linear directional coupler has also been investigated by using the su(2) Lie algebra [15]. The nonclassical properties of light in these devices are important in quantum communication and in high-precision measurement, such as the detection of gravitational waves. The su(2) Lie algebra is concerned with a rotational transformation [11,16], and the functions of the beam splitter, interferometer, and directional coupler are expressed as rotations in abstract space.

The phase operators for the su(1,1) and su(2) Lie algebras, which are generalizations of the harmonic oscillator phase [17], have been studied from mathematical viewpoints [18]. The phase operator plays an important role in investigating nonclassical properties of lights. Optical phenomena are also described by the Lie algebras such as nilpotent Heisenberg-Wyle algebra and symplectic algebra [19]. Digital or analog signal processing, Fourier optics, and so on are successfully described by the Lie algebras.

The Liouville-space formulation is a powerful method for describing physical systems [20]. Thermofield dynamics, which is a real-time quantum-field theory with finite temperature, has been constructed within the framework of the Liouville-space formulation [21]. Thermofield dynamics has recently been generalized to nonequilibrium conditions, thus enabling us to treat nonequilibrium dissipative processes [22]. When we investigate a dissipative process caused by a thermal reservoir, nonequilibrium thermofield dynamics (or the Liouville-space formulation) makes the manipulation of time-evolution equations much easier than it is with the usual methods such as the damping theory [23]. The combination of the Lie algebra and the Liouville-space methods is therefore useful to us. I have recently shown that we can investigate a wider class of physical systems by the su(2) or su(1,1) Lie algebras by describing them in the Liouville space [24]. Furthermore, the dissipative dynamics of nonlinear optical systems is also described by the Lie algebra within the framework of the Liouville-space formulation (see below) [24,25]. A photon echo phenomenon in a localized electron-phonon system, for example, can be treated by the su(1,1) Lie algebra. The photon-counting processes in the Srinivas and Davies model [26] are also investigated by using the su(1,1) Lie algebra. Furthermore, this paper will show that the electron-counting probability can be considered by modifying the Srinivas and Davies model and using su(2) Lie algebra. These problems are the main subject of this paper.

The paper is organized as follows: Section II briefly summarizes the decomposition formulas (or Baker-Campbell-Hausdorff formulas) for su(2) and su(1,1) Lie algebras. This section also explains the formulation of the Liouville space based on the tilde conjugation of operators [21]. Section III derives the generalized normal-order and antinormal-order decomposition formulas for exponential functions of the generators of su(2)

1050-2947/93/47(6)/5093(27)/$06.00  

and su(1,1) Lie algebras. These formulas are useful when we investigate nonlinear optical processes. The expectation values of the generalized formulas are calculated for typical states in quantum optics. These include the vacuum state, the Glauber coherent state, and the SU(2) and SU(1,1) generalized coherent states. The canonical average is also calculated. Section IV uses the su(1,1) Lie algebra in the Liouville space to evaluate the optical processes in a localized electron-phonon system. After describing the general procedure, this section calculates the absorption line shape and photon echo signal that occur in a localized electron-phonon system. Section V investigates quantum counting processes. The photon-counting probability is calculated by using the su(1,1) Lie algebra, and the electron-counting probability by using the su(2) Lie algebra. Using the Liouville formulation simplifies calculation of the quantum counting probability. This section also uses the su(1,1) Lie algebra to investigate quantum-nondemolition (QND) measurement of photon numbers in the four-wave-mixing model. Section VI summarizes this paper.

## II. MATHEMATICAL BASIS

### A. su(1,1) and su(2) algebras

This subsection first briefly summarizes the decomposition formulas of exponential functions of the generators of the su(1,1) and su(2) Lie algebras. These formulas are called Baker-Campbell-Hausdorff formulas, and they are used to derive several useful relations of the su(1,1) and su(2) Lie algebras frequently used to investigate properties of quantum optical systems. Let us consider operators $K_+$, $K_-$, and $K_0$ satisfying the commutation relations

$$[K_-,K_+]=2\sigma K_0 , \tag{2.1}$$

$$[K_0,K_\pm]=\pm K_\pm , \tag{2.2}$$

with $\sigma=\pm 1$. When $\sigma=1$, $\{K_+,K_-,K_0\}$ are the generators of the su(1,1) Lie algebra, and when $\sigma=-1$, they are the generators of the su(2) Lie algebra. In the following sections, we will consider quantities expressed as

$$\prod_{k=1}^{n} \exp[a_+(k)K_+ + a_-(k)K_- + a_0(k)K_0] ,$$

where $a_\pm(k)$ and $a_0(k)$ are $c$-number functions. Such quantities appear frequently when we investigate the dynamics of quantum optical systems, especially nonlinear optical processes with dissipation. We also have to treat such a quantity in the quantum counting theory.

According to the decomposition formulas for the generators of su(1,1) and su(2) Lie algebras, we can get the normal-order and antinormal-order decomposition formulas, respectively, as follows:

$$\exp(a_+K_+ + a_0K_0 + a_-K_-)$$

$$=\exp(A_+K_+)\exp[\ln(A_0)K_0]\exp(A_-K_-) \tag{2.3}$$

$$=\exp(B_-K_-)\exp[\ln(B_0)K_0]\exp(B_+K_+) . \tag{2.4}$$

Here, $a_+$, $a_-$, and $a_0$ are arbitrary $c$ numbers, and $A_+$, $A_-$, and $A_0$ as well as $B_+$, $B_-$, and $B_0$ are expressed in terms of $a_+$, $a_-$, and $a_0$, as follows:

$$A_\pm=\frac{\frac{a_\pm}{\phi}\sinh\phi}{\cosh\phi-\frac{a_0}{2\phi}\sinh\phi} , \tag{2.5}$$

$$A_0=\left[\cosh\phi-\frac{a_0}{2\phi}\sinh\phi\right]^2 \tag{2.6}$$

and

$$B_\pm=\frac{\frac{a_\pm}{\phi}\sinh\phi}{\cosh\phi+\frac{a_0}{2\phi}\sinh\phi} , \tag{2.7}$$

$$B_0=\left[\cosh\phi+\frac{a_0}{2\phi}\sinh\phi\right]^2 , \tag{2.8}$$

with

$$\phi^2=\left[\frac{a_0}{2}\right]^2-\sigma a_+a_- . \tag{2.9}$$

These formulas are proven by a parameter differentiation method [6]. Here we define normal-order expansion of $F(K_+,K_-,K_0)$ by

$$F(K_+,K_-,K_0)=\sum_l\sum_m\sum_n f_{lmn}^{(N)}K_+^lK_0^mK_-^n \tag{2.10}$$

and antinormal-order expansion by

$$F(K_+,K_-,K_0)=\sum_l\sum_m\sum_n f_{lmn}^{(A)}K_-^lK_0^mK_+^n , \tag{2.11}$$

where $f_{lmn}^{(N)}$ and $f_{lmn}^{(A)}$ are expansion coefficients.

It is easily found from (2.5)–(2.8) that $\{A_+,A_-,A_0\}$ and $\{B_+,B_-,B_0\}$ are related to each other by the following relationships:

$$A_0=\frac{B_0}{(1-\sigma B_+B_-B_0)^2} , \tag{2.12}$$

$$A_\pm=\frac{B_\pm B_0}{1-\sigma B_+B_-B_0} \tag{2.13}$$

and

$$B_0=\frac{(A_0-\sigma A_+A_-)^2}{A_0} , \tag{2.14}$$

$$B_\pm=\frac{A_\pm}{A_0-\sigma A_+A_-} . \tag{2.15}$$

It should be noted that we can use the formulas (2.12)–(2.15) to rewrite the antinormal order into normal order, and vice versa. We frequently need to do this when we investigate the dynamics of quantum optical systems.

The su(1,1) Lie algebra can be represented in terms of boson annihilation and creation operators. In the one-mode bosonic realization, the generators $\{K_+,K_-,K_0\}$ are expressed as

$$K_+=\frac{1}{2}(a^\dagger)^2 , \tag{2.16}$$

$$K_-=\frac{1}{2}a^2 , \tag{2.17}$$

$$K_0=\frac{1}{2}(a^\dagger a+\frac{1}{2}) . \tag{2.18}$$

Here, $a$ and $a^\dagger$ are boson annihilation and creation operators satisfying $[a,a^\dagger]=1$. In this realization, the Casimir operator $C_1$ becomes

$$C_1=K_0^2-\frac{1}{2}(K_+K_-+K_-K_+)=-\frac{3}{16} . \tag{2.19}$$

Thus the Bargmann index is $k=\frac{1}{4}$ or $\frac{3}{4}$. For $k=\frac{1}{4}$, the basis for the irreducible unitary representation space is a set of states with an even boson number, and for $k=\frac{3}{4}$, it is a set of states with an odd boson number [10]. The one-mode bosonic realization is used to describe the degenerate parametric amplifier [2].

The two-mode bosonic realization gives us

$$K_+=a^\dagger b^\dagger , \tag{2.20}$$

$$K_-=ab , \tag{2.21}$$

$$K_0=\frac{1}{2}(a^\dagger a+b^\dagger b+1) , \tag{2.22}$$

with $[a,a^\dagger]=[b,b^\dagger]=1$. In this case, the Casimir operator $C_2$ is given by

$$C_2=\frac{1}{4}(a^\dagger a-b^\dagger b+1)(a^\dagger a-b^\dagger b-1) . \tag{2.23}$$

This realization conserves the boson-number difference between the two modes. In quantum optics, the two-mode bosonic realization is used to describe the nondegenerate parametric amplifier [3]. When we describe physical systems in the Liouville space, we can investigate many kinds of phenomena in terms of the su(1,1) Lie algebra. For example, the linear dissipative process and the photon-counting process are described by this algebra (see Secs. IV and V).

The su(2) algebra, on the other hand, can be expressed in terms of two boson operators $a$ ($a^\dagger$) and $b$ ($b^\dagger$), as follows:

$$K_+=a^\dagger b , \tag{2.24}$$

$$K_-=ab^\dagger , \tag{2.25}$$

$$K_0=\frac{1}{2}(a^\dagger a-b^\dagger b) . \tag{2.26}$$

This realization is used in describing the beam splitter [13], the interferometer [14], and the linear directional coupler [15]. The Casimir operator becomes

$$C=\frac{a^\dagger a+b^\dagger b}{2}\left[\frac{a^\dagger a+b^\dagger b}{2}+1\right] . \tag{2.27}$$

The total boson number is conserved, and this realization is nothing but Schwinger's realization of angular momentum [27].

We also have the following fermionic realization:

$$K_+=d^\dagger c^\dagger , \tag{2.28}$$

$$K_-=cd , \tag{2.29}$$

$$K_0=\frac{1}{2}(c^\dagger c+d^\dagger d-1) , \tag{2.30}$$

where $c$ ($d$) and $c^\dagger$ ($d^\dagger$) are fermionic annihilation and creation operators that satisfy

$$[c,c^\dagger]_+=[d,d^\dagger]_+=1 .$$

This fermionic realization is used to describe electron-counting processes (see Sec. V). Furthermore, the su(2) algebra is also realized by the Pauli matrices as

$$K_+=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\quad K_-=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\quad K_0=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{bmatrix} . \tag{2.31}$$

The realizations of the su(1,1) and su(2) Lie algebras described in this subsection appear frequently in quantum optics and give us compact descriptions of various kinds of phenomena.

### B. Liouville-space formulation

This subsection briefly reviews the Liouville-space formulation (or thermofield dynamics) [21,22]. The Liouville space $\mathcal{L}$ can be constructed as a direct product of the two independent Hilbert spaces, $\mathcal{L}=\mathcal{H}\otimes\tilde{\mathcal{H}}$. Here, $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are the ordinary Hilbert spaces. We denote as $A$ an arbitrary operator acting on any vector in $\mathcal{H}$. Then any operator $\tilde{A}$ defined on $\tilde{\mathcal{H}}$ is given by the tilde conjugation of $A$ [21]. The tilde conjugation is defined by

$$(A_1A_2)^\sim=\tilde{A}_1\tilde{A}_2 , \tag{2.32}$$

$$(A^\dagger)^\sim=(\tilde{A})^\dagger , \tag{2.33}$$

$$(a_1A_1+a_2A_2)^\sim=a_1^*\tilde{A}_1+a_2^*\tilde{A}_2 , \tag{2.34}$$

$$\tilde{\tilde{A}}=\sigma A , \tag{2.35}$$

where $a_1$ and $a_2$ are arbitrary $c$ numbers, and $\sigma=1$ for a bosonic operator $A$ and $\sigma=-1$ for a fermionic operator $A$. Bosonic and fermionic tilde operators are, respectively, assumed to commute and to anticommute with a respective nontilde operator.

For simplicity, we consider a single-mode bosonic system, so the Liouville space is spanned by vectors belonging to a complete orthonormal set,

$$S=\{|m,n\rangle=|m\rangle\otimes|\tilde{n}\rangle \mid |m\rangle\in\mathcal{H},\ |\tilde{n}\rangle\in\tilde{\mathcal{H}},\ m,n=0,1,2,\ldots,\infty\} \tag{2.36}$$

that satisfies

$$\langle n_1,m_1|m_2,n_2\rangle=\delta_{m_1m_2}\delta_{n_1n_2} , \quad (2.37)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|m,n\rangle\langle n,m|=1 , \quad (2.38)$$

where $|m\rangle$ and $|\tilde{n}\rangle$ are the number eigenstates, $N|m\rangle=m|m\rangle$, and $\tilde{N}|\tilde{n}\rangle=n|\tilde{n}\rangle$.

In the Liouville space, we introduce a state vector defined by

$$|1\rangle=\sum_{n=0}^{\infty}|n,n\rangle . \quad (2.39)$$

This state vector satisfies

$$a|1\rangle=\tilde{a}^{\dagger}|1\rangle, \quad a^{\dagger}|1\rangle=\tilde{a}|1\rangle , \quad (2.40)$$

where $a$ and $a^{\dagger}$ are annihilation and creation operators and where $\tilde{a}$ and $\tilde{a}^{\dagger}$ are their tilde conjugates; these operators are defined by

$$a|m,n\rangle=\sqrt{m}\,|m-1,n\rangle , \quad (2.41)$$

$$a^{\dagger}|m,n\rangle=\sqrt{m+1}|m+1,n\rangle \quad (2.42)$$

and

$$\tilde{a}|m,n\rangle=\sqrt{n}\,|m,n-1\rangle , \quad (2.43)$$

$$\tilde{a}^{\dagger}|m,n\rangle=\sqrt{n+1}|m,n+1\rangle , \quad (2.44)$$

with $a|0,n\rangle=\tilde{a}|n,0\rangle=0$ for all $n$. It is easily seen from (2.41)–(2.44) that $[a,a^{\dagger}]=[\tilde{a},\tilde{a}^{\dagger}]=1$ and $[a,\tilde{a}]=[a,\tilde{a}^{\dagger}]=0$. Note that $|1\rangle$ is a tilde invariant state, $|\tilde{1}\rangle=|1\rangle$ [22]. For fermion annihilation and creation operators $c$ and $c^{\dagger}$ and their tilde conjugates $\tilde{c}$ and $\tilde{c}^{\dagger}$, (2.39) and (2.40) should be modified to

$$|1\rangle=|0,0\rangle+|1,1\rangle \quad (2.45)$$

and

$$c|1\rangle=\tilde{c}|1\rangle, \quad c^{\dagger}|1\rangle=-\tilde{c}|1\rangle . \quad (2.46)$$

The generalization of these relations to systems with many fermions is straightforward.

Any state vector in the Liouville space $\mathcal{L}$ corresponds to an operator in the usual Hilbert space $\mathcal{H}$. For example, a state vector $|m,n\rangle$ in $\mathcal{L}$ has the same meaning as the operator $|m\rangle\langle n|$ in $\mathcal{H}$. The correspondence between a vector of the Liouville space and an operator of the Hilbert space is derived from the following rules:

$$a|m,n\rangle \Longleftrightarrow a|m\rangle\langle n| , \quad (2.47)$$

$$a^{\dagger}|m,n\rangle \Longleftrightarrow a^{\dagger}|m\rangle\langle n| \quad (2.48)$$

and

$$\tilde{a}|m,n\rangle \Longleftrightarrow |m\rangle\langle n|a^{\dagger} , \quad (2.49)$$

$$\tilde{a}^{\dagger}|m,n\rangle \Longleftrightarrow |m\rangle\langle n|a . \quad (2.50)$$

Using this correspondence rule, we have in general

$$|A\rangle=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}|m,n\rangle \Longleftrightarrow A=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|m\rangle A_{mn}\langle n| . \quad (2.51)$$

An arbitrary state $|\Psi\rangle$ in the Liouville space $\mathcal{L}$ can be expanded as follows:

$$|\Psi\rangle=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}f_{mn}|m,n\rangle . \quad (2.52)$$

Hence, for $A=A(a,a^{\dagger})$, using (2.39) and (2.52) gives us

$$\langle 1|A(a,a^{\dagger})|\Psi\rangle=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle n|A(a,a^{\dagger})|m\rangle f_{mn} . \quad (2.53)$$

When there exists an operator $F$ acting on a vector in the Hilbert space $\mathcal{H}$, and its matrix element is given by $f_{mn}=\langle m|F|n\rangle$, then we have

$$\langle 1|A(a,a^{\dagger})|\Psi\rangle=\mathrm{Tr}[A(a,a^{\dagger})F] , \quad (2.54)$$

where Tr indicates the trace operation on the Hilbert space $\mathcal{H}$. It should be noted that in the Liouville space $\mathcal{L}$, a scalar product with $\langle 1|$ is equivalent to the trace operation in the Hilbert space $\mathcal{H}$. Thus, if we put $F=\rho$, where $\rho$ is a density operator of the system, we find that a quantum statistical average of $A$ is calculated as the matrix element in the Liouville space $\mathcal{L}$:

$$\langle A(a,a^{\dagger})\rangle=\langle 1|A(a,a^{\dagger})|\rho\rangle , \quad (2.55)$$

with

$$|\rho\rangle=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\rho_{mn}|m,n\rangle=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|m,n\rangle\langle m|\rho|n\rangle .$$

It is found that $|\rho\rangle$ is a tilde invariant state, $|\tilde{\rho}\rangle=|\rho\rangle$ [22]. Note that the thermal average is obtained if we put in (2.52) [or (2.54)],

$$f_{mn}=\delta_{mn}\frac{1}{Z}\exp(-\beta\omega n) \quad (2.56)$$

or

$$F=\frac{1}{Z}\exp(-\beta\omega a^{\dagger}a) , \quad (2.57)$$

where $\beta$ is a reciprocal of temperature and $Z=1+\bar{n}$ with $\bar{n}=(e^{\beta\omega}-1)^{-1}$.

A quantum-statistical-mechanical system is described by a density matrix $\rho(t)$ defined on $\mathcal{H}$. The dynamics of the system is governed by the Liouville–von Neumann equation

$$\frac{\partial}{\partial t}\rho(t)=-i[H,\rho(t)]=-iL\rho(t) , \quad (2.58)$$

where $H$ is the Hamiltonian of the system and $L$ is the Liouvillian operator. In the Liouville space, the system is described by the state vector $|\rho(t)\rangle$ corresponding to $\rho(t)$ defined on the Hilbert space $\mathcal{H}$ and the time evolution of $|\rho(t)\rangle$ is determined by

$$\frac{\partial}{\partial t}|\rho(t)\rangle=-i\hat{H}|\rho(t)\rangle , \quad (2.59)$$

with $\hat{H}=H-\tilde{H}$, where $\tilde{H}$ is the tilde conjugate of $H$. When $H=H(a,a^{\dagger})$, we have $\tilde{H}=H^{*}(\tilde{a},\tilde{a}^{\dagger})$. The relationship between (2.58) and (2.59) is derived by using the mapping rule (2.47)–(2.50). It is easily seen from the correspondence rules that $H|\rho(t)\rangle \Longrightarrow H\rho(t)$ and $\tilde{H}|\rho(t)\rangle \Longrightarrow \rho(t)H$. In the Liouville space $\mathcal{L}$, the Liouville–von Neumann equation takes the same form as the Schrödinger equation. This is true even if there is dissipation. Section IV will use the description of dynamics in the Liouville space to investigate nonlinear optical processes.

## III. GENERAL FORMULAS FOR LIE ALGEBRAS

### A. Derivation of general formulas

In this section we derive general formulas for calculating an expectation value of the product of exponential functions of the generators of the su(1,1) and su(2) Lie algebras. We would like to calculate the following quantity:

$$G = \langle \Psi | \mathcal{F} | \Phi \rangle \ , \tag{3.1}$$

with

$$\mathcal{F} = \exp[V_n]\exp[V_{n-1}] \cdots \exp[V_1] \ , \tag{3.2}$$

and

$$V_k = a_+(k)K_+ + a_0(k)K_0 + a_-(k)K_- \ , \tag{3.3}$$

where $\{K_+, K_-, K_0\}$ are generators of su(1,1) or su(2) algebras and $\{a_+(k), a_-(k), a_0(k)\}$ are arbitrary $c$-number functions. In (3.1), $|\Psi\rangle$ and $|\Phi\rangle$ are arbitrary states in the Hilbert space $\mathcal{H}$ or in the Liouville space $\mathcal{L}$. This subsection will first derive the generalized normal- and antinormal-order decomposition formulas (or the generalized Baker-Campbell-Hausdorff formulas) by using (2.3)–(2.9) and (2.12)–(2.15). Section III B will calculate the average value of $\mathcal{F}$ for typical states in quantum optics, such as the Glauber coherent state and the generalized SU(1,1) coherent states.

Now let us derive the generalized normal-order decomposition formulas of $\mathcal{F}$ for the su(1,1) and the su(2) Lie algebras. First, decompose the last exponential in (3.2), $\exp[V_1]$, into the normal order and the other $n-1$ exponentials,

$$\exp[V_n], \exp[V_{n-1}], \ldots, \exp[V_2] \ ,$$

into the antinormal order. Using (2.3)–(2.8), we obtain

$$\begin{aligned}\mathcal{F} = {} & \exp[\tilde{a}_-(n)K_-]\exp\{\ln[\tilde{a}_0(n)]K_0\}\exp[\tilde{a}_+(n)K_+] \cdots \exp[\tilde{a}_-(2)K_-]\exp\{\ln[\tilde{a}_0(2)]K_0\}\exp[\tilde{a}_+(2)K_+] \\ & \times \exp[\tilde{a}_+(1)K_+]\exp\{\ln[\tilde{a}_0(1)]K_0\}\exp[\tilde{a}_-(1)K_-] \ ,\end{aligned} \tag{3.4}$$

where $\tilde{a}_+(k)$, $\tilde{a}_-(k)$, and $\tilde{a}_0(k)$ are given by

$$\tilde{a}_\pm(k) = \frac{\dfrac{a_\pm(k)}{\phi(k)}\sinh\phi(k)}{\cosh\phi(k) + \dfrac{a_0(k)}{2\phi(k)}\sinh\phi(k)} \ , \tag{3.5}$$

$$\tilde{a}_0(k) = \left[\cosh\phi(k) + \frac{a_0(k)}{2\phi(k)}\sinh\phi(k)\right]^2 \ , \tag{3.6}$$

for $k > 1$, and

$$\tilde{a}_\pm(1) = \frac{\dfrac{a_\pm(1)}{\phi(1)}\sinh\phi(1)}{\cosh\phi(1) - \dfrac{a_0(1)}{2\phi(1)}\sinh\phi(1)} \ , \tag{3.7}$$

$$\tilde{a}_0(1) = \left[\cosh\phi(1) - \frac{a_0(1)}{2\phi(1)}\sinh\phi(1)\right]^{-2} \ . \tag{3.8}$$

For all $k$, $\phi(k)$ is defined by

$$\phi(k) = \left\{\left[\frac{a_0(k)}{2}\right]^2 - \sigma a_+(k)a_-(k)\right\}^{1/2} \ , \tag{3.9}$$

with $\sigma = 1$ for the su(1,1) Lie algebra and $\sigma = -1$ for the su(2) Lie algebra.

Consider the quantity

$$\exp[x_-K_-]\exp[\ln(x_0)K_0]\exp[x_+K_+]\exp[y_+K_+]\exp[\ln(y_0)K_0]\exp[y_-K_-] \ . \tag{3.10}$$

These exponentials can be rearranged as follows:

$$\begin{aligned}&\exp[x_-K_-]\exp[\ln(x_0)K_0]\exp[x_+K_+]\exp[y_+K_+]\exp[\ln(y_0)K_0]\exp[y_-K_-]\\&\quad=\exp[x_-K_-]\exp[\ln(x_0)K_0]\exp[(x_++y_+)K_+]\exp[\ln(y_0)K_0]\exp[y_-K_-]\\&\quad=\exp[\tilde{x}_+K_+]\exp[\ln(\tilde{x}_0)K_0]\exp[\tilde{x}_-K_-]\exp[\ln(y_0)K_0]\exp[y_-K_-]\ ,\end{aligned} \tag{3.11}$$

where the formulas given by the relationships (2.12) and (2.13) have been used in the last equality to rewrite the antinormal order into the normal order. In (3.11), the values of $\tilde{x}_+$, $\tilde{x}_-$, and $\tilde{x}_0$ are given by

$$\tilde{x}_+=\frac{(x_++y_+)x_0}{1-\sigma(x_++y_+)x_-x_0}\ , \tag{3.12}$$

$$\tilde{x}_-=\frac{x_-x_0}{1-\sigma(x_++y_+)x_-x_0}\ , \tag{3.13}$$

$$\tilde{x}_0=\frac{x_0}{[1-\sigma(x_++y_+)x_-x_0]^2}\ . \tag{3.14}$$

Using such a rearrangement repeatedly, we can rewrite (3.4) such that the generator $K_+$ appears only in the leftmost exponential on the right-hand side. Hence, we obtain

$$\begin{aligned}\mathcal{F}=&\exp[A_+(n)K_+]\exp\{\ln[A_0(n)]K_0\}\exp[A_-(n)K_-]\exp\{\ln[A_0(n-1)]K_0\}\exp[A_-(n-1)K_-]\\&\times\cdots\times\exp\{\ln[A_0(1)]K_0\}\exp[A_-(1)K_-]\ ,\end{aligned} \tag{3.15}$$

where $A_+(k)$, $A_-(k)$, and $A_0(k)$ are given by

$$A_+(k)=\frac{\left[\cosh\phi(k)+\dfrac{a_0(k)}{2\phi(k)}\sinh\phi(k)\right]A_+(k-1)+\dfrac{a_+(k)}{\phi(k)}\sinh\phi(k)}{\cosh\phi(k)-\dfrac{a_0(k)}{2\phi(k)}\sinh\phi(k)-\sigma\dfrac{a_-(k)}{\phi(k)}\sinh[\phi(k)]A_+(k-1)}\ , \tag{3.16}$$

$$A_-(k)=\frac{\dfrac{a_-(k)}{\phi(k)}\sinh\phi(k)}{\cosh\phi(k)-\dfrac{a_0(k)}{2\phi(k)}\sinh\phi(k)-\sigma\dfrac{a_-(k)}{\phi(k)}\sinh[\phi(k)]A_+(k-1)}\ , \tag{3.17}$$

$$A_0(k)=\frac{1}{\left[\cosh\phi(k)-\dfrac{a_0(k)}{2\phi(k)}\sinh\phi(k)-\sigma\dfrac{a_-(k)}{\phi(k)}\sinh[\phi(k)]A_+(k-1)\right]^2}\ , \tag{3.18}$$

for $n\geq k>1$, and

$$A_\pm(1)=\frac{\dfrac{a_\pm(1)}{\phi(1)}\sinh\phi(1)}{\cosh\phi(1)-\dfrac{a_0(1)}{2\phi(1)}\sinh\phi(1)}\ , \tag{3.19}$$

$$A_0(1)=\left[\cosh\phi(1)-\frac{a_0(1)}{2\phi(1)}\sinh\phi(1)\right]^{-2}\ . \tag{3.20}$$

We can finally obtain the generalized normal-order decomposition formula as follows:

$$\mathcal{F}=\exp\{A_+(n)K_+\}\exp\left\{\ln\left[\prod_{k=1}^{n}A_0(k)\right]K_0\right\}\exp\left\{\sum_{l=1}^{n}A_-(l)\left[\prod_{k=1}^{l-1}A_0(k)\right]K_-\right\}\ . \tag{3.21}$$

In deriving this formula, we have repeatedly used the relation

$$\exp(x_-K_-)\exp[\ln(x_0)K_0]=\exp[\ln(x_0)K_0]\exp(x_-x_0K_-)\ . \tag{3.22}$$

For $n=1$, (3.21) reduces to the ordinary Baker-Hausdorff formula.

Next, we will derive the generalized antinormal-order decomposition formula for $\mathcal{F}$. In this case, we decompose the last exponential $\exp[V_1]$ in (3.2) into the antinormal order and the other $n-1$ exponentials $\exp[V_n],\ldots,\exp[V_2]$ into the normal order, and we make a rearrangement such that the generator $K_-$ appears only in the leftmost exponential on the right-hand side of (3.2). Using the same procedure to derive the generalized normal-order decomposition formula, we can obtain

$$\mathcal{T}= \exp[B_-(n)K_-]\exp\{\ln[B_0(n)]K_0\}\exp[B_+(n)K_+]\exp\{\ln[B_0(n-1)]K_0\}\exp[B_+(n-1)K_+] \times \cdots \times\exp\{\ln[B_0(1)]K_0\}\exp[B_+(1)K_+] , \tag{3.23}$$

where $B_-(k)$, $B_+(k)$, and $B_0(k)$ are defined by

$$B_-(k)=\frac{\left[\cosh\phi(k)-\frac{a_0(k)}{2\phi(k)}\sinh\phi(k)\right]B_-(k-1)+\frac{a_-(k)}{\phi(k)}\sinh\phi(k)}{\cosh\phi(k)+\frac{a_0(k)}{2\phi(k)}\sinh\phi(k)-\sigma\frac{a_+(k)}{2\phi(k)}\sinh[\phi(k)]B_-(k-1)} , \tag{3.24}$$

$$B_+(k)=\frac{\frac{a_+(k)}{\phi(k)}\sinh\phi(k)}{\cosh\phi(k)+\frac{a_0(k)}{2\phi(k)}\sinh\phi(k)-\sigma\frac{a_+(k)}{2\phi(k)}\sinh[\phi(k)]B_-(k-1)} , \tag{3.25}$$

$$B_0(k)=\left[\cosh\phi(k)+\frac{a_0(k)}{2\phi(k)}\sinh\phi(k)-\sigma\frac{a_+(k)}{2\phi(k)}\sinh[\phi(k)]B_-(k-1)\right]^2 , \tag{3.26}$$

for $n\geq k>1$, and

$$B_\pm(1)=\frac{\frac{a_\pm(1)}{\phi(1)}\sinh\phi(1)}{\cosh\phi(1)+\frac{a_0(1)}{2\phi(1)}\sinh\phi(1)} , \tag{3.27}$$

$$B_0(1)=\left[\cosh\phi(1)+\frac{a_0(1)}{2\phi(1)}\sinh\phi(1)\right]^2 . \tag{3.28}$$

Using the relation

$$\exp(x_+K_+)\exp[\ln(x_0)K_0] =\exp[\ln(x_0)K_0]\exp\left[\frac{x_+}{x_0}K_+\right] , \tag{3.29}$$

repeatedly, we can get the generalized antinormal-order decomposition formula for $\mathcal{T}$ as follows:

$$\mathcal{T}= \exp\{B_-(n)K_-\}\exp\left\{\ln\left[\prod_{k=1}^{n}B_0(k)\right]K_0\right\} \times\exp\left\{\sum_{l=1}^{n}B_+(l)\left[\prod_{k=1}^{l-1}B_0(k)\right]^{-1}K_+\right\} . \tag{3.30}$$

For $n=1$, (3.30) reduces to the ordinary Baker-Campbell-Hausdorff formula.

Finally, we will rewrite (3.21) and (3.30) into more compact forms. When we define $\overline{A}_\pm(n)$, $\overline{A}_0(n)$ and $\overline{B}_\pm(n)$, $\overline{B}_0(n)$ by

$$\overline{A}_+(n)=A_+(n) , \tag{3.31}$$

$$\overline{A}_0(n)=\prod_{k=1}^{n}A_0(k) , \tag{3.32}$$

$$\overline{A}_-(n)=\sum_{l=1}^{n}A_-(l)\prod_{k=1}^{l-1}A_0(k) \tag{3.33}$$

and

$$\overline{B}_+(n)=\sum_{l=1}^{n}B_+(l)\left[\prod_{k=1}^{l-1}B_0(l)\right]^{-1} , \tag{3.34}$$

$$\overline{B}_0(n)=\prod_{k=1}^{n}B_0(k) , \tag{3.35}$$

$$\overline{B}_-(n)=B_0(n) , \tag{3.36}$$

where $A_\pm(n)$ and $A_0(n)$ are given by (3.16)–(3.20) and $B_\pm(n)$, $B_0(n)$, are given by (3.24)–(3.28), then (3.21) and (3.30) are expressed as

$$\mathcal{T}=\prod_{k=1}^{n\;>}\exp[a_+(k)K_++a_0(k)K_0+a_-(k)K_-]$$

$$=\exp[\overline{A}_+(n)K_+]\exp\{\ln[\overline{A}_0(n)]K_0\}\exp[\overline{A}_-(n)K_-] \tag{3.37}$$

$$=\exp[\overline{B}_-(n)K_-]\exp\{\ln[\overline{B}_0(n)]K_0\}\exp[\overline{B}_+(n)K_+] , \tag{3.38}$$

where we have introduced the ordered product as

$$\prod_{k=1}^{n\;>}f(k)=f(n)f(n-1)\cdots f(2)f(1) . \tag{3.39}$$

We find from (3.16)–(3.20) and (3.24)–(3.28) that $\overline{A}_\pm(n)$ and $\overline{A}_0(n)$ as well as $\overline{B}_\pm(n)$ and $\overline{B}_0(n)$ satisfy the following recurrence relations:

$$\bar{A}_{+}(n)=\frac{\frac{a_{+}(n)}{\phi(n)}\sinh\phi(n)+\left[\cosh\phi(n)+\frac{a_0(n)}{2\phi(n)}\sinh\phi(n)\right]\bar{A}_{+}(n-1)}{\cosh\phi(n)-\frac{a_0(n)}{2\phi(n)}\sinh\phi(n)-\sigma\frac{a_{-}(n)}{\phi(n)}\sinh[\phi(n)]\bar{A}_{+}(n-1)}\ , \tag{3.40}$$

$$\bar{A}_{-}(n)=\bar{A}_{-}(n-1)+\frac{\frac{a_{-}(n)}{\phi(n)}\sinh[\phi(n)]\bar{A}_0(n-1)}{\cosh\phi(n)-\frac{a_0(n)}{2\phi(n)}\sinh\phi(n)-\sigma\frac{a_{-}(n)}{\phi(n)}\sinh[\phi(n)]\bar{A}_{+}(n-1)}\ , \tag{3.41}$$

$$\bar{A}_0(n)=\frac{\bar{A}_0(n-1)}{\left[\cosh\phi(n)-\frac{a_0(n)}{2\phi(n)}\sinh\phi(n)-\sigma\frac{a_{-}(n)}{\phi(n)}\sinh[\phi(n)]\bar{A}_{+}(n-1)\right]^2} \tag{3.42}$$

and

$$\bar{B}_{+}(n)=\bar{B}_{+}(n-1)+\frac{\frac{1}{\bar{B}_0(n-1)}\frac{a_{+}(n)}{\phi(n)}\sinh\phi(n)}{\cosh\phi(n)+\frac{a_0(n)}{2\phi(n)}\sinh\phi(n)-\sigma\frac{a_{+}(n)}{\phi(n)}\sinh[\phi(n)]\bar{B}_{-}(n-1)}\ , \tag{3.43}$$

$$\bar{B}_{-}(n)=\frac{\frac{a_{-}(n)}{\phi(n)}\sinh\phi(n)+\left[\cosh\phi(n)+\frac{a_0(n)}{2\phi(n)}\sinh\phi(n)\right]\bar{B}_{-}(n-1)}{\cosh\phi(n)+\frac{a_0(n)}{2\phi(n)}\sinh\phi(n)-\sigma\frac{a_{-}(n)}{\phi(n)}\sinh[\phi(n)]\bar{B}_{-}(n-1)}\ , \tag{3.44}$$

$$\bar{B}_0(n)=\bar{B}_0(n-1)\left[\cosh\phi(n)+\frac{a_0(n)}{2\phi(n)}\sinh\phi(n)-\sigma\frac{a_{-}(n)}{\phi(n)}\sinh[\phi(n)]\bar{B}_{-}(n-1)\right]^2\ , \tag{3.45}$$

with $\bar{A}_{\pm}(0)=0$, $\bar{A}_0(0)=1$, $\bar{B}_{\pm}(0)=0$, $\bar{B}_0(0)=1$, and $\phi(n)$ being given by (3.9).

The formulas (3.21) and (3.30) [or (3.37) and (3.38)] are useful in calculating the average value of (3.2), and we frequently treat such quantities when we investigate the dynamics of quantum optical systems. In Secs. IV and V we will use these formulas to calculate the absorption spectrum and photon echo signal in a localized electron-phonon system and to investigate quantum counting processes.

### B. Calculation of expectation values

Now we calculate the matrix element of $\mathcal{F}$ defined by (3.2) in terms of the generalized decomposition formulas (3.21) and (3.30) [or (3.37) and (3.38)], and we write $G=\langle\Psi|\mathcal{F}|\Phi\rangle$. First, we assume that $|\Phi\rangle$ is a vacuum state of the one-mode (or two-mode) bosonic realization of the su(1,1) Lie algebra. We denote both the one-mode and two-mode vacuum states as $|0\rangle$ satisfying $a|0\rangle=0$ or $a|0\rangle=b|0\rangle=0$. Since the one-mode and two-mode bosonic realizations are given by (2.16)–(2.18) and (2.20)–(2.22), we can easily see that $K_{-}|0\rangle=0$ and $K_0|0\rangle=c|0\rangle$, where $c=\frac{1}{4}$ for the one-mode bosonic realization and $c=\frac{1}{2}$ for the two-mode realization. Thus from (3.37), we have

$$G=\bar{A}_0(n)^{1/4}\sum_{k=0}^{\infty}\left[\frac{(2k-1)!!}{(2k)!!}\right]^{1/2}\bar{A}_{+}(n)^k\langle\Psi|2k\rangle \tag{3.46}$$

for the one-mode bosonic realization and

$$G=\bar{A}_0(n)^{1/2}\sum_{k=0}^{\infty}\bar{A}_{+}(n)^k\langle\Psi|k,k\rangle \tag{3.47}$$

for the two-mode bosonic realization. Here, $|n\rangle$ and $|n,n\rangle$ are number eigenstates defined by

$$|n\rangle=\frac{1}{\sqrt{n!}}a^{\dagger n}|0\rangle\ , \tag{3.48}$$

$$|m,n\rangle=\frac{1}{\sqrt{m!n!}}a^{\dagger m}b^{\dagger n}|0\rangle\ . \tag{3.49}$$

And when $|\Psi\rangle$ is also a vacuum state, (3.46) and (3.47) reduce to

$$G=\langle 0|\mathcal{F}|0\rangle=\bar{A}_0(n)^c\ , \tag{3.50}$$

with $c=\frac{1}{4}$ or $\frac{1}{2}$.

Next we consider the Glauber coherent state $|\alpha\rangle$ [28]. First assume the one-mode bosonic realization of the su(1,1) Lie algebra and calculate $G=\langle\Psi|\mathcal{F}|\alpha\rangle$. The coherent state $|\alpha\rangle$ is defined by

$$|\alpha\rangle = D(\alpha)|0\rangle\ , \qquad (3.51)$$

where $|0\rangle$ is a vacuum state and $D(\alpha)=\exp(\alpha a^\dagger - \alpha^* a)$ is a displacement operator. From (2.16)–(2.18) and (3.51), we have

$$\exp[A_- K_-]|\alpha\rangle = \exp[\tfrac{1}{2}\alpha^2 A_-]|\alpha\rangle\ , \qquad (3.52)$$

$$\exp[\ln(A_0)K_0]|\alpha\rangle = A_0^{1/4}\exp[-\tfrac{1}{2}|\alpha|^2(1-|A_0|)]|A_0^{1/2}\alpha\rangle\ , \qquad (3.53)$$

with

$$|A_0^{1/2}\alpha\rangle = D(A_0^{1/2}\alpha)|0\rangle\ .$$

Using the normal-order decomposition formula of $\mathcal{T}$ given by (3.37), we can obtain

$$G = A_0(n)^{1/4}\exp\{-\tfrac{1}{2}|\alpha|^2[1-|\overline{A}_0(n)|] + \tfrac{1}{2}\alpha^2 \overline{A}_-(n)\} \times \langle\Psi|\exp[\overline{A}_+(n)K_+]|f_C(\alpha)\rangle\ , \qquad (3.54)$$

where $|f_C(\alpha)\rangle$ is defined by

$$|f_C(\alpha)\rangle = \exp[f_C(\alpha)a^\dagger - f_C^*(\alpha)a]|0\rangle\ , \qquad (3.55)$$

with $f_C(\alpha)$ being given by

$$f_C(\alpha) = \alpha\overline{A}_0(n)^{1/2}\ . \qquad (3.56)$$

Note that

$$a|f_C(\alpha)\rangle = f_C(\alpha)|f_C(\alpha)\rangle\ .$$

When $|\Psi\rangle$ is also a coherent state $|\alpha\rangle$, (3.54) is

$$G = \langle\alpha|\mathcal{T}|\alpha\rangle = \overline{A}_0(n)^{1/4}\exp\{\tfrac{1}{2}\alpha^{*2}\overline{A}_+(n) + \tfrac{1}{2}\alpha^2\overline{A}_-(n) - |\alpha|^2[1-\overline{A}_0(n)^{1/2}]\}\ . \qquad (3.57)$$

When we consider the two-mode bosonic realization for the su(1,1) Lie algebra, we have the two-mode Glauber coherent state given by

$$|\alpha,\beta\rangle = D_A(\alpha)D_B(\beta)|0\rangle = |\alpha\rangle\otimes|\beta\rangle\ , \qquad (3.58)$$

with

$$D_A(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$$

and

$$D_B(\beta) = \exp(\beta b^\dagger - \beta^* b)\ .$$

Using the same procedure we used to obtain (3.54), we can get

$$G = \langle\Psi|\mathcal{T}|\alpha,\beta\rangle = \overline{A}_0(n)^{1/2}\exp\{-\tfrac{1}{2}(|\alpha|^2+|\beta|^2)[1-|\overline{A}_0(n)|] + \alpha\beta\overline{A}_-(n)\} \times \langle\Psi|\exp[\overline{A}_+(n)K_+]|f_C(\alpha),f_C(\beta)\rangle\ , \qquad (3.59)$$

where $f_C(x)$ is given by (3.56). And when $|\Psi\rangle = |\alpha,\beta\rangle$, (3.59) becomes

$$G = \langle\beta,\alpha|\mathcal{T}|\alpha,\beta\rangle = \overline{A}_0(n)^{1/2}\exp\{-(|\alpha|^2+|\beta|^2)[1-\overline{A}_0(n)^{1/2}] + \alpha\beta\overline{A}_-(n) + \alpha^*\beta^*\overline{A}_+(n)\}\ . \qquad (3.60)$$

We will consider the two kinds of the generalized SU(1,1) coherent states. One, introduced first by Barut and Girardello [9], is defined as an eigenstate of $K_- = ab$ and is given by

$$|z,m\rangle_{\rm BG} = \left[\frac{|z|^m}{I_m(2|z|)}\right]^{1/2}\sum_{k=0}^{\infty}\frac{z^k}{\sqrt{k!(k+m)!}}|k+m,k\rangle\ , \qquad (3.61)$$

with

$$|m+n,m\rangle = \frac{a^{\dagger m+n}b^{\dagger m}}{\sqrt{(m+n)!m!}}|0,0\rangle\ , \qquad (3.62)$$

where $I_n(x)$ is the $n$th-order modified Bessel function defined by

$$I_\nu(x) = \left[\frac{x}{2}\right]^\nu \sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+\nu+1)}\left[\frac{x}{2}\right]^{2n}\ . \qquad (3.63)$$

We can see from the definition that

$$K_-|z,m\rangle_{\rm BG} = z|z,m\rangle_{\rm BG}\ .$$

This state is also called the charged boson coherent state or the pair coherent state [9,12]. The other generalized SU(1,1) coherent state, introduced by Perelomov [10], is constructed by means of the generalized displacement operator $D(z) = \exp[zK_+ - z^*K_-]$:

$$|\mu,m\rangle_{\rm P} = \exp[zK_+ - z^*K_-]|m,0\rangle = (1-|\mu|^2)^{(1+m)/2}\exp(\mu K_+)|m,0\rangle = (1-|\mu|^2)^{(1+m)/2}\sum_{k=0}^{\infty}\left[\frac{(m+k)!}{m!k!}\right]^{1/2} \times \mu^k|m+k,k\rangle\ , \qquad (3.64)$$

with $\mu = (z/|z|)\tanh|z|$. Using the same procedure we used to calculate $G = \langle\Psi|\mathcal{T}|\Phi\rangle$ with the Glauber coherent states, we obtain

$$G_1 = \langle\Psi|\mathcal{T}|z,m\rangle_{\rm BG} = \left[\frac{\overline{A}_0(n)^{m+1}I_m(2|z\overline{A}_0(n)|)}{|\overline{A}_0(n)|^m I_m(2|z|)}\right]^{1/2} \times \exp[\overline{A}_-(n)z]\langle\Psi|\exp[\overline{A}_+(n)K_+]|f_{\rm BG}(z),m\rangle_{\rm BG}\ , \qquad (3.65)$$

$$G_2 = {}_{\rm BG}\langle m,z|\mathcal{T}|z,m\rangle_{\rm BG} = \overline{A}_0(n)^{1/2}\exp[\overline{A}_+(n)z^* + \overline{A}_-(n)z] \times \frac{I_m(2|z|\overline{A}_0(n)^{1/2})}{I_m(2|z|)} \qquad (3.66)$$

for the Barut-Girardello coherent state, and we obtain

$$G_3=\langle\Psi|\mathcal{F}|\mu,m\rangle_{\rm P}$$
$$=\left\{\frac{(1-|\mu|^2)\bar{B}_0(n)}{1-|[\mu+\bar{B}_+(n)]\bar{B}_0(n)|^2}\right\}^{(m+1)/2}$$
$$\times\langle\Psi|\exp[\bar{B}_-(n)K_-]|f_{\rm P}(\mu),m\rangle_{\rm P}\ , \quad (3.67)$$

$$G_4={}_{\rm P}\langle m,\mu|\mathcal{F}|\mu,m\rangle_{\rm P}$$
$$=\left\{\frac{(1-|\mu|^2)\bar{B}_0(n)^{1/2}}{1-[\mu^*+\bar{B}_-(n)][\mu+\bar{B}_+(n)]\bar{B}_0(n)}\right\}^{m+1}, \quad (3.68)$$

for the Perelomov coherent state. Here, $|\Psi\rangle$ is an arbitrary state, and $f_{\rm BG}(z)$ and $f_{\rm P}(\mu)$ are defined by

$$f_{\rm BG}(z)=z\bar{A}_0(n)\ , \quad (3.69)$$

$$f_{\rm P}(\mu)=[\mu+\bar{B}_+(n)]\bar{B}_0(n)\ . \quad (3.70)$$

In deriving (3.65) and (3.66), we have used the generalized normal-order and antinormal-order decomposition formulas, (3.37) and (3.38), and we have assumed (2.20)–(2.22).

Next let us consider the su(2) Lie algebra. Using the same method we used to derive (3.68), we can obtain $G$ for the Bloch state $|\mu;k\rangle_{\rm B}$ [16] as

$$G={}_{\rm B}\langle k;\mu|\mathcal{F}|\mu;k\rangle_{\rm B}$$
$$=\left\{\frac{1+[\mu+\bar{B}_+(n)][\mu^*+\bar{B}_-(n)]\bar{B}_0(n)}{(1+|\mu|^2)\bar{B}_0(n)^{1/2}}\right\}^k, \quad (3.71)$$

where we have assumed (2.24)–(2.26) and the Bloch state $|\mu;k\rangle_{\rm B}$ is defined by

$$|\mu;k\rangle_{\rm B}=\exp[zK_+-z^*K_-]|0,k\rangle$$
$$=\frac{1}{(1+|\mu|^2)^{k/2}}\exp[\mu K_+]|0,k\rangle$$
$$=\frac{1}{(1+|\mu|^2)^{k/2}}\sum_{m=0}^{k}\left[\frac{k!}{m!(k-m)!}\right]^{1/2}\mu^m|m,k-m\rangle\ , \quad (3.72)$$

with $\mu=(z/|z|)\tan|z|$ and with $|m,n\rangle$ being given by (3.62).

Next we consider the $\frac{1}{2}$-spin realization for the su(2) algebra. We calculate the matrix element of $\mathcal{F}$ with respect to the eigenstates of $\sigma_z$, $|+\rangle$ and $|-\rangle$, that satisfy $\sigma_z|+\rangle=\frac{1}{2}|+\rangle$ and $\sigma_z|-\rangle=-\frac{1}{2}|-\rangle$. The matrix elements $G_{ij}=\langle i|\mathcal{F}|j\rangle$ become

$$G(n)=\begin{bmatrix}\langle+|G(n)|+\rangle & \langle+|G(n)|-\rangle\\ \langle-|G(n)|+\rangle & \langle-|G(n)|-\rangle\end{bmatrix}$$
$$=\begin{bmatrix}\bar{A}_0(n)^{1/2}+\bar{A}_+(n)\bar{A}_-(n)\bar{A}_0(n)^{-1/2} & \bar{A}_+(n)\bar{A}_0(n)^{-1/2}\\ \bar{A}_-(n)\bar{A}_0(n)^{-1/2} & \bar{A}_0(n)^{-1/2}\end{bmatrix} \quad (3.73)$$
$$=\begin{bmatrix}\bar{B}_0(n)^{1/2} & \bar{B}_+(n)\bar{B}_0(n)^{1/2}\\ \bar{B}_-(n)\bar{B}_0(n)^{1/2} & \bar{B}_0(n)^{-1/2}+\bar{B}_+(n)\bar{B}_-(n)\bar{B}_0(n)^{1/2}\end{bmatrix}. \quad (3.74)$$

We can calculate $G=\langle\Psi|\mathcal{F}|\Phi\rangle$ in another realization for the su(2) algebra by using the same procedure we used for the su(1,1) algebra.

Before closing this section, we calculate the thermal average of $\mathcal{F}$. This average is calculated by $\langle 1|\mathcal{F}|0(\beta)\rangle$ in the Liouville space (see Sec. II), where $|0(\beta)\rangle$ is the thermal vacuum state with $\alpha=1$ in thermofield dynamics [22] and is defined by

$$|0(\beta)\rangle=\frac{1}{\bar{n}+1}\sum_{n=0}^{\infty}\left[\frac{\bar{n}}{\bar{n}+1}\right]^n|n,n\rangle\ , \quad (3.75)$$

with $\bar{n}$ being a boson distribution function and with $|m,n\rangle=|m\rangle\otimes|\tilde{n}\rangle$. Then the generators of the su(1,1) Lie algebra are expressed as

$$K_-=a\tilde{a}\ , \quad (3.76)$$

$$K_+=a^\dagger\tilde{a}^\dagger\ , \quad (3.77)$$

$$K_0=\tfrac{1}{2}(a^\dagger a+\tilde{a}^\dagger\tilde{a}+1)\ , \quad (3.78)$$

where $a$ and $a^\dagger$ are boson annihilation and creation operators and $\tilde{a}$ and $\tilde{a}^\dagger$ are their tilde conjugates. The thermal vacuum state is proportional to the Perelomov coherent state:

$$|0(\beta)\rangle=\frac{1}{\sqrt{2\bar{n}+1}}\left|\frac{\bar{n}}{\bar{n}+1},0\right\rangle_{\rm P}\ . \quad (3.79)$$

Thus, by using (3.67) with

$$\langle\Psi|=\langle 1|=\sum_{n=0}^{\infty}\langle n,n|\ ,$$

we can obtain the thermal average as

$$G_\beta = \langle 1|\mathcal{F}|0(\beta)\rangle = \frac{1}{\sqrt{2\bar{n}+1}}\left\langle 1\left|\mathcal{F}\right|\frac{\bar{n}}{\bar{n}+1},0\right\rangle_{\rm P} = \frac{\bar{B}_0(n)^{1/2}}{1+\bar{n}-[\bar{n}+(\bar{n}+1)\bar{B}_+(n)][1+\bar{B}_-(n)]\bar{B}_0(n)} . \tag{3.80}$$

Such a calculation of $G_\beta$ will appear frequently in the following sections.

## IV. APPLICATION TO OPTICAL PROCESSES

### A. General treatment

Within the framework of the Liouville-space formulation, this section will use the generalized decomposition formulas obtained in Sec. III to investigate linear and nonlinear optical processes. To see how the generalized decomposition formulas are used to investigate quantum optical processes, we will consider a system made up of an electronic subsystem and a bosonic subsystem, and we will treat an external field classically. In the Liouville space, the time evolution of a state $|\Psi(t)\rangle$ of the system interacting with an external field is governed by

$$\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle = -i[\hat{\mathcal{H}}+\hat{\mathcal{V}}(t)]|\Psi(t)\rangle , \tag{4.1}$$

where $\hat{\mathcal{H}}$ is the time-evolution generator of a matter system (electronic and bosonic systems) and $\hat{\mathcal{V}}(t)$ is defined as $\mathcal{V}(t)-\tilde{\mathcal{V}}(t)$, in which $\mathcal{V}(t)$ is an interaction Hamiltonian between the matter and the external field and $\tilde{\mathcal{V}}(t)$ is the tilde conjugate of $\mathcal{V}(t)$ [29]. When we solve (4.1) by the perturbative method, we obtain

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty}(-i)^n\int_0^{t_{n+1}}dt_n\int_0^{t_n}dt_{n-1}\cdots\int_0^{t_2}dt_1\exp[-i\hat{\mathcal{H}}\Delta t_{n+1,n}]\hat{\mathcal{V}}(t_n) \times\exp[-i\hat{\mathcal{H}}\Delta t_{n,n-1}]\hat{\mathcal{V}}(t_{n-1})\cdots\exp[-i\hat{\mathcal{H}}\Delta t_{2,1}]\hat{\mathcal{V}}(t_1) \times\exp[-i\hat{\mathcal{H}}\Delta t_{1,0}]|\Psi(0)\rangle , \tag{4.2}$$

where we have put $\Delta t_{i,j}=t_i-t_j$, with $t_{n+1}=t$ and $t_0=0$.

We assume that the initial state of the system can be expressed as a direct product of the states of two subsystems, $|\Psi(0)\rangle=|\Psi_E\rangle\otimes|\Psi_B\rangle$, where $|\Psi_E\rangle$ represents the state of the electronic system and where $|\Psi_B\rangle$ represents the state of the bosonic system. Furthermore, the initial electronic state is assumed to be expanded as follows:

$$|\Psi_E\rangle=\sum_{\mathbf{k}}a_{\mathbf{k}}|\mathbf{k}\rangle , \tag{4.3}$$

with

$$\mathbf{k}=(k_1,k_2,\ldots,k_{\max}),\quad k_j=1,2,\ldots,n_{\max} . \tag{4.4}$$

Let us assume, for example, the two-level electronic system. Since in the Liouville space an arbitrary state of this system can be expressed as a linear combination of

$$\{|--\rangle,|-+\rangle,|+-\rangle,|++\rangle\} ,$$

we have $\mathbf{k}=(k_1,k_2)$ and $k_j=\pm$. Note that in the usual Hilbert space the density matrix of the system is expressed in terms of

$$\{|-\rangle\langle-|,|-\rangle\langle+|,|+\rangle\langle-|,|+\rangle\langle+|\} .$$

We assume that the external field changes only the electronic state, according to the following relation:

$$\hat{\mathcal{V}}(t)|\mathbf{k}\rangle=\sum_{\mathbf{k}'}a_{\mathbf{k}',\mathbf{k}}(t)|\mathbf{k}'\rangle . \tag{4.5}$$

Hence we have

$$\hat{\mathcal{V}}(t_n)\hat{\mathcal{V}}(t_{n-1})\cdots\hat{\mathcal{V}}(t_1)|\Psi_E\rangle = \sum_{\mathbf{k}_{n+1}}\sum_{\mathbf{k}_n}\cdots\sum_{\mathbf{k}_1}a_{\mathbf{k}_{n+1},\mathbf{k}_n}(t_n)a_{\mathbf{k}_n,\mathbf{k}_{n-1}}(t_{n-1}) \times\cdots a_{\mathbf{k}_1,\mathbf{k}_0}(t_0)|\mathbf{k}_{n+1}\rangle , \tag{4.6}$$

with $a_{\mathbf{k}_1,\mathbf{k}_0}(t_0)=a_{\mathbf{k}_1}$.

We denote as $\hat{\mathcal{H}}_{\mathbf{k}}$ the time-evolution generator in the $k$th subspace, which is the time-evolution generator of the bosonic system (or the reservoir) under the condition that the electronic system is in the $k$th state $|\mathbf{k}\rangle$. Thus we assume that

$$\exp[-i\hat{\mathcal{H}}\Delta t_{i,j}]|\mathbf{k}\rangle=|\mathbf{k}\rangle\exp[-i\hat{\mathcal{H}}_{\mathbf{k}}\Delta t_{i,j}] . \tag{4.7}$$

Therefore, from (4.2) and (4.6) we obtain

$$|\Psi(t)\rangle=\sum_{n=0}^{\infty}(-i)^n\int_0^{t_{n+1}}dt_n\int_0^{t_n}dt_{n-1}\cdots\int_0^{t_2}dt_1\sum_{\mathbf{k}_{n+1}}\sum_{\mathbf{k}_n}\cdots\sum_{\mathbf{k}_1}|\mathbf{k}_{n+1}\rangle a_{\mathbf{k}_{n+1},\mathbf{k}_n}(t_n)a_{\mathbf{k}_n,\mathbf{k}_{n-1}}(t_{n-1}) \times\cdots a_{\mathbf{k}_2,\mathbf{k}_1}(t_1)a_{\mathbf{k}_1,\mathbf{k}_0}(t_0)\exp[-i\hat{\mathcal{H}}_{\mathbf{k}_{n+1}}\Delta t_{n+1,n}] \times\exp[-i\hat{\mathcal{H}}_{\mathbf{k}_n}\Delta t_{n,n-1}] \times\cdots\exp[-i\hat{\mathcal{H}}_{\mathbf{k}_1}\Delta t_{1,0}]|\Psi_B\rangle . \tag{4.8}$$

Since we are interested in information only about the state of the electronic system, we take the partial average over the bosonic subspace. Thus, we obtain

$$|\Psi_E(t)\rangle = \sum_{n=0}^{\infty}(-i)^n \int_0^{t_{n+1}} dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 \sum_{\mathbf{k}_{n+1}} \sum_{\mathbf{k}_n} \cdots \sum_{\mathbf{k}_1} |\mathbf{k}_{n+1}\rangle a_{\mathbf{k}_{n+1},\mathbf{k}_n}(t_n) a_{\mathbf{k}_n,\mathbf{k}_{n-1}}(t_{n-1}) \times \cdots a_{\mathbf{k}_2,\mathbf{k}_1}(t_1) a_{\mathbf{k}_1,\mathbf{k}_0}(t_0) \mathcal{G}(t_{n+1},t_n,\ldots,t_1) , \tag{4.9}$$

where $\mathcal{G}(t_{n+1},t_n,\ldots,t_1)$ is given by

$$\mathcal{G}(t_{n+1},t_n,\ldots,t_1) = \langle \exp[-i\hat{\mathcal{H}}_{\mathbf{k}_{n+1}}\Delta t_{n+1,n}] \exp[-i\hat{\mathcal{H}}_{\mathbf{k}_n}\Delta t_{n,n-1}] \cdots \exp[-i\hat{\mathcal{H}}_{\mathbf{k}_1}\Delta t_{1,0}] \rangle_B , \tag{4.10}$$

and $\langle\ \rangle_B$ indicates the average over the bosonic subsystem.

We assume that the time-evolution generator $\hat{\mathcal{H}}_{\mathbf{k}}$ of the $k$th bosonic subsystem is expressed in terms of the generators of the su(1,1) Lie algebra. Hence we can write down $\hat{\mathcal{H}}_{\mathbf{k}}$ as

$$-i\hat{\mathcal{H}}_{\mathbf{k}_j}\Delta t_{j,j-1} = z(j)C + a_+(j)K_+ + a_0(j)K_0 + a_-(j)K_- , \tag{4.11}$$

where $C$ is an operator commuting with $K_i$ ($i=\pm,0$). This means that $\mathcal{G}(t_{n+1},t_n,\ldots,t_1)$ can be written as

$$\mathcal{G}(t_{n+1},t_n,\ldots,t_1) = \left\langle \exp\left[\sum_{m=0}^{n} z(m+1,m)C\right] \mathcal{T} \right\rangle_B . \tag{4.12}$$

Here, $\mathcal{T}$ is the same form as that given by (3.2), so the quantity $\mathcal{G}(t_{n+1},t_n,\ldots,t_1)$ can be calculated by using the generalized decomposition formulas derived in the previous sections. When the initial state of the bosonic subsystem is the vacuum state, for example, we can get

$$|\Psi_E(t)\rangle = \sum_{n=0}^{\infty}(-i)^n \int_0^{t_{n+1}} dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 \sum_{\mathbf{k}_{n+1}} \sum_{\mathbf{k}_n} \cdots \sum_{\mathbf{k}_1} |\mathbf{k}_{n+1}\rangle a_{\mathbf{k}_{n+1},\mathbf{k}_n}(t_n) a_{\mathbf{k}_n,\mathbf{k}_{n-1}}(t_{n-1}) \times \cdots a_{\mathbf{k}_2,\mathbf{k}_1}(t_1) a_{\mathbf{k}_1,\mathbf{k}_0}(t_0) [A_0(n+1)A_0(n)A_0(n-1) \times \cdots A_0(2)A_0(1)]^s \times \langle \exp[ZC + A_+(n+1)K_+] \rangle_B , \tag{4.13}$$

with $Z=\sum_{m=0}^{n+1} z(m)$ and $s$ being a certain constant. The average value can be calculated once $C$ and $K_+$ are specified explicitly. The generalized decomposition formulas in Sec. III can be used to calculate all quantities appearing in (4.13): $s$, $A_0(k)$, and $A_+(k)$ ($k=1,2,\ldots,n+1$). We can use this procedure to obtain an explicit form of $|\Psi_E\rangle$ similar to (4.13) even when the bosonic subsystem is in the one-mode (or two-mode) coherent state, the SU(1,1) generalized coherent state, or the thermal equilibrium state.

We can use this result to investigate various kinds of optical processes. In Secs. IV B and IV C, we will calculate the absorption line shape and photon echo signal in a localized electron-phonon system interacting with an intense coherent field.

### B. Absorption line shape

This subsection first presents a model to be considered here and in Sec. IV C. The localized electron-phonon system considered here consists of four subsystems: a photon system (an external field), a localized electron system assumed to be a two-level system, an interaction mode, and a thermal reservoir. The interaction mode corresponds to the adiabatic potential for the localized electron [30]. The thermal reservoir, which is assumed to include all phonon modes except the interaction mode, ensures that the interaction mode is in thermal equilibrium. The Hamiltonian of the system is therefore written as

$$H_T = H_E + H_P + H_R + H_{EP} + H_{PR} + V(t) . \tag{4.14}$$

The Hamiltonian $H_E$ of the localized electron system and the Hamiltonian $H_P$ of the interaction mode are given by

$$H_E = \varepsilon_+ c_+^\dagger c_+ + \varepsilon_- c_-^\dagger c_- \tag{4.15}$$

and

$$H_P = \omega a^\dagger a , \tag{4.16}$$

where $c_+$ ($c_-$) and $c_+^\dagger$ ($c_-^\dagger$) are annihilation and creation operators of the electron in the upper (lower) electronic state whose energy is $\varepsilon_+$ ($\varepsilon_-$), where $a$ and $a^\dagger$ are the boson annihilation and creation operators of the interaction mode, and where $\omega$ is the frequency of the interaction mode.

The interaction $H_{EP}$ between the localized electron and the interaction mode is assumed to be a mutual phase modulation [31]:

$$H_{EP} = g c_+^\dagger c_+ a^\dagger a . \tag{4.17}$$

The coupling constant $g$ in (4.17) represents the change in the curvature of the adiabatic potential due to the transition of the electron between the upper and lower states. This interaction Hamiltonian usually includes the linear terms with respect to the interaction mode, such as $g_1 c_+^\dagger c_+ (a+a^\dagger)$, which indicates the shift of center of the adiabatic potential. This linear term, however, can be removed by a unitary transformation. Although this transformation causes the operator $c_+^\dagger c_+$ to appear in other terms, there is no technical problem because $c_+^\dagger c_+$ commutes with all terms in the Hamiltonian [31]. We can therefore neglect the linear term for the purpose of this paper, although this term is physically important in some cases.

The interaction $H_{PR}$ between the interaction mode and the thermal reservoir is assumed to be linear dissipative coupling [31,32], and the Hamiltonian $H_R$ of the thermal reservoir need not be specified explicitly.

The interaction between the localized electron and the external field is assumed to be

$$V(t) = -\mu E^*(t) c_-^\dagger c_+ - \mu^* E(t) c_+^\dagger c_- \ , \tag{4.18}$$

where $\mu$ is the off-diagonal element of the electric dipole moment operator and $E(t)$ is the positive frequency part of the $c$-number field. We have used the rotating-wave approximation in (4.18).

Next we write the equation of motion for a state vector of the system in the Liouville space. When we eliminate the information about the thermal reservoir by using the projection operator under the van Hove limit (or the Markovian approximation), we can obtain the equation of motion for a state vector $|W(t)\rangle$ of the relevant system (the electronic system and the interaction mode):

$$\frac{\partial}{\partial t}\left|W(t)\right\rangle = -i[\hat{\mathcal{H}} + \hat{\mathcal{V}}(t)]|W(t)\rangle \ . \tag{4.19}$$

Here, $\hat{\mathcal{V}}(t) = V(t) - \tilde{V}(t)$ and $\hat{\mathcal{H}}$ is defined by

$$\hat{\mathcal{H}} = H - \tilde{H} + i\hat{\Pi} \ , \tag{4.20}$$

with $H = H_E + H_{EP} + H_P$ and $\tilde{H}$ being the tilde conjugate of $H$. The damping operator $\hat{\Pi}$ is expressed as

$$\begin{aligned}\hat{\Pi} = -\kappa[\,&(2\bar{n}+1)(a^\dagger a + \tilde{a}^\dagger \tilde{a}) - 2(\bar{n}+1)a\tilde{a} \\ &- 2\bar{n}a^\dagger \tilde{a}^\dagger + 2\bar{n}\,] \ ,\end{aligned} \tag{4.21}$$

where $\bar{n} = (e^{\beta\omega}-1)^{-1}$ and $\kappa$ is a damping constant determined by the correlation function of reservoir variables. This constant characterizes the linear dissipative process of the system. In the usual Hilbert space, (4.21) takes the well-known form as

$$\begin{aligned}\Pi W(t) = \kappa(\,&(\bar{n}+1)\{[aW(t),a^\dagger] + [a, W(t)a^\dagger]\} \\ &+ \bar{n}\{[a^\dagger, W(t)a] + [a^\dagger W(t), a]\}) \ ,\end{aligned} \tag{4.22}$$

where $W(t)$ is the density matrix of the relevant system. When we define the generators of su(1,1) Lie algebra by (3.76)–(3.78), (4.21) can be written as

$$\hat{\Pi} = 2\kappa[(\bar{n}+1)K_- + \bar{n}K_+ - (2\bar{n}+1)K_0 + \tfrac{1}{2}] \ . \tag{4.23}$$

In the Liouville space the complete orthonormal basis for the electronic system is given by

$$S_E = \{|--\rangle, |+-\rangle, |+-\rangle, |++\rangle\} \ ,$$

with $|ij\rangle = |i\rangle \otimes |\tilde{j}\rangle$. Here, $|i\rangle = |\pm\rangle$ is the upper or lower electronic state and $|\tilde{i}\rangle$ is the tilde conjugate of $|i\rangle$. In this case, the term $a_{\mathbf{k},\mathbf{k}'}$ appearing in (4.5) is in the matrix form

$$[a_{\mathbf{k},\mathbf{k}'}(t)] = \begin{bmatrix} 0 & -\mu^* E(t) & \mu E^*(t) & 0 \\ -\mu E^*(t) & 0 & 0 & \mu E^*(t) \\ \mu^* E(t) & 0 & 0 & -\mu^* E(t) \\ 0 & \mu^* E(t) & -\mu E^*(t) & 0 \end{bmatrix} , \tag{4.24}$$

with

$$\mathbf{k} = (i,j) = (--), (+-), (-+), (++) \ .$$

From (4.5), we have

$$\hat{\mathcal{V}}(t)|i,j\rangle = \sum_{k=\pm}\sum_{l=\pm} a_{ij,kl}(t)|k,l\rangle \ . \tag{4.25}$$

Using (4.9), we obtain the electronic state $|\Psi_E(t)\rangle$ at the nontrivial lowest order:

$$\begin{aligned}|\Psi_E(t)\rangle &= \langle 1_B | W(t)\rangle \\ &= \int_0^{t_3=t} dt_2 \int_0^{t_2} dt_1 |\mu|^2 \{\, |++\rangle [E^*(t_2)E(t_1)U_1(t_3,t_2,t_1,t_0) + E(t_2)E^*(t_1)U_2(t_3,t_2,t_1,t_0)] \\ &\qquad - |--\rangle [E^*(t_2)E(t_1)U_3(t_3,t_2,t_1,t_0) + E(t_2)E^*(t_1)U_4(t_3,t_2,t_1,t_0)]\} \ ,\end{aligned} \tag{4.26}$$

where $U_j(t_3,t_2,t_1,t_0)$ is given by

$$U_1(t_3,t_2,t_1,t_0) = \langle \exp[-i\hat{\mathcal{H}}_{++}\Delta t_{32}]\exp[-i\hat{\mathcal{H}}_{+-}\Delta t_{21}]\exp[-i\hat{\mathcal{H}}_{--}\Delta t_{10}]\rangle_B \ , \tag{4.27}$$

$$U_2(t_3,t_2,t_1,t_0) = \langle \exp[-i\hat{\mathcal{H}}_{++}\Delta t_{32}]\exp[-i\hat{\mathcal{H}}_{-+}\Delta t_{21}]\exp[-i\hat{\mathcal{H}}_{--}\Delta t_{10}]\rangle_B \ , \tag{4.28}$$

$$U_3(t_3,t_2,t_1,t_0)=\langle \exp[-i\hat{\mathcal{H}}_{--}\Delta t_{32}]\exp[-i\hat{\mathcal{H}}_{+-}\Delta t_{21}]\exp[-i\hat{\mathcal{H}}_{--}\Delta t_{10}]\rangle_B \ , \tag{4.29}$$

$$U_4(t_3,t_2,t_1,t_0)=\langle \exp[-i\hat{\mathcal{H}}_{--}\Delta t_{32}]\exp[-i\hat{\mathcal{H}}_{-+}\Delta t_{21}]\exp[-i\hat{\mathcal{H}}_{--}\Delta t_{10}]\rangle_B \ , \tag{4.30}$$

with $\hat{\mathcal{H}}_{ij}$'s defined by

$$\hat{\mathcal{H}}_{--}=i\kappa+\omega\hat{N}+2i\kappa[(\bar{n}+1)K_-+\bar{n}K_+-(2\bar{n}+1)K_0] \ , \tag{4.31}$$

$$\hat{\mathcal{H}}_{++}=i\kappa+(\omega+g)\hat{N}+2i\kappa[(\bar{n}+1)K_-+\bar{n}K_+-(2\bar{n}+1)K_0] \ , \tag{4.32}$$

$$\hat{\mathcal{H}}_{+-}=\Delta\varepsilon-\tfrac{1}{2}g+i\kappa+(\omega+\tfrac{1}{2}g)\hat{N}+2i\kappa\left[(\bar{n}+1)K_-+\bar{n}K_+-\left[2\bar{n}+1+i\frac{g}{2\kappa}\right]K_0\right] \ , \tag{4.33}$$

$$\hat{\mathcal{H}}_{-+}=-\Delta\varepsilon+\tfrac{1}{2}g+i\kappa+(\omega+\tfrac{1}{2}g)\hat{N}+2i\kappa\left[(\bar{n}+1)K_-+\bar{n}K_+-\left[2\bar{n}+1-i\frac{g}{2\kappa}\right]K_0\right] \ . \tag{4.34}$$

Here, $\hat{N}=a^\dagger a-\tilde{a}^\dagger\tilde{a}$ and $\Delta\varepsilon=\varepsilon_+-\varepsilon_-$. Note that the average over the interaction mode, $\langle\ \rangle_B$, means the matrix element $\langle 1_B||\Psi_B\rangle$, where $\langle 1_B|=\sum_{n=0}^{\infty}\langle n,n|=\langle 0,0|\exp[a\tilde{a}]$ [see (2.39)], and that the state $|\Psi_B\rangle$ is an initial state of the interaction mode. Since using (2.40) gives us $\langle 1_B|\hat{\mathcal{H}}_{++}=\langle 1_B|\hat{\mathcal{H}}_{--}=0$, (4.26) becomes

$$|\Psi_E(t)\rangle=\int_0^{t_3=t}dt_2\int_0^{t_2}dt_1 2|\mu|^2(|++\rangle-|--\rangle)\mathrm{Re}\{E^*(t_2)E(t_1)G(t_3,t_2,t_1)\exp[-i(t_2-t_1)(\Delta\varepsilon-\tfrac{1}{2}g)+\kappa t_2]\} \ , \tag{4.35}$$

with

$$G(t_3,t_2,t_1)=\langle \exp[-i\hat{\mathcal{H}}_{+-}\Delta t_{21}]\exp[-i\hat{\mathcal{H}}_{--}\Delta t_{10}]\rangle_B \ . \tag{4.36}$$

We assume that at an initial time ($t=0$), the interaction mode is in a thermal equilibrium state. Hence we have

$$|\Psi_B\rangle=\frac{1}{\bar{n}+1}\sum_{n=0}^{\infty}e^{-\beta\omega n}|n,n\rangle \ . \tag{4.37}$$

It is therefore convenient to introduce new operators by the Bogoliubov transformation as follows:

$$\gamma=\frac{\bar{n}+1}{\sqrt{2\bar{n}+1}}a-\frac{\bar{n}}{\sqrt{2\bar{n}+1}}\tilde{a}^\dagger \ , \tag{4.38}$$

$$\gamma^\dagger=\frac{\bar{n}+1}{\sqrt{2\bar{n}+1}}a^\dagger-\frac{\bar{n}}{\sqrt{2\bar{n}+1}}\tilde{a} \tag{4.39}$$

and

$$\tilde{\gamma}=\frac{\bar{n}+1}{\sqrt{2\bar{n}+1}}\tilde{a}-\frac{\bar{n}}{\sqrt{2\bar{n}+1}}a^\dagger \ , \tag{4.40}$$

$$\tilde{\gamma}^\dagger=\frac{\bar{n}+1}{\sqrt{2\bar{n}+1}}\tilde{a}^\dagger-\frac{\bar{n}}{\sqrt{2\bar{n}+1}}a \ . \tag{4.41}$$

It is easily seen from (2.41)–(2.44) that $[\gamma,\gamma^\dagger]=[\tilde{\gamma},\tilde{\gamma}^\dagger]=1$. It is important to note that

$$\gamma|\Psi_B\rangle=\tilde{\gamma}|\Psi_B\rangle=0 \ . \tag{4.42}$$

This means that $\gamma$ and $\tilde{\gamma}$ are the annihilation operators for the thermal equilibrium state. Thus if we express all quantities appearing in (4.36) in terms of $\gamma$, $\gamma^\dagger$, $\tilde{\gamma}$, and $\tilde{\gamma}^\dagger$, the thermal average $\langle\ \rangle_B$ is equivalent to the vacuum average.

We rewrite (4.36) in terms of (4.33) and (4.31), and we have

$$G=\Big\langle \exp\Big\{2\kappa(t_2-t_1)\left[1-\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}\right]\Lambda_- -2\kappa(t_2-t_1)\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}\Lambda_+ -2\kappa(t_2-t_1)\left[1+\frac{ig}{2\kappa}\frac{(\bar{n}+1)^2+\bar{n}^2}{2\bar{n}+1}\right]\Lambda_0\Big\} \times\exp\{2\kappa t_1(\Lambda_--\Lambda_0)\}\Big\rangle_B \ , \tag{4.43}$$

where $\Lambda_+$, $\Lambda_0$, and $\Lambda_-$ are the su(1,1) generators defined by

$$\Lambda_+=\gamma^\dagger\tilde{\gamma}^\dagger \ , \tag{4.44}$$

$$\Lambda_-=\gamma\tilde{\gamma} \ , \tag{4.45}$$

$$\Lambda_0=\tfrac{1}{2}(\gamma^\dagger\gamma+\tilde{\gamma}^\dagger\tilde{\gamma}+1) \ . \tag{4.46}$$

In deriving (4.43), we have used $\langle 1_B|\hat{N}=0$ and $[\hat{N},K_i]=0$ ($i=\pm,0$).

We find that (4.43) is equivalent to (3.1) with $n=2$ in (3.2) and $|\Phi\rangle$ is a vacuum state and $\langle\Psi|=\langle 1_B|$. In this case,

$$\{a_\pm(k),a_0(k)|k=1,2\}$$

are given by

$$a_+(1)=2\kappa t_1 \ , \tag{4.47}$$

$$a_-(1)=0 \ , \tag{4.48}$$

$$a_0(1)=-2\kappa t_1 \tag{4.49}$$

and

$$a_-(2)=2\kappa(t_2-t_1)\left[1-\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}\right], \quad (4.50)$$

$$a_+(2)=-2\kappa(t_2-t_1)\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}, \quad (4.51)$$

$$a_0(2)=-2\kappa(t_2-t_1)\left[1+\frac{ig}{2\kappa}\frac{(\bar{n}+1)^2+\bar{n}^2}{2\bar{n}+1}\right]. \quad (4.52)$$

Thus from (3.16)–(3.20), we obtain

$$A_-(1)=1-e^{-2\kappa t_1}, \quad (4.53)$$

$$A_+(1)=0, \quad (4.54)$$

$$A_0(1)=e^{-2\kappa t_1} \quad (4.55)$$

and

$$A_+(2)=\frac{-2\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}\sinh[\kappa p(t_2-t_1)]}{p\cosh[\kappa p(t_2-t_1)]+\left[1+\frac{ig}{2\kappa}\frac{(\bar{n}+1)^2+\bar{n}^2}{2\bar{n}+1}\right]\sinh[\kappa p(t_2-t_1)]}, \quad (4.56)$$

$$A_-(2)=\frac{2\left[1-\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}\right]\sinh[\kappa p(t_2-t_1)]}{p\cosh[\kappa p(t_2-t_1)]+\left[1+\frac{ig}{2\kappa}\frac{(\bar{n}+1)^2+\bar{n}^2}{2\bar{n}+1}\right]\sinh[\kappa p(t_2-t_1)]}, \quad (4.57)$$

$$A_0(2)=\left\{\frac{p}{p\cosh[\kappa p(t_2-t_1)]+\left[1+\frac{ig}{2\kappa}\frac{(\bar{n}+1)^2+\bar{n}^2}{2\bar{n}+1}\right]\sinh[\kappa p(t_2-t_1)]}\right\}^2, \quad (4.58)$$

with

$$p=\left[\left[1+\frac{ig}{2\kappa}\right]^2+4\bar{n}\frac{ig}{2\kappa}\right]^{1/2}. \quad (4.59)$$

Since $\gamma|\Psi_B\rangle=\tilde{\gamma}|\Psi_B\rangle=0$, we have $\Lambda_-|\Psi_B\rangle=0$ and $\Lambda_0|\Psi_B\rangle=\frac{1}{2}|\Psi_B\rangle$. We find from (3.21) that

$$G=e^{-\kappa t_1}\frac{\exp[-\kappa p(t_2-t_1)]}{1-r\{1-\exp[-2\kappa p(t_2-t_1)]\}}, \quad (4.60)$$

with

$$r=\frac{1}{2}\left\{1-\frac{1}{p}\left[1+(2\bar{n}+1)\frac{ig}{2\kappa}\right]\right\}. \quad (4.61)$$

Finally, when we substitute (4.60) into (4.35), the electronic state at the second order with respect to the external field $E(t)$ becomes

$$|\Psi_E(t)\rangle=\int_0^t dt_2\int_0^{t_2}dt_1 2|\mu|^2(|++\rangle-|--\rangle)\mathrm{Re}\left\{E^*(t_2)E(t_1)e^{-i\Delta\varepsilon(t_2-t_1)}\frac{\exp[2\kappa\bar{n}s(t_2-t_1)]}{1-r\{1-\exp[-2\kappa p(t_2-t_1)]\}}\right\}, \quad (4.62)$$

where we put

$$s=(1/2\bar{n})(1+ig/2\kappa-p).$$

For a monochromatic external field $E(t)=E_0e^{-i\Omega t}$, the absorption line shape is given by

$$I(\Omega)=\lim_{t\to\infty}\frac{d}{dt}\langle++|\Psi_E(t)\rangle. \quad (4.63)$$

Therefore, we obtain

$$I(\Omega)=\frac{I_0}{\pi}\mathrm{Re}\int_0^\infty dt\, e^{i(\Omega-\Delta\varepsilon)t}\frac{\exp[2\kappa\bar{n}st]}{1-r[1-\exp(-2\kappa pt)]}$$
$$=\frac{I_0}{\pi}\mathrm{Im}\sum_{n=0}^{\infty}\frac{r^n}{(r-1)^{n+1}}\times\frac{1}{\Omega-\Delta\varepsilon+\frac{1}{2}g+i\kappa[(2n+1)p-1]}, \quad (4.64)$$

where $I_0$ is an unimportant numerical factor and Re(Im) indicates taking the real (imaginary) part.

At the weak coupling limit ($g/\kappa \ll 1$), (4.64) is simplified as

$$I(\Omega)=\frac{I_0}{\pi}\int_0^\infty dt\,\cos[(\Omega-\Delta\varepsilon-g\bar{n})t] \times\exp\left\{\bar{n}(\bar{n}+1)\left[\frac{g}{2\kappa}\right]^2 \times[1-2\kappa t-e^{-2\kappa t}]\right\} . \tag{4.65}$$

This result is equivalent to that obtained for the model in which the upper level in the two-level electronic system is modulated by fluctuation subject to the Gauss-Markov stochastic process. In the fast modulation limit ($\kappa t \gg 1$), $I(\Omega)$ becomes the Lorentzian line shape:

$$I(\Omega)=\frac{I_0}{\pi}\frac{\Gamma}{(\Omega-\Delta\varepsilon-g\bar{n})^2+\Gamma^2} , \tag{4.66}$$

with $\Gamma=2\kappa\bar{n}(\bar{n}+1)(g/2\kappa)^2$. In the slow modulation limit ($\kappa t \ll 1$), on the other hand, $I(\Omega)$ is the Gaussian line shape:

$$I(\Omega)=\frac{I_0}{\sqrt{2\pi\delta^2}}\exp\left\{-\frac{(\Omega-\Delta\varepsilon-g\bar{n})^2}{2\delta^2}\right\} , \tag{4.67}$$

with $\delta=g\sqrt{\bar{n}(\bar{n}+1)}$. These results are well known in the Anderson-Kubo model of the motional narrowing [33].

### C. Photon echo

We will now use the generalized decomposition formulas to investigate the photon echo signal [34] from the localized electron-phonon system. The model Hamiltonian of the system is given by (4.14) with (4.15)–(4.18). In this case, the external field in (4.18) is assumed to be a pulse signal as follows:

$$E(t)=E_1(t)\exp(i\mathbf{k}_1\cdot\mathbf{x}-i\Omega t)+E_2(t)\exp(i\mathbf{k}_2\cdot\mathbf{x}-i\Omega t) , \tag{4.68}$$

where $E_1(t)$ and $E_2(t)$ are the envelope functions of the pulse. Furthermore, we assume that in comparison with the characteristic time of dynamics of the system, the pulse width is extremely short. We can therefore reasonably assume that

$$E_1(t)=\theta_1\delta(t) , \tag{4.69}$$

$$E_2(t)=\theta_2\delta(t-t_s) , \tag{4.70}$$

where $t_s$ is the pulse separation and $\theta_j$ ($j=1,2$) is the area of the pulse. This means that the first pulse is imposed at $t=0$ and the second at $t=t_s$. The photon echo signal is expected to appear at $t=2t_s$.

Since the lowest-order terms that contribute to the photon echo are the third-order terms with a phase factor of

$$\exp[\pm(2\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{x}] ,$$

we consider only these terms. Therefore, by using the same procedure we used to calculate the absorption line shape, from (4.9) we can get the electronic state of the two-level system:

$$|\Psi_E(t)\rangle=G(t,t_s)|+-\rangle+G^*(t,t_s)|-+\rangle , \tag{4.71}$$

with

$$G(t,t_s)=\langle\exp[-i(t-t_s)\hat{\mathcal{H}}_{+-}]\exp[-it_s\hat{\mathcal{H}}_{-+}]\rangle_B , \tag{4.72}$$

where $\hat{\mathcal{H}}_{+-}$ and $\hat{\mathcal{H}}_{-+}$ are defined by

$$\hat{\mathcal{H}}_{+-}=2i\kappa\left[(\bar{n}+1)K_-+\bar{n}K_+-\left[2\bar{n}+1+\frac{ig}{2\kappa}\right]K_0+\tfrac{1}{2}\right] \tag{4.73}$$

and

$$\hat{\mathcal{H}}_{-+}=2i\kappa\left[(\bar{n}+1)K_-+\bar{n}K_+-\left[2\bar{n}+1-\frac{ig}{2\kappa}\right]K_0+\tfrac{1}{2}\right] . \tag{4.74}$$

In deriving (4.71), we have ignored an unimportant numerical factor and we have integrated with respect to the intermediate times. The slowly varying amplitude of the polarization of the localized electron is given by $I(t)=|G(t,t_s)|$.

Since the interaction mode is in the thermal equilibrium state, we express $I(t)$ in terms of $\gamma$, $\gamma^\dagger$, $\tilde{\gamma}$, and $\tilde{\gamma}^\dagger$ defined by (4.38)–(4.41). Thus we have

$$I(t)=e^{\kappa t}\left|\left\langle\exp\left\{2\kappa(t-t_s)\left[\left[1-\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}\right]\Lambda_--\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}\Lambda_+-\left[1+\frac{ig}{2\kappa}\frac{(\bar{n}+1)^2+\bar{n}^2}{2\bar{n}+1}\right]\Lambda_0\right]\right\}\right.\right.$$
$$\left.\left.\times\exp\left\{2\kappa t_s\left[\left[1+\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}\right]\Lambda_-+\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}\Lambda_+-\left[1-\frac{ig}{2\kappa}\frac{(\bar{n}+1)^2+\bar{n}^2}{2\bar{n}+1}\right]\Lambda_0\right]\right\}\right\rangle_B\right| . \tag{4.75}$$

This is the same form as (3.1) with (3.2) and the terms $\{a_+(k),a_0(k),a_-(k)|k=1,2\}$ in (3.2) are expressed as

$$a_-(1)=2\kappa t_s\left[1+\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}\right] , \tag{4.76}$$

$$a_+(1)=2\kappa t_s\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1} , \tag{4.77}$$

$$a_0(1)=-2\kappa t_s\left[1-\frac{ig}{2\kappa}\frac{(\bar{n}+1)^2+\bar{n}^2}{2\bar{n}+1}\right] \tag{4.78}$$

and

$$a_-(2)=2\kappa(t-t_s)\left[1-\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}\right], \tag{4.79}$$

$$a_+(2)=-2\kappa(t-t_s)\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}, \tag{4.80}$$

$$a_0(2)=-2\kappa(t-t_s)\left[1+\frac{ig}{2\kappa}\frac{(\bar{n}+1)^2+\bar{n}^2}{2\bar{n}+1}\right]. \tag{4.81}$$

When we substitute (4.76)–(4.81) into (3.16)–(3.20), we have

$$A_0(1)=\left[\frac{p^*}{\sinh(\kappa p^* t_s)}\right]^2\left[1-\frac{ig}{2\kappa}\frac{(\bar{n}+1)^2+\bar{n}^2}{2\bar{n}+1}+p^*\coth(\kappa p^* t_s)\right]^{-2}, \tag{4.82}$$

$$A_-(1)=2\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}\left[1-\frac{ig}{2\kappa}\frac{(\bar{n}+1)^2+\bar{n}^2}{2\bar{n}+1}+p^*\coth(\kappa p^* t_s)\right]^{-2}, \tag{4.83}$$

$$A_+(1)=2\left[1+\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}\right]\left[1-\frac{ig}{2\kappa}\frac{(\bar{n}+1)^2+\bar{n}^2}{2\bar{n}+1}+p^*\coth(\kappa p^* t_s)\right]^{-2} \tag{4.84}$$

and

$$A_0(2)=\left[\Xi(t,t_s)\frac{p}{\sinh[\kappa p(t-t_s)]}\right]^2,$$

$$A_-(2)=\frac{2}{\Xi(t,t_s)}\left[1-\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}\right], \tag{4.85}$$

$$A_+(2)=2\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}\frac{1}{\Xi(t,t_s)}\left[1+\frac{1+\frac{ig}{2\kappa}\frac{(\bar{n}+1)^2+\bar{n}^2}{2\bar{n}+1}-p\coth[\kappa p(t-t_s)]}{1-\frac{ig}{2\kappa}\frac{(\bar{n}+1)^2+\bar{n}^2}{2\bar{n}+1}+p\coth(\kappa p^* t_s)}\right],$$

where $\Xi(t,t_s)$ is defined by

$$\Xi(t,t_s)=1+\frac{ig}{2\kappa}\frac{(\bar{n}+1)^2+\bar{n}^2}{2\bar{n}+1}+p\coth[\kappa p(t-t_s)]-\frac{\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}\left[1-\frac{ig}{2\kappa}\frac{\bar{n}(\bar{n}+1)}{2\bar{n}+1}\right]\frac{4}{\sinh(\kappa p^* t_s)}}{1-\frac{ig}{2\kappa}\frac{(\bar{n}+1)^2+\bar{n}^2}{2\bar{n}+1}+p^*\coth(\kappa p^* t_s)} \tag{4.86}$$

and $p$ is given by (4.59). After some calculations, we can obtain

$$I(t)=e^{\kappa t}\left|[A_0(2)A_0(1)]^{1/2}\frac{1}{1-A_+(2)}\right|$$

$$=\left|\frac{1}{Z(t,t_s)}\exp[2\kappa\bar{n}s(t-t_s)+2\kappa\bar{n}s^* t_s]\right|, \tag{4.87}$$

where

$$s=(1/2\bar{n})(1+ig/2\kappa-p)$$

and $Z(t,t_s)$ is defined by

$$Z(t,t_s)=1-rf(t-t_s)-r^*f^*(t_s)+\left[1-\frac{s(1-s^*\bar{n})}{s^*(1-s\bar{n})}\left[1-\frac{1}{r}\right]\right]\times|r|^2f(t-t_s)f^*(t_s), \tag{4.88}$$

with

$$f(t)=1-e^{-2\kappa pt}. \tag{4.89}$$

Let us consider the physical meaning of (4.87) [31]. When we take a short-time limit ($\kappa t_s<\kappa t\ll 1$), (4.87) is simplified to

$$I(t)=\exp[-\tfrac{1}{2}g^2\bar{n}(\bar{n}+1)(t-2t_s)^2]. \tag{4.90}$$

This clearly shows that the photon echo signal with the Gaussian profile appears at $t=2t_s$. Note that (4.90) is independent of the dissipative coefficient $\kappa$. This indicates that the effect of the thermal reservoir on the photon echo signal can be neglected at the short-time limit. When we put $\hat{\Omega}=ga^{\dagger}a$, we can rewrite (4.90) as

$$I(t)=\exp[-\tfrac{1}{2}\langle(\Delta\hat{\Omega})^2\rangle_B(t-2t_s)^2] , \tag{4.91}$$

with $\Delta\hat{\Omega}=\hat{\Omega}-\langle\hat{\Omega}\rangle_B$. The Heisenberg equations of motion for the localized electron then become

$$\frac{d}{dt}c_+=-i(\varepsilon_+ +\hat{\Omega})c_+ , \tag{4.92}$$

$$\frac{d}{dt}c_-=-i\varepsilon_- c_- . \tag{4.93}$$

From these equations, we find that $\hat{\Omega}$ is the frequency modulation for the localized electron in the excited state. Thus from (4.92) the width of the echo signal in the short-time region is equal to a reciprocal of the magnitude of fluctuation of the modulation frequency.

At the weak coupling limit ($g/\kappa \ll 1$), if we take up to the second-order cumulant with respect to $g/\kappa$, (4.87) becomes

$$I(t)=\exp\left\{\bar{n}(\bar{n}+1)\left[\frac{g}{2\kappa}\right]^2 \times[-2\kappa t+f_0(t-t_s)+f_0(t_s) +f_0(t-t_s)f_0(t_s)]\right\} , \tag{4.94}$$

where $f_0(t)=1-e^{-2\kappa t}$. This result is equivalent to that obtained for the two-level model with a stochastic frequency modulation where the upper level is modulated by the Gauss-Markov fluctuation [35]. When we express the frequency modulation as $\omega(t)$ subject to the Gauss-Markov process, we can get (4.94) if $\omega(t)$ satisfies

$$\langle\omega(t)\rangle_s=0 , \tag{4.95}$$

$$\langle\omega(t)\omega(s)\rangle_s=\left[\frac{g}{2\kappa}\right]^2\bar{n}(\bar{n}+1)\exp(-2\kappa|t-s|) , \tag{4.96}$$

where $\langle\ \rangle_s$ is the stochastic average. Note that (4.94) shows the photon echo at $t=2t_s$ if the dissipative effect of the thermal reservoir is not large.

In a previous paper [31] we derived (4.87) and (4.88) by using the boson coherent-state representation of the antinormal ordering [36] for bosonic operators of the interaction mode, and we used the numerical calculation of (4.87) to analyze the photon echo signal. The analytical calculation, however, is more complicated than that given in this section. This shows that the Lie-algebra method with the generalized decomposition formulas makes the calculation simple and systematic.

By calculating the absorption line shape and the photon echo signal in the localized electron-phonon system, we have seen usefulness of the generalized decomposition formulas derived in Sec. III. Using the procedure described above, we can also apply them to the investigation of the higher-order optical processes. When we investigate the resonant Raman scattering, for example, we have to calculate the following quantity:

$$\begin{aligned}|\Psi_E\rangle=|\mu|^4(|--\rangle-|++\rangle)\int_0^{t_5=t}dt_4\int_0^{t_4}dt_3\int_0^{t_3}dt_2\int_0^{t_2}dt_1\{&E^*(t_4)E(t_3)E^*(t_2)E(t_1)[G(+-,--,+-)\\&+G(+-,++,+-)]\\&+E^*(t_4)E(t_3)E(t_2)E^*(t_1)[G(+-,--,-+)\\&+G(+-,++,-+)]\\&+E(t_4)E^*(t_3)E^*(t_2)E(t_1)[G(-+,--,+-)\\&+G(-+,++,+-)]\\&+E(t_4)E^*(t_3)E(t_2)E^*(t_1)[G(-+,--,-+)\\&+G(-+,++,-+)]\} ,\end{aligned} \tag{4.97}$$

with

$$G(ij,kl,mn)=\langle\exp[-i\hat{\mathcal{H}}_{ij}\Delta t_{43}]\exp[-i\hat{\mathcal{H}}_{kl}\Delta t_{32}] \times\exp[-i\hat{\mathcal{H}}_{mn}\Delta t_{21}]\rangle_B . \tag{4.98}$$

The generalized decomposition formulas can be used to calculate (4.97) and (4.98).

## V. QUANTUM COUNTING PROCESSES

### A. General treatment

Section IV, using the Lie-algebra method in the Liouville space, considered quantum-statistical properties of matter (a localized electron-phonon system) and treated light as a classical system. In this section we will see that the Lie-algebra method in the Liouville space is also a powerful tool for dealing with nonclassical properties of light. We will consider the photon-counting model proposed by Srinivas and Davies [26], which is based on the theory of the quantum Markov processes [37]. In their model, the counting process and the time evolution of a state under the influence of the photon counter are described by superoperators in the usual Hilbert space. It is therefore convenient to describe the model in the Liouville space $\mathcal{L}$. This subsection presents the general treatment of photon-counting processes in terms of the Lie-algebra method, and Sec. V B uses the method to investigate the quantum-nondemolition measurement of a photon number in the four-wave-mixing model. Section V C considers the electron-counting process [38]. The photon-counting process is described by the su(1,1) Lie algebra and the electron-counting process by the su(2) Lie algebra.

According to Srinivas and Davies [26], a photon-counting measurement in which $m$ photons are counted during time $t$ is characterized by the operation $\hat{\mathcal{N}}_t(m)$, which is a linear positive transformation. Thus in the Liouville space, the state of the system after $m$ photons were counted can be expressed as

$$|\psi_m(t)\rangle = \frac{\hat{\mathcal{N}}_t(m)|\psi\rangle}{\langle 1|\hat{\mathcal{N}}_t(m)|\psi\rangle} , \tag{5.1}$$

where $|\psi\rangle$ is the state of the system before the counting. The linear positive transformation $\hat{\mathcal{N}}_t(m)$ in the Liouville space is given by

$$\hat{\mathcal{N}}_t(m) = \int_0^t dt_m \int_0^{t_m} dt_{m-1} \cdots \int_0^{t_2} dt_1 S(t-t_m) J S(t_m - t_{m-1}) \cdots S(t_2 - t_1) J S(t_1) , \tag{5.2}$$

where $J$ represents the one-count process and means that one photon is taken out of the cavity when the counter registers a photon, and where $S(t_i - t_j)$ describes the time evolution of the system with no count in the interval between $t_i$ and $t_j$. It has been shown [26] that $S(t)$ is a nonunitary operator, satisfying

$$S(t_1+t_2) = S(t_1)S(t_2) , \tag{5.3}$$

$$S(0) = 1 . \tag{5.4}$$

Using (5.2)–(5.4), we find that $\hat{\mathcal{N}}_t(m)$ can be expressed as

$$\hat{\mathcal{N}}_t(m) = \frac{1}{m!} S(t) \int_0^t dt_m \int_0^t dt_{m-1} \cdots \int_0^t dt_1 T[J(t_m)J(t_{m-1}) \cdots J(t_1)] , \tag{5.5}$$

where $J(t) = S(t)^{-1} J S(t)$ and $T$ is the time-ordered product. Hence, we can easily see from (5.5) that

$$\hat{\mathcal{N}}_t(m) = \frac{1}{m!} \frac{\partial^m}{\partial \mu^m} \hat{\mathcal{N}}(t;\mu) \bigg|_{\mu=0} , \tag{5.6}$$

with

$$\hat{\mathcal{N}}(t;\mu) = S(t) T \exp\left[\mu \int_0^t d\tau J(\tau)\right] = T \exp\left[\int_0^t d\tau \dot{S}(\tau) S(\tau)^{-1} + \mu t J\right] . \tag{5.7}$$

This exponential function can be calculated by using the Lie algebra. Thus from (5.1), (5.2), (5.6), and (5.7), the state of the system after the counting measurement is given by

$$|\psi_m(t)\rangle = \frac{|\Psi_m(t)\rangle}{\langle 1|\Psi_m(t)\rangle} , \tag{5.8}$$

with

$$|\Psi_m(t)\rangle = \frac{1}{m!} \frac{\partial^m}{\partial \mu^m} |\Psi(t;\mu)\rangle \bigg|_{\mu=0} \tag{5.9}$$

and

$$|\Psi(t;\mu)\rangle = \hat{\mathcal{N}}(t;\mu)|\psi\rangle . \tag{5.10}$$

When we do not read the result shown by the photon counter, the state of the system becomes

$$|\psi(t)\rangle = \sum_{m=0}^{\infty} |\Psi_m(t)\rangle = |\Psi(t;1)\rangle , \tag{5.11}$$

where we have used (5.7) and (5.10). The time evolution of (5.11) is determined by

$$\frac{\partial}{\partial t} \Big|\psi(t)\Big\rangle = [\dot{S}(t)S(t)^{-1} + J]|\psi(t)\rangle . \tag{5.12}$$

If $S(t)$ is expressed as $\exp[tY]$, we obtain

$$\frac{\partial}{\partial t} \Big|\Psi(t)\Big\rangle = (Y+J)|\Psi(t)\rangle . \tag{5.13}$$

This equation describes the relaxation process of the system. The relaxation is caused by continuous measurement with the photon counter. In this case, since the photon counter absorbs photons but cannot emit, it is equivalent to a thermal reservoir with a temperature of $T=0$. For free photons, (5.13) becomes the master equation for a damped harmonic oscillator.

The probability $P_m(t)$ that the counter registers $m$ counts during time $t$ is given by

$$P_m(t) = \langle 1|\Psi_m(t)\rangle = \frac{1}{m!} \left\langle 1 \left| \frac{\partial^m}{\partial \mu^m} \hat{\mathcal{N}}(t;\mu) \right| \psi \right\rangle \bigg|_{\mu=0} . \tag{5.14}$$

We rewrite this equation as

$$P_m(t) = \frac{1}{m!} \frac{\partial^m}{\partial \mu^m} P(t;\mu) \bigg|_{\mu=0} , \tag{5.15}$$

with $P(t;\mu) = \langle 1|\Psi(t;\mu)\rangle$. The $k$th moment of the photon number registered by the counter during time $t$ is calculated as

$$\overline{n^k} = \sum_{n=0}^{\infty} n^k P_n(t) . \tag{5.16}$$

Then from (5.15) and (5.16), we obtain

$$\overline{n}=\frac{\partial}{\partial\mu}P(t;\mu+1)\bigg|_{\mu=0}, \tag{5.17}$$

$$\overline{n^2}=\overline{n}+\frac{\partial^2}{\partial\mu^2}P(t;\mu+1)\bigg|_{\mu=0}. \tag{5.18}$$

Thus according to the relations

$$\left[\frac{\partial}{\partial\mu}P(t;\mu+1)\bigg|_{\mu=0}\right]^2 \gtreqless \frac{\partial^2}{\partial\mu^2}P(t;\mu+1)\bigg|_{\mu=0}, \tag{5.19}$$

the statistics of the photon number recorded by the counter is characterized by a sub-Poisson, Poisson, or super-Poisson distribution [39].

Next, we obtain the elementary probability density

$$P_m(t,t_m,t_{m-1},\ldots,t_2,t_1)$$

that the counter registers photons at

$$t=t_m,t_{m-1},\ldots,t_2,t_1$$

and none in the rest of the interval $[0,t)$ [26]. By using the functional derivative, we can write this probability density as

$$P_m(t,t_m,t_{m-1},\ldots,t_2,t_1) =\frac{1}{m!}\frac{\delta^m}{\delta\mu(t_m)\delta\mu(t_{m-1})\cdots\delta\mu(t_1)} \times P(t;[\mu(t)])\bigg|_{[\mu(\tau)]=0}, \tag{5.20}$$

where $P(t;[\mu(\tau)])=\langle 1|\hat{\mathcal{N}}(t;[\mu(\tau)])|\psi\rangle$ and $\hat{\mathcal{N}}(t;[\mu(\tau)])$ is defined by

$$\hat{\mathcal{N}}(t;[\mu(\tau)])=S(t)T\exp\left[\int_0^t d\tau\,\mu(\tau)J(\tau)\right] =T\exp\left\{\int_0^t d\tau[\dot{S}(\tau)S(\tau)^{-1}+\mu(\tau)J]\right\}, \tag{5.21}$$

where $\mu(t)$ is a $c$-number function.

When we consider a single-mode free photon as an example, $S(t)$ and $J$ are given by

$$S(t)=\exp[-\lambda t(K_0-\tfrac{1}{2})], \tag{5.22}$$

$$J=\lambda K_-, \tag{5.23}$$

where $K_0=\frac{1}{2}(a^\dagger a+\tilde{a}^\dagger\tilde{a}+1)$, $K_-=a\tilde{a}$, and $K_+=a^\dagger\tilde{a}^\dagger$ are the generators of the su(1,1) Lie algebra. The parameter $\lambda$ characterizes the measurement performed by the photon counter, and its inverse is a measure of the average time that elapses before the counter registers the presence of the photon. Using (5.7) and (5.10), we can get

$$|\Psi(t;\mu)\rangle=\exp[-\lambda t(K_0-\tfrac{1}{2})+\mu\lambda K_-]|\psi\rangle =\exp[-\lambda t(K_0-\tfrac{1}{2})]\exp[\mu(1-e^{-\lambda t})K_-]|\psi\rangle, \tag{5.24}$$

where we have used the normal-order decomposition of the su(1,1) Lie algebra. Thus from (5.8), we obtain

$$|\Psi_m(t)\rangle=\frac{(1-e^{-\lambda t})^m}{m!}\exp[-\lambda t(K_0-\tfrac{1}{2})]K_-^m|\psi\rangle. \tag{5.25}$$

The probability $P(t,m)$ that $m$ photons are registered by the photon counter during time $t$ becomes

$$P(t,m)=\sum_{n=m}^{\infty}\binom{n}{m}\mu(t)^{n-m}[1-\mu(t)]^m\langle n,n|\psi\rangle, \tag{5.26}$$

where $\mu(t)=1-e^{-\lambda t}$ is the effective quantum efficiency of the counter [26]. Note that $\sum_{m=0}^{\infty}P(t,m)=1$ is satisfied. When $\lambda t\ll 1$, this reduces to the quantum Mandel formula [40].

Note that (5.12) becomes

$$\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle=\tfrac{1}{2}\gamma[2a\tilde{a}-a^\dagger a-\tilde{a}^\dagger\tilde{a}]|\Psi(t)\rangle, \tag{5.27}$$

which is a master equation of the damped harmonic oscillator in contact with the thermal reservoir of $T=0$ in the Liouville space.

When the initial state (the premeasurement state) of the photon is a coherent state which in the Liouville space is given by

$$|\psi\rangle=|\alpha\rangle\otimes|\tilde{\alpha}\rangle=e^{-|\alpha|^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^m\alpha^{*n}}{\sqrt{m!n!}}|m,n\rangle, \tag{5.28}$$

(5.24) becomes

$$|\Psi(t;\mu)\rangle=\exp[\mu(1-e^{-\lambda t})|\alpha|^2]|\alpha(t),\alpha(t)\rangle, \tag{5.29}$$

with $\alpha(t)=\alpha e^{-(1/2)\lambda t}$. Thus, from (5.9) we obtain

$$|\Psi_m(t)\rangle=\frac{|\alpha|^{2m}}{m!}(1-e^{-\lambda t})^m|\alpha(t),\alpha(t)\rangle, \tag{5.30}$$

and the normalized photon state is given by

$$|\psi_m(t)\rangle=\frac{|\Psi_m(t)\rangle}{\langle 1|\Psi_m(t)\rangle}=|\alpha(t),\alpha(t)\rangle. \tag{5.31}$$

This shows that except for decreasing the amplitude, the coherent state does not change in the photon-counting measurement. On the other hand, when the initial state is a thermal state (or a chaotic state) given by

$$|\psi\rangle=\frac{1}{1+\overline{n}}\sum_{k=0}^{\infty}\left[\frac{\overline{n}}{1+\overline{n}}\right]^k|k,k\rangle, \tag{5.32}$$

we obtain from (5.8)–(5.10),

$$|\psi_m(t)\rangle=\frac{1}{(1+\overline{n})P(t,m)} \times\sum_{k=m}^{\infty}\binom{k}{m}\left[\frac{\overline{n}}{1+\overline{n}}\right]^k\mu(t)^m \times[1-\mu(t)]^{k-m}|k-m,k-m\rangle \tag{5.33}$$

and

$$P(t,m)=\frac{1}{1+\mu(t)\bar{n}}\left[\frac{\mu(t)\bar{n}}{1+\mu(t)\bar{n}}\right]^m . \quad (5.34)$$

These results show that the combination of the Lie-algebra method and the Liouville-space formulation greatly simplify the calculations required for analysis of photon-counting processes.

### B. Quantum-nondemolition measurement

This subsection uses the Lie-algebra method in the Liouville space to analyze nondemolition (QND) measurement of a photon number. In particular, we will investigate the state reduction by the QND measurement. Consider two coupled harmonic oscillators whose Hamiltonian is given by

$$\mathcal{H}=\mathcal{H}_S+\mathcal{H}_D+\mathcal{H}_{\rm int} , \quad (5.35)$$

with

$$\mathcal{H}_S=\omega_S c^\dagger c , \quad (5.36)$$

$$\mathcal{H}_D=\omega_D a^\dagger a , \quad (5.37)$$

$$\mathcal{H}_{\rm int}=g c^\dagger c(a e^{i\omega t}+a^\dagger e^{-i\omega t}) , \quad (5.38)$$

where $c$ ($c^\dagger$) and $a$ ($a^\dagger$) are annihilation (creation) operators of photons with frequencies $\omega_S$ and $\omega_D$, and $\omega$ is the frequency of a classical external field for which it is assumed that $\omega=\omega_D$. This model describes the four-wave mixing in which one mode is a highly excited field to be treated classically [41]. We refer to the $(c,c^\dagger)$ oscillator as the relevant system and to the $(a,a^\dagger)$ oscillator as the measurement device. The variables of the measurement device are measured by using some apparatus such as a photon counter and balanced homodyne detector in order to get information about the relevant system.

From (5.36)–(5.38) we can easily see that

$$[\mathcal{H}_S,c^\dagger c]=[\mathcal{H}_{\rm int},c^\dagger c]=0 , \quad (5.39)$$

$$[\mathcal{H}_{\rm int},a^\dagger a]\neq 0 . \quad (5.40)$$

Thus, $c^\dagger c$ satisfies the condition for the QND variable, and $a^\dagger a$ is the readout variable that gives us information about the QCD variable $c^\dagger c$ and that is measured by photon counting [42].

Let us consider the photon counting for the device system. In this model, the nonunitary time-evolution generator with no count, $S(t)$, and the one-count operation $J$ for the readout variables in the interaction representation in the Liouville space are given by

$$S(t)=\exp[-it(\mathcal{H}_{\rm int}-\tilde{\mathcal{H}}_{\rm int})-\tfrac{1}{2}\lambda t(a^\dagger a+\tilde{a}^\dagger\tilde{a})] , \quad (5.41)$$

$$J=\lambda a\tilde{a} . \quad (5.42)$$

Then the positive linear transformation described by (5.7) becomes

$$\hat{\mathcal{N}}(t;\mu)=\exp[-t\mathcal{L}] , \quad (5.43)$$

with

$$\mathcal{L}=igA(a+a^\dagger)-ig\tilde{A}(\tilde{a}+\tilde{a}^\dagger)+\tfrac{1}{2}\lambda(a^\dagger a+\tilde{a}^\dagger\tilde{a}^\dagger)-\mu\lambda a\tilde{a} , \quad (5.44)$$

where we have put $A=c^\dagger c$.

To eliminate the linear term with respect to the device variables, we perform a nonunitary transformation as follows:

$$a=a_- - i\frac{2g}{\lambda}A , \quad (5.45)$$

$$a^\dagger=a_+ - i\frac{2g}{\lambda}(A-2\mu\tilde{A}) \quad (5.46)$$

and

$$\tilde{a}=\tilde{a}_- + i\frac{2g}{\lambda}\tilde{A} , \quad (5.47)$$

$$\tilde{a}^\dagger=\tilde{a}_+ + i\frac{2g}{\lambda}(\tilde{A}-2\mu A) . \quad (5.48)$$

Note that

$$[a_-,a_+]=[\tilde{a}_-,\tilde{a}_+]=1 .$$

Hence we have

$$\mathcal{L}=\frac{2g^2}{\lambda}(A^2-2\mu A\tilde{A}+\tilde{A}^2)+\tfrac{1}{2}(a_+a_-+\tilde{a}_+\tilde{a}_-)-\mu\lambda a_-\tilde{a}_- . \quad (5.49)$$

When we define the generators of the su(1,1) Lie algebra as

$$\mathcal{K}_0=\tfrac{1}{2}(a_+a_-+\tilde{a}_+\tilde{a}_-+1) , \quad (5.50)$$

$$\mathcal{K}_+=a_+\tilde{a}_+ , \quad (5.51)$$

$$\mathcal{K}_-=a_-\tilde{a}_- , \quad (5.52)$$

(5.49) is expressed as

$$\mathcal{L}=\frac{2g^2}{\lambda}(A^2-2\mu A\tilde{A}+\tilde{A}^2)+\lambda(\mathcal{K}_0-\tfrac{1}{2})-\mu\lambda\mathcal{K}_- . \quad (5.53)$$

Thus we obtain the expression for $\hat{\mathcal{N}}(t;\mu)$:

$$\hat{\mathcal{N}}(t;\mu)=\exp\left[-\frac{2g^2t}{\lambda}(A^2-2\mu A\tilde{A}+\tilde{A}^2)\right]\times\exp[-\lambda t(\mathcal{K}_0-\tfrac{1}{2})+\mu\lambda t\mathcal{K}_-] . \quad (5.54)$$

The second exponential in (5.54) has the same form that it does in the corresponding equation for free photons, and we can solve it in the same way.

We assume that the device system is in the vacuum state at the initial time ($t=0$). Then we have

$$|\Psi\rangle=|\Psi_S\rangle\otimes|0\rangle , \quad (5.55)$$

where $|\Psi_S\rangle$ is the initial state of the relevant system. Note that from $a|0\rangle=\tilde{a}|0\rangle=0$, we have $a_-|0\rangle=i(2g/\lambda)A|0\rangle$ and $\tilde{a}_-|0\rangle=-i(2g/\lambda)\tilde{A}|0\rangle$. Since we would like to know information only about the relevant system, we eliminate the device variables from

$$|\Psi(t;\mu)\rangle = \hat{\mathcal{N}}(t;\mu)|\psi_S\rangle \otimes |0\rangle \ .$$

By tracing out the device variables, we obtain

$$|\Psi_S(t;\mu)\rangle = \langle 1_D|e^{-i\mathcal{L}}|\psi_S\rangle \otimes |0\rangle$$
$$= \exp\left\{-\frac{2g^2}{\lambda}(A^2+\tilde{A}^2)\left[t-\frac{2}{\lambda}(1-e^{-\lambda t/2})\right]+\left[\frac{2g}{\lambda}\right]^2 A\tilde{A}(1-e^{-\lambda t/2})^2 + \mu\frac{4g^2}{\lambda}A\tilde{A}\left[t-\frac{(1-e^{-\lambda t/2})(3-e^{-\lambda t/2})}{\lambda}\right]\right\}\Big|\psi_S\Big\rangle , \tag{5.56}$$

where $\langle 1_D| = \langle 0|\exp[a\tilde{a}]$ corresponds to the trace operation over the device variables in the usual Hilbert space. Thus the state of the relevant system after the photon counter for the device variable $a^\dagger a$ has registered $m$ photons during time $t$ is given by

$$|\psi_S(t)\rangle = G_S(t;m)|\psi_S\rangle$$
$$= \frac{1}{m!F_m(t)}\left[\frac{2g}{\lambda}\right]^{2m}(A\tilde{A})^m f(t)^m \exp\left\{-\frac{1}{2}\left[\frac{2g}{\lambda}\right]^2[(A^2+\tilde{A}^2)f(t)+(A-\tilde{A})^2(1-e^{-\lambda t/2})^2]\right\}\Big|\psi_S\Big\rangle , \tag{5.57}$$

with

$$F_m(t) = \frac{1}{m!}\sum_{n=0}^{\infty}\left[\frac{2gn}{\lambda}\right]^{2m} f(t)^m \exp\left[-\left[\frac{2gn}{\lambda}\right]^2 f(t)\right]\langle n,n|\psi_S\rangle \tag{5.58}$$

and

$$f(t) = \lambda t - (1-e^{-\lambda t/2})(3-e^{-\lambda t/2}) , \tag{5.59}$$

where $\langle n,n|\psi_S\rangle$ is identical with the diagonal element of the density matrix $\langle n|\rho_S|n\rangle$ in the usual Hilbert space $\mathcal{H}$. Note that (5.58) satisfies $\sum_{m=0}^{\infty} F_m(t) = 1$. The function $F_m(t)$ is the probability that $m$ photons of the device system are registered during time $t$. Note that this probability is expressed as

$$F_m(t) = \frac{1}{m!}\left\langle\left[\frac{2gf(t)^{1/2}}{\lambda}A\right]^{2m} \times \exp\left[-\left[\frac{2gf(t)^{1/2}}{\lambda}A\right]^2\right]\right\rangle_S , \tag{5.60}$$

where $\langle\ \rangle_S = \langle 1_S|\,|\psi_S\rangle$ is the average with the premeasurement state of the relevant system.

From (5.57) the probability $P_S(t,n)$ that the relevant system is in the $n$-photon state is found to be

$$P_S(t,n) = \frac{1}{m!F_m(t)}\left[\frac{2gn}{\lambda}\right]^{2m} f(t)^m \times \exp\left[-\left[\frac{2gn}{\lambda}\right]^2 f(t)\right]P_S(0,n) , \tag{5.61}$$

where $P_S(0,n)$ is the initial photon-number distribution in the relevant system. This result (5.61) is identical with that obtained by Milburn and Walls [41], who derived the probability distribution (5.61) directly without obtaining the reduced state vector of the relevant system (5.57). It should be noted that (5.57) has more information than does (5.61): this Lie-algebra method gives us the reduced time-evolution generator $G_S(t;m)$ of the relevant system.

If, although the counter interacts with the device, we do not read the photon counter for the device variables, the state of the relevant system reduces to

$$|\psi_S(t)\rangle = \exp\left\{-\frac{1}{2}\left[\frac{2g}{\lambda}\right]^2[\lambda t - 2(1-e^{-\lambda t/2})] \times (A-\tilde{A})^2\right\}\Big|\psi_S\Big\rangle , \tag{5.62}$$

and the equation of motion becomes

$$\frac{\partial}{\partial t}\Big|\psi_S(t)\Big\rangle = -\frac{2g^2}{\lambda}(1-e^{-\lambda t/2})(A-\tilde{A})^2|\psi_S(t)\rangle \ . \tag{5.63}$$

Since $A = c^\dagger c$, (5.62) and (5.63) describe the pure phase-relaxation process of the relevant system [43]. Note that the photon number of the relevant system, $c^\dagger c$, is conserved in this relaxation process: This is consistent with the QND measurement of a photon number. Now we expand $|\psi_S\rangle$ in terms of the phase eigenstates $\{|\phi,m\rangle\rangle\}$ [44] as follows:

$$|\psi_S\rangle = \sum_{m=0}^{\infty}\int_{\phi_0-\pi}^{\phi_0+\pi} d\phi\, f(t;\phi,m)|\phi,m\rangle\rangle , \tag{5.64}$$

where the phase eigenstate $|\phi,m\rangle\rangle$ is defined by

$$|\phi,m\rangle\rangle = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}|n,m\rangle\rangle e^{-in\phi} \tag{5.65}$$

and

$$|n,m\rangle\rangle = \theta(n)|m+n,m\rangle + \theta(-n-1)|m,m-n\rangle , \tag{5.66}$$

with $\theta(n)=1$ for $n\geq 0$ and $\theta(n)=0$ for $n<0$. In (5.64), $\phi_0$ is an arbitrary real number that determines the $2\pi$-phase window. The details of $\{|n,m\rangle\rangle\}$ and $\{|\phi,m\rangle\rangle\}$ are given in Ref. [44]. Using (5.64), from (5.63) we can obtain

$$\frac{\partial}{\partial t}f(t;\phi,m)=D(t)\frac{\partial^2}{\partial\phi^2}f(t;\phi,m) , \tag{5.67}$$

with the diffusion coefficient

$$D(t)=\frac{2g^2}{\lambda}(1-e^{-\lambda t/2}) . \tag{5.68}$$

This equation determines the phase diffusion (or the decay of coherence) due to the QND measurement of a photon number.

In this discussion, we have considered the vacuum state for the measurement device. If we express $(a_-,a_+)$ and $(\tilde{a}_-,\tilde{a}_+)$ in terms of the annihilation and creation operators for the thermal state [see (4.38)–(4.41)], we can also treat the thermal state of the device.

### C. Electron counting probability

The quantum counting probability and intensity correlation function are useful for investigating the properties of an electron beam. Saito *et al.* have recently obtained the electron-counting probability by using a method similar to that used for deriving the Mandel formula in quantum optics, and they have shown that the probability distribution is sub-Poissonian and that there is antibunching correlation of electrons [38]. In this subsection we will use the Lie-algebra method to investigate the electron-counting process by modifying Srinivas and Davies' model for the photon-counting process [45]. When we consider an electron, we have to treat it as a quantum field $\psi$. Here we assume $\psi(x)=\sum_k f(x;k)a_k$, where $f(x;k)$ is an expansion coefficient satisfying the wave equation and where $a_k$, the annihilation operator of an electron with momentum $k$, satisfies $[a_k,a_l^\dagger]_+=\delta_{kl}$. We have omitted electron spin because we will consider only free electrons. It is easy to include a spin freedom.

As Srinivas and Davies did for photon counting, we assume the following two operations $J$ and $S(t)$ for the electron-counting process in the Liouville space as follows:

$$S(t)=\exp\left[-\tfrac{1}{2}t\sum_k \lambda_k(a_k^\dagger a_k+\tilde{a}_k^\dagger\tilde{a}_k)\right] , \tag{5.69}$$

$$J=-\sum_k \lambda_k a_k\tilde{a}_k , \tag{5.70}$$

where $\{\lambda_k\}$ characterizes the measurement performed by the electron counter and the minus sign in $J$ is due to the anticommutation relation of electrons. The meaning of $J$ is that one electron is taken out of the system when the counter registers an electron. In specifying these operations, we have used the interaction representation. When we define $J_\pm^{(k)}$ and $J_0^{(k)}$ as

$$J_-^{(k)}=a_k\tilde{a}_k , \tag{5.71}$$

$$J_+^{(k)}=-a_k^\dagger\tilde{a}_k^\dagger , \tag{5.72}$$

$$J_0^{(k)}=\tfrac{1}{2}(a_k^\dagger a_k+\tilde{a}_k^\dagger\tilde{a}-1) , \tag{5.73}$$

$\{J_-^{(k)},J_+^{(k)},J_0^{(k)}\}$ are the generators of the su(2) Lie algebras. Then $J$ and $S(t)$ are expressed in terms of su(2) generators as

$$S(t)=\exp\left[-t\sum_k \lambda_k(J_0^{(k)}+\tfrac{1}{2})\right] , \tag{5.74}$$

$$J=-\sum_k \lambda_k J_-^{(k)} . \tag{5.75}$$

We note that the photon-counting process is described by the su(1,1) Lie algebra and the electron-counting process by the su(2) Lie algebra.

Using the same procedure as that used to obtain the photon-counting probability, we obtain the following probability $P_m(t)$ that $m$ electrons are registered by the counter during time $t$:

$$P_m(t)=\frac{1}{m!}\frac{\partial^m}{\partial\mu^m}P(t;\mu)\bigg|_{\mu=0} , \tag{5.76}$$

with

$$P(t;\mu)=\langle 1|\hat{\mathcal{N}}(t;\mu)|\psi\rangle , \tag{5.77}$$

$$\begin{aligned}\hat{\mathcal{N}}(t;\mu)&=\exp\left[-t\sum_k \lambda_k(J_0^{(k)}+\mu J_-^{(k)}+\tfrac{1}{2})\right]\\&=\exp\left[-t\sum_k \lambda_k(J_0^{(k)}+\tfrac{1}{2})\right]\\&\quad\times\exp\left[-\mu\sum_k(1-e^{-\lambda_k t})J_-^{(k)}\right] .\end{aligned} \tag{5.78}$$

Here, $|\psi\rangle$ is the initial state of electrons and $\mu$ is an ordinary real number but not a Gaussmann number. In deriving (5.78) we have used the normal-order decomposition formula for the su(2) Lie algebra.

Since for electrons, the state $\langle 1|$ is given by

$$\langle 1|=\prod_k [{}_k\langle 0,0|+{}_k\langle 1,1|] \tag{5.79}$$

(where $|0,0\rangle_k$ is the vacuum state of an electron with momentum $k$ and where $|1,0\rangle_k=a_k^\dagger|0,0\rangle_k$, $|0,1\rangle_k=\tilde{a}_k^\dagger|0,0\rangle_k$, and $|1,1\rangle_k=a_k^\dagger\tilde{a}_k^\dagger|0,0\rangle_k$), (5.77) is calculated to be

$$P(t;\mu)=\prod_k \{{}_k\langle 0,0|+[1-\xi_k(t)+\mu\xi_k(t)]_k\langle 1,1|\}|\psi\rangle , \tag{5.80}$$

where $\xi_k(t)=1-e^{-\lambda_k t}$ is the effective quantum efficiency of the counter for the electron with momentum $k$. In the Liouville space, we assume that an initial state $|\psi\rangle$ for electrons is expressed as

$$\begin{aligned}|\psi\rangle=\prod_k [&a_{00}(k)|0,0\rangle_k+a_{01}(k)|0,1\rangle_k\\&+a_{10}(k)|1,0\rangle+a_{11}(k)|1,1\rangle_k] ,\end{aligned} \tag{5.81}$$

where $a_{ij}(k)$ is an expansion coefficient. The normalization condition $\langle 1|\psi\rangle=1$ gives us the relationship

$$a_{00}(k)+a_{11}(k)=1 . \tag{5.82}$$

For the chaotic state of electrons, we have $a_{00}(k)=1-\bar{n}_k$, $a_{11}(k)=\bar{n}_k$, and $a_{10}(k)=a_{01}(k)=0$, where $\bar{n}_k$ is the Fermion distribution function. The chaotic state is determined by maximizing an entropy of the system. Thus (5.80) becomes

$$P(t;\mu)=\prod_k [1-(1-\mu)\xi_k(t)a_{11}(k)] . \tag{5.83}$$

This result shows that the electron-counting process in the modified Srinivas-Davies model is characterized by the effective quantum efficiency $\{\xi_k(t)\}$ and the occupation probability of electrons $\{a_{11}(k)\}$.

Let us calculate the average number of electrons registered by the counter and its fluctuation. It is easily found from (5.76) and (5.83) that $\bar{n}$ and $\overline{n^2}$ are given by

$$\bar{n}=\sum_k \xi_k(t)a_{11}(k) \tag{5.84}$$

and

$$\overline{n^2}=\bar{n}+\sum_k \sum_{\substack{l\\(l\neq k)}} \xi_k(t)\xi_l(t)a_{11}(k)a_{11}(l) . \tag{5.85}$$

From these equations we obtain the fluctuation $\Delta n^2$ of the electron number registered by the counter:

$$\begin{aligned}\Delta n^2&=\overline{n^2}-\bar{n}^2\\&=\bar{n}-\sum_k [\xi_k(t)a_{11}(k)]^2<\bar{n} .\end{aligned} \tag{5.86}$$

Note that

$$0\le\sum_k [\xi_k(t)a_{11}(k)]^2<\bar{n}$$

because

$$0<\xi_k(t)<1 \quad (0<t<\infty)$$

and $0\le a_{11}(k)\le 1$ for all $k$. This result shows that the statistics of the electron number recorded by the counter is described by a sub-Poisson distribution. The sub-Poissonian distribution often indicates the antibunching correlation of electrons, and Silverman also showed the antibunching correlation by calculating the intensity correlation function [46]. The sub-Poissonian distribution of electrons seems to be a consequence of the Pauli exclusion principle [47], but our result does not exclude the possibility of a Poisson or super-Poisson distribution of the electron number when there is a certain correlation of electrons.

When the effective quantum efficiency of the counter is extremely low [$\xi_k(t)\ll 1$], (5.83) can be approximate as

$$P(t;\mu)\approx[1-\tfrac{1}{2}(\mu-1)^2(\bar{n}-\Delta n)]e^{(\mu-1)\bar{n}} . \tag{5.87}$$

Thus the probability $P_m(t)$ that the counter registers $m$ electrons during time $t$ is given by

$$P_m(t)\approx\frac{1}{m!}\bar{n}^m e^{-\bar{n}}\{1-\tfrac{1}{2}\kappa[\bar{n}^2-2\bar{n}m+m(m-1)]\} , \tag{5.88}$$

with $\kappa=(\bar{n}-\Delta n^2)/\bar{n}^2>0$. The factor in braces represents the deviation from the Poisson distribution, ensuring the sub-Poisson distribution. This approximate distribution was first derived by Saito *et al.* [38], who derived it only for the chaotic initial state.

Next we consider electron counting by two electron counters that register the electrons arriving at different times. In this setup, we can directly observe the second-order coherence proposed by Hanbury-Brown and Twiss [48]. The one-count process for the two counters is specified by two operations:

$$J_1=-\sum_k \lambda_k^{(1)}J_-^{(k)} , \tag{5.89}$$

$$J_2=-\sum_k \lambda_k^{(2)}J_-^{(k)} , \tag{5.90}$$

where $\{\lambda_k^{(1)}\}$ and $\{\lambda_k^{(2)}\}$ characterize the measurements performed by the two electron counters, 1 and 2. The nonunitary time evolution with no count is given by

$$S(t)=\exp\left\{-t\sum_k [\lambda_k^{(1)}+\lambda_k^{(2)}](J_0^{(k)}+\tfrac{1}{2})\right\} . \tag{5.91}$$

Using the same procedure we used to derive (5.76)–(5.78), we obtain the probability distribution $P_{n_1n_2}(t)$ that $n_1$ electrons are registered by counter 1 and $n_2$ electrons are registered by the counter 2:

$$P_{n_1n_2}(t)=\frac{1}{n_1!n_2!}\frac{\partial^{n_1+n_2}}{\partial\mu^{n_1}\partial\mu_2^{n_2}}P(t;\mu_1,\mu_2)\bigg|_{\mu_1=\mu_2=0} , \tag{5.92}$$

$$P(t;\mu_1,\mu_2)=\langle 1|\hat{\mathcal{N}}(t;\mu_1,\mu_2)|\psi\rangle , \tag{5.93}$$

$$\begin{aligned}\hat{\mathcal{N}}(t;\mu_1,\mu_2)&=\exp\left\{-t\sum_k [\lambda_k^{(1)}+\lambda_k^{(2)}](J_0^{(k)}+\tfrac{1}{2})-t\sum_k [\mu_1\lambda_k^{(1)}+\mu_2\lambda_k^{(2)}]J_-^{(k)}\right\}\\&=\exp\left\{-t\sum_k [\lambda_k^{(1)}+\lambda_k^{(2)}](J_0^{(k)}+\tfrac{1}{2})\right\}\exp\left\{-\sum_k \frac{\mu_1\lambda_k^{(1)}+\mu_2\lambda_k^{(2)}}{\lambda_k^{(1)}+\lambda_k^{(2)}}\bar{\xi}_k(t)J_-^{(k)}\right\} ,\end{aligned} \tag{5.94}$$

with $\bar{\xi}_k(t)=1-e^{-(\lambda_k^{(1)}+\lambda_k^{(2)})t}$. From (5.79) and (5.81) we obtain

$$P(t;\mu_1,\mu_2)=\prod_k \left\{1-\left[1-\frac{\mu_1\lambda_k^{(1)}+\mu_2\lambda_k^{(2)}}{\lambda_k^{(1)}+\lambda_k^{(2)}}\right]\bar{\xi}_k(t)a_{11}(k)\right\} . \quad (5.95)$$

Since the moment $\overline{n_1^k n_2^l}$ recorded by the two counters is given by

$$\overline{n_1^k n_2^l}=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} n_1^k n_2^l P_{n_1 n_2}(t) , \quad (5.96)$$

$\bar{n}_j$, $\overline{n_j^2}$ $(j=1,2)$, and $\overline{n_1 n_2}$ are

$$\bar{n}_j=\frac{\partial}{\partial\mu_j}P(t;\mu_1+1,\mu_2+1)\bigg|_{\mu_1=\mu_2=0} , \quad (5.97)$$

$$\overline{n_j^2}=\bar{n}_j+\frac{\partial^2}{\partial\mu_j^2}P(t;\mu_1+1,\mu_2+1)\bigg|_{\mu_1=\mu_2=0} , \quad (5.98)$$

$$\overline{n_1 n_2}=\frac{\partial^2}{\partial\mu_1\partial\mu_2}P(t;\mu_1+1,\mu_2+1)\bigg|_{\mu_1=\mu_2=0} . \quad (5.99)$$

Substituting (5.95) into (5.97)–(5.99), we obtain

$$\bar{n}_j=\sum_k \frac{\lambda_k^{(j)}}{\lambda_k^{(1)}+\lambda_k^{(2)}}\bar{\xi}_k(t)a_{11}(k) , \quad (5.100)$$

$$\overline{n_j^2}=\bar{n}_j+\sum_k\sum_{l\,(\neq k)}\frac{\lambda_k^{(j)}\lambda_l^{(j)}}{(\lambda_k^{(1)}+\lambda_k^{(2)})(\lambda_l^{(1)}+\lambda_l^{(2)})}\times\bar{\xi}_k(t)\bar{\xi}_l(t)a_{11}(k)a_{11}(l) , \quad (5.101)$$

$$\overline{n_1 n_2}=\sum_k\sum_{l\,(\neq k)}\frac{\lambda_k^{(1)}\lambda_l^{(2)}}{(\lambda_k^{(1)}+\lambda_k^{(2)})(\lambda_l^{(1)}+\lambda_l^{(2)})}\times\bar{\xi}_k(t)\bar{\xi}_l(t)a_{11}(k)a_{11}(l) . \quad (5.102)$$

When we define the $2\times 2$ symmetric matrix $\kappa_{ij}$ as

$$\kappa_{ij}=\frac{1}{\bar{n}_i\bar{n}_j}\sum_k\frac{\lambda_k^{(i)}\lambda_k^{(j)}}{(\lambda_k^{(1)}+\lambda_k^{(2)})^2}[\bar{\xi}_k(t)a_{11}(k)]^2 , \quad (5.103)$$

from (5.100)–(5.102) we have

$$\frac{\overline{n_j^2}-\bar{n}_j^2}{\bar{n}_j}=1-\kappa_{jj}\bar{n}_j , \quad (5.104)$$

$$\frac{\overline{n_1 n_2}}{\bar{n}_1\bar{n}_2}=1-\kappa_{12} . \quad (5.105)$$

We find from (5.104) that the statistics of the electron number recorded by each counter is characterized by the sub-Poisson probability distribution. We can also see from (5.105) that since $\kappa_{ij}$ is positive definite, the electron-number correlation $\overline{n_1 n_2}$ in the two-counter measurement is smaller than the noncorrelated value $\bar{n}_1\bar{n}_2$. This difference reflects the antibunching correlation of electrons.

When the quantum efficiency is extremely low $[\bar{\xi}_k(t)\ll 1]$, (5.95) becomes

$$P(t;\mu_1,\mu_2)\approx\left\{1-\tfrac{1}{2}\sum_k\left[1-\frac{\mu_1\lambda_k^{(1)}+\mu_2\lambda_k^{(2)}}{\lambda_k^{(1)}+\lambda_k^{(2)}}\right]^2\times[\bar{\xi}_k(t)a_{11}(k)]^2\right\}\times e^{-(1-\mu_1)\bar{n}_1-(1-\mu_2)\bar{n}_2} , \quad (5.106)$$

and the probability $P_{m_1 m_2}(t)$ is given by

$$P_{m_1 m_2}(t)=\frac{\bar{n}_1^{m_1}\bar{n}_2^{m_2}}{m_1!m_2!}e^{-\bar{n}_1-\bar{n}_2}\{1-\tfrac{1}{2}\kappa_{11}[\bar{n}_1^2-2m_1\bar{n}_1+m_1(m_1-1)]-\kappa_{12}(\bar{n}_1-m_1)(\bar{n}_2-m_2)-\tfrac{1}{2}\kappa_{22}[\bar{n}_2^2-2m_2\bar{n}_2+m_2(m_2-1)]\} . \quad (5.107)$$

This result is identical to that obtained by Saito *et al.* for the chaotic initial state of electrons [38].

We have derived the electron-counting probability and have seen that the statistics of the electron number are described by a sub-Poisson distribution. We have also seen that the coincidence probability for two counters shows the antibunching correlation of electrons. The analysis in this section shows that the combination of the Lie-algebra method and the Liouville space formulation gives us a systematic way to treat quantum counting processes.

## VI. SUMMARY

This paper has presented the Lie-algebra formulation for investigating properties of quantum optical processes in the Liouville space. It has derived generalized decomposition formulas that enable us to calculate the expectation values of quantities such as

$$G(n)=\prod_{k=1}^{n}\exp[a_+(k)K_+ + a_0(k)K_0 + a_-(k)K_-] ,$$

where $K_\pm$ and $K_0$ are the generators of the su(1,1) or su(2) Lie algebras. We have calculated the average value $\langle G(n)\rangle$ for typical states in quantum optics, such as the vacuum state, the Glauber coherent state, and the SU(1,1) generalized coherent states.

This paper has shown how the generalized decomposition formulas are used to investigate quantum optical processes, and it has shown that describing the quantum optical systems in the Liouville space enables us to use the Lie-algebra method to investigate a wider class of physical phenomena. Using the generalized decomposition formulas, we have investigated the absorption line shape and the photon echo phenomenon in the localized electron-phonon system. Although these are usually solved by transforming the operator equation into a complex $c$-number differential equation based on the coherent-state representation, the Lie-algebra approach make it possible to solve operator-value equations directly.

We have also seen here that photon-counting processes can be well described by the su(1,1) Lie algebra, and we have investigated the quantum-nondemolition measurement of a photon number in the four-wave-mixing model. When we use the Srinivas and Davies model of the photon-counting processes, in the Liouville space the one-count process and the nonunitary time-evolution operator with no count are expressed in terms of the generators of the su(1,1) Lie algebra. Thus the photon-counting processes can be handled by the Lie-algebra method. This paper has also used the su(2) Lie algebra to derive the electron-counting probability for the modified Srinivas and Davies model. The result shows that the statistics of the electron number counted by the detector is described by a sub-Poisson distribution.

This paper has considered the relaxation processes for boson systems, but is is also possible to treat relaxation processes for fermion systems by using Lie algebra in the Liouville space. The relaxation process for a fermion system can be described by the following operator [22]:

$$\hat{\Pi}_F = -\kappa[(1-2f_F)(c^\dagger c + \tilde{c}^\dagger \tilde{c}) - 2(1-f_F)\tilde{c}c + 2f_F \tilde{c}^\dagger c^\dagger + 2f_F] , \tag{6.1}$$

where $c$ and $c^\dagger$ are fermion annihilation and creation operators, where $\tilde{c}$ and $\tilde{c}^\dagger$ are their tilde conjugates, where $\kappa$ is a damping constant, and $f_F$ is a fermion distribution function. For simplicity, we have omitted the momentum suffix. When we define the generators of the su(2) Lie algebra as follows:

$$K_- = c\tilde{c} , \tag{6.2}$$

$$K_+ = \tilde{c}^\dagger c^\dagger , \tag{6.3}$$

$$K_0 = \tfrac{1}{2}(c^\dagger c + \tilde{c}^\dagger \tilde{c} - 1) , \tag{6.4}$$

(6.1) is expressed as

$$\hat{\Pi}_F = -2\kappa[(1-2f_F)K_0 + (1-f_F)K_- + f_F K_+ + \tfrac{1}{2}] . \tag{6.5}$$

Thus the relaxation process for a fermion system is described by the su(2) Lie algebra, whereas the relaxation process for a boson system is described by the su(1,1) Lie algebra.

Finally, we should note that the su(1,1) and su(2) Lie algebras appear in models of quantum optics other than those kinds of models treated in this paper. The generalized decomposition formulas for the su(1,1) and su(2) Lie algebras derived here, and their combination with the Liouville-space formulation, are therefore very useful and convenient for investigating quantum optical processes.

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[1] K. Wódkiewicz and J. H. Eberly, J. Opt. Soc. Am. B **2**, 458 (1985); P. K. Aravind, *ibid.* **7**, 1545 (1988).

[2] R. Loudon and P. L. Knight, J. Mod. Opt. **34**, 709 (1987).

[3] C. M. Caves and B. L. Schumaker, Phys. Rev. A **31**, 3068 (1985); **31**, 3093 (1985); B. L. Schumaker, Phys. Rep. **135**, 317 (1986).

[4] F. Ciocci, G. Dattoli, A. Renieri, and A. Torre, Phys. Rep. **141**, 1 (1985); C. C. Gerry, Phys. Rev. A **31**, 2721 (1985); G. Dattoli, S. Solimeno, and A. Torre, *ibid.* **35**, 1668 (1987); **34**, 2646 (1986); G. Dattoli and A. Torre, *ibid.* **37**, 1571 (1988); F. M. Fernández, *ibid.* **40**, 41 (1989).

[5] J. Wei and E. Norman, J. Math. Phys. **4**, 575 (1963).

[6] R. M. Wilcox, J. Math. Phys. **8**, 962 (1967).

[7] W. Magnus, Pure Appl. Math. **7**, 649 (1954).

[8] C. C. Gerry, Phys. Rev. A **37**, 2683 (1988); V. Bužek, *ibid.* **39**, 3196 (1989); C. C. Gerry and R. F. Welch, J. Opt. Soc. Am. B **8**, 868 (1991); **9**, 290 (1992).

[9] A. O. Barut and L. Girardello, Commun. Math. Phys. **21**, 41 (1971); D. Bhaumik, K. Bhaumik, and B. Dutta-Roy, J. Phys. A **9**, 1507 (1976).

[10] A. M. Perelomov, Commun. Math. Phys. **26**, 222 (1972); Sov. Phys. Usp. **20**, 703 (1977).

[11] A. M. Perelomov, Generalized Coherent States and Their Applications (Springer-Verlag, New York, 1985).

[12] G. S. Agarwal, J. Opt. Soc. Am. B **5**, 1940 (1988); N. Grønbech-Jensen, P. L. Christiansen, and P. S. Ramanujan, *ibid.* **6**, 2423 (1989); A. Joshi and R. R. Puri, Phys. Rev. A **42**, 4336 (1990); C. C. Gerry, J. Opt. Soc. Am. B **8**, 685 (1991).

[13] S. Prasad, M. O. Scully, and W. Martienssen, Opt. Commun. **62**, 139 (1987); B. Huttner and Y. Ben-Aryeh, Phys. Rev. A **38**, 204 (1988); R. A. Campos, B. E. Saleh, and M. C. Teich, *ibid.* **40**, 1371 (1989).

[14] B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A **33**, 4033 (1986).

[15] W. K. Lai, V. Bužek, and P. L. Knight, Phys. Rev. A **43**, 6323 (1991).

[16] F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A **6**, 2211 (1972).

[17] L. Susskind and J. Glogower, J. Phys. **1**, 49 (1964); P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40**, 411 (1968); E. C. Lerner, H. W. Huang, and G. M. Walters, J. Math. Phys. **11**, 1679 (1970).

[18] C. C. Gerry, Phys. Rev. A **38**, 1734 (1988); A. Vourdas, *ibid.* **41**, 1653 (1990); **43**, 1564 (1991); D. Ellinas, J. Math. Phys. **32**, 135 (1991); J. Mod. Phys. **38**, 2393 (1991); A. Vourdas, Phys. Rev. A **45**, 1943 (1992).

[19] *Lie Methods in Optics,* edited by J. Sánchez Mondragón and K. B. Wolf (Springer-Verlag, New York, 1986).

[20] Y. Nambu, Prog. Theor. Phys. **4**, 331 (1949); U. Fano, Rev. Mod. Phys. **29**, 74 (1957); J. A. Crawford, Nuovo Cimento **5**, 689 (1958); Il. Prigogine, C. George, F. Henin, and L. Rosenfeld, Chem. Scr. **4**, 5 (1973); M. Schmutz, Z. Phys. B **30**, 97 (1978).

[21] Y. Takahashi and H. Umezawa, Collect. Phenom. **2**, 55 (1975); H. Umezawa, H. Matsumoto, and M. Tachiki, *Thermo Field Dynamics and Condensed States* (North-Holland, Amsterdam, 1982); N. P. Landsman and Ch. G.

Weert, Phys. Rep. **145**, 141 (1987).

[22] T. Arimitsu and H. Umezawa, Prog. Theor. Phys. **74**, 429 (1985); **77**, 32 (1987); **77**, 57 (1987); I. Hardman, U. Umezawa, and Y. Yamanaka, J. Math. Phys. **28**, 2952 (1987); H. Umezawa and Y. Yamanaka, Adv. Phys. **37**, 531 (1988).

[23] R. Zwanzig, J. Chem. Phys. **33**, 1338 (1960); F. Shibata, Y. Takahashi, and N. Hashitsume, J. Stat. Phys. **17**, 171 (1977); S. Chaturvedi and F. Shibata, Z. Phys. B **35**, 297 (1979); F. Shibata and T. Arimitsu, J. Phys. Soc. Jpn. **47**, 891 (1980).

[24] M. Ban, J. Math. Phys. **33**, 3213 (1992).

[25] S. Chaturvedi and V. Srinivasan, J. Mod. Opt. **38**, 777 (1991); Phys. Rev. A **43**, 4054 (1991).

[26] M. D. Srinivas and E. B. Davies, Opt. Acta **28**, 981 (1981); **29**, 235 (1982); N. Imoto and M. Ueda, Phys. Rev. A **41**, 4127 (1990).

[27] J. Schwinger, in *Quantum Theory of Angular Momentum,* edited by L. C. Biedenharn and H. Van Dam (Academic, New York, 1965).

[28] R. J. Glauber, Phys. Rev. **131**, 2766 (1963); *Coherent States,* edited by J. R. Klauder and B. Skagerstam (World Scientific, Singapore, 1985).

[29] M. Ban and T. Arimitsu, Physica A **146**, 89 (1987).

[30] Y. Toyozawa, J. Phys. Soc. Jpn. **21**, 1663 (1966).

[31] M. Ban and T. Arimitsu, J. Phys. Soc. Jpn. **55**, 1759 (1986).

[32] J. Schwinger, J. Math. Phys. **2**, 407 (1961); R. P. Feynman and F. L. Vernon, Jr., Am. Phys. **24**, 118 (1963).

[33] P. W. Anderson, J. Phys. Soc. Jpn. **9**, 316 (1954); K. Kubo, *ibid.* **9**, 935 (1954); R. Kubo and T. Toyabe, in *Magnetic Resonance and Relaxation,* edited by R. Blinc (North-Holland, Amsterdam, 1967), p. 810.

[34] L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New York, 1975); Y. R. Shen, *The Principles of Nonlinear Optics* (Wiley, New York, 1984).

[35] E. Hanamura, J. Phys. Soc. Jpn. **52**, 2258 (1983).

[36] G. S. Agarwal and E. Wolf, Phys. Rev. D **2**, 2161 (1970); **2**, 2187 (1970); **2**, 2206 (1970).

[37] E. B. Davies, Commun. Math. Phys. **17**, 277 (1969); **19**, 83 (1970); **22**, 51 (1971); M. D. Srinivas, J. Math. Phys. **18**, 2138 (1977); **19**, 1705 (1978).

[38] S. Saito, J. Endo, T. Kodama, A. Tonomura, A. Fukuhara, and K. Ohbayashi, Phys. Lett. A **162**, 442 (1992).

[39] M. C. Teich and B. E. Saleh, in *Photon Bunching and Antibunching,* Progress in Optics Vol. XXVI (Elsevier, Amsterdam, 1988).

[40] R. J. Grauber, Phys. Rev. **130**, 2529 (1963); L. Mandel, E. C. G. Sudarshan, and E. Wolf, Proc. Phys. Soc. London **84**, 135 (1964); P. L. Kelley and W. H. Kleiner, Phys. Rev. **136**, 316 (1964).

[41] G. J. Milburn and D. F. Walls, Phys. Rev. A **28**, 2646 (1983); **30**, 56 (1984).

[42] N. Imoto, H. A. Haus, and Y. Yamamoto, Phys. Rev. A **32**, 2287 (1985); Y. Yamamoto, S. Machida, S. Saito, N. Imoto, T. Yanagawa, M. Kitagawa, and G. Björk, *Quantum Mechanical Limit in Optical Precision Measurement and Communication,* Progress in Optics Vol. XXVIII (Elsevier, Amsterdam, 1990).

[43] T. Arimitsu, M. Ban, and F. Shibata, Physica **123**, 131 (1984); M. Ban and T. Arimitsu, Physica A **129**, 455 (1985).

[44] M. Ban, Phys. Lett. A **155**, 397 (1991); Physica A **179**, 103 (1991); J. Math. Phys. **32**, 3077 (1991); Found. Phys. Lett. **5**, 297 (1992); J. Opt. Soc. Am. B **9**, 1189 (1992); Opt. Commun. **94**, 231 (1992).

[45] M. Ban, Phys. Lett. A **172**, 337 (1993).

[46] M. P. Silverman, Nuovo Cimento B **97**, 200 (1987).

[47] E. M. Purcell, Nature **178**, 1446 (1956).

[48] R. Hanbury-Brown and R. Q. Twiss, Nature **177**, 27 (1956).