Quantum noise in two- and three-level models of the laser

R. B. Levien,* M. J. Collett, and D. F. Walls
Department of Physics, University of Auckland, Auckland, New Zealand
(Received 18 May 1992; revised manuscript received 25 January 1993)

We compare the quantum fluctuations in two commonly used models of the laser, the Haken-Lamb model [H. Haken, *Laser Theory* (Springer-Verlag, Berlin, 1984)] and the Lax-Louisell model [W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973)]. In the Haken-Lamb model the laser medium is taken to consist of two-level atoms whereas in the Lax-Louisell model it consists of three-level atoms. We compare the predictions of both models for the output statistics of the field in the limit where the atomic variables may be adiabatically eliminated. Our comparison is valid from near threshold to far above threshold. While the three-level model is the more realistic of the two, we find that the simpler two-level model gives good agreement for the amplitude noise, provided this decouples from the phase noise in a given system.

PACS number(s): 42.55.−f, 42.50.Lc

I. INTRODUCTION

The quantum theory of the laser was developed by several groups in the 1960s [1–4]. The models differ in their basic assumptions. In the Lax-Louisell [2,3] model the laser medium is taken to be an ensemble of three-level atoms whereas in the Haken-Lamb [1,4] model the laser medium is taken to be an ensemble of two-level atoms. In order to achieve population inversion the atoms are coupled to an inverted pumping reservoir such that the excitation rate ω_{12} is greater than the deexitation rate ω_{21}. Because a real laser must have more than two levels in order to achieve a population inversion (the inversion is put in "by hand" in the Haken-Lamb model), the Lax-Louisell mode is the more realistic of the two. The aim of this paper is to find under what circumstances the simpler Haken-Lamb model is an acceptable approximation, in the adiabatic limit.

The predictions of both models are identical for the classical dynamics. The difference between the models lie in their predictions for the quantum fluctuations. In the Haken-Lamb model, the diffusion coefficients in the Fokker-Planck equation grow linearly with the pump rate, whereas in the Lax-Louisell model, they approach a limiting value.

A recent investigation into generating squeezed light from a laser with an intracavity second-order nonlinearity $\chi^{(2)}$ found different results depending on the laser model chosen [5]. We wish to give a detailed comparison of the two models, which we shall designate Haken's and Louisell's for simplicity, in the limit where the time scale for the atomic decay is much greater than that of the cavity field. Therefore we can eliminate the atomic variables adiabatically. For the three-level model the adiabatic elimination was done by Louisell (Chap. 10 of Ref. [2]) and we shall restate his results valid from near threshold to far above threshold. In the two-level model the adiabatic elimination has only been done in the near-threshold regime [1,6,7]. We wish to extend this to the region far above threshold so that a comparison may be made with the predictions of the three-level model for both near and far above threshold. We note that Schack, Sizmann, and Schenzle [5] used the near-threshold expansion of the two-level model and Louisell's full expansion of the three-level model in their comparison.

II. THE TWO-LEVEL MODEL

We use Haken's model of the laser [1]. The laser medium consists of N two-level atoms, of transition frequency ω_L, coupled to a single cavity mode at frequency $\omega=\omega_L+\Delta$. The lower lasing level of the jth atom is denoted by $|1\rangle_j$ and the upper lasing level by $|2\rangle_j$. Then the Hamiltonian for the system may be written

$$H=H_{\rm sys}+H_{\rm res}+H_{\rm int} , \tag{2.1}$$

where $H_{\rm sys}$ contains the free parts of the field and atomic Hamiltonians, and their interaction:

$$H_{\rm sys}=\hbar\omega a^\dagger a+\frac{\hbar\omega_L}{2}S_z+ig\hbar(a^\dagger S_- - aS_+) , \tag{2.2}$$

and there are separate reservoirs for the atoms and the field:

$$H_{\rm int}=\hbar\sum_{j=1}^{N}(\Gamma_p^j\sigma_z^j)+\hbar\sum_{j=1}^{N}(\Gamma_a^j\sigma_+^j+\Gamma_a^{j\dagger}\sigma_-^j) +\hbar(\Gamma a^\dagger+\Gamma^\dagger a) , \tag{2.3}$$

with

$$S_z=\sum_{j=1}^{N}\sigma_z^j , \tag{2.4}$$

$$S_\pm=\sum_{j=1}^{N}\sigma_\pm^j e^{\pm i\mathbf{k}\cdot\mathbf{x}_j} . \tag{2.5}$$

The operators $\sigma_z^j,\sigma_\pm^j$ are the usual atomic Pauli operators for the jth atom:

$$\sigma_z^j=\tfrac{1}{2}(|2\rangle_j\langle 2|_j-|1\rangle_j\langle 1|_j) , \tag{2.6}$$

1050-2947/93/47(6)/5030(7)/$06.00

$$\sigma_+^j = |2\rangle_j \langle 1|_j \ , \quad (2.7)$$

$$\sigma_-^j = |1\rangle_j \langle 2|_j \ . \quad (2.8)$$

Thus $2\sigma_z^j$ is the inversion and σ_+^j and σ_-^j are the raising and lowering operators, for the jth atom. S_z is the total atomic inversion and $S_\pm$ can be thought of as the macroscopic dipole moment of all the atoms. $\mathbf{k}$ is the wave vector for the cavity mode and $\mathbf{x}_j$ is the position of the jth atom. For the interaction between the atoms and the cavity mode in Eq. (2.2), the dipole and rotating-wave approximations have been used. g is the dipole coupling constant. In (2.3) the assumption of homogeneous broadening has been made; that is, to model the atomic loss and pumping mechanisms, each of the atomic operators is coupled to independent heat baths. Γ_p^j represents energy loss by collisional or phase-damping processes for the jth atom, while Γ_a^j represents spontaneous emission and incoherent pumping.

The derivation of the Langevin equations via a master equation and Fokker-Planck equation for the generalized P function, in the Markov and rotating-wave approximations, has been well treated by other authors [1,7,6] so here we simply state the Langevin equations:

$$\dot{\alpha} = -(\gamma + i\Delta)\alpha + gv + \Gamma_\alpha \ , \quad (2.9)$$

$$\dot{v} = -\gamma_\perp v + gD\alpha + \Gamma_v \ , \quad (2.10)$$

$$\dot{D} = \gamma_{12} N - \gamma_\parallel D - 2g(v^\dagger \alpha + v\alpha^\dagger) + \Gamma_D \ , \quad (2.11)$$

where

$$\langle \Gamma_\alpha(t)\Gamma_{\alpha^\dagger}(t')\rangle = \langle \Gamma_{\alpha^\dagger}(t)\Gamma_\alpha(t')\rangle = 2\gamma\bar{n}\delta(t-t') \ , \quad (2.12)$$

$$\langle \Gamma_v(t)\Gamma_v(t')\rangle = \langle \Gamma_{v^\dagger}(t)\Gamma_{v^\dagger}(t')\rangle^\dagger = 2gv\alpha\delta(t-t') \ , \quad (2.13)$$

$$\langle \Gamma_{v^\dagger}(t)\Gamma_v(t')\rangle = \langle \Gamma_v(t)\Gamma_{v^\dagger}(t')\rangle = \left[\omega_{12}N + \frac{\gamma_p}{2}(N+D)\right]\delta(t-t') \ , \quad (2.14)$$

$$\langle \Gamma_D(t)\Gamma_D(t')\rangle = [2(\gamma_\parallel N - \gamma_{12} D) - 4g(v^\dagger\alpha + v\alpha^\dagger)]\delta(t-t') \ , \quad (2.15)$$

$$\langle \Gamma_D(t)\Gamma_v(t')\rangle = \langle \Gamma_D(t)\Gamma_{v^\dagger}(t')\rangle^\dagger = -2\omega_{12} v\delta(t-t') \ . \quad (2.16)$$

α is the internal amplitude for the cavity mode, corresponding to the operator a. v is the collective atomic dipole moment, corresponding to the operator S_-, while D is the total atomic inversion, corresponding to the operator $2S_z$. We note that in the generalized P representation, the variable pairs $(\alpha,\alpha^\dagger)$ and $(v,v^\dagger)$ are not constrained to be complex conjugates. We have defined

$$\gamma_\parallel \equiv \omega_{12} + \omega_{21} \ , \quad (2.17)$$

$$\gamma_\perp \equiv \frac{\omega_{12} + \omega_{21} + \gamma_p}{2} \ , \quad (2.18)$$

$$\gamma_{12} \equiv \omega_{12} - \omega_{21} \ , \quad (2.19)$$

where ω_{21} is the loss rate $|2\rangle \to |1\rangle$ due to spontaneous emission, ω_{12} is the incoherent pumping rate $|1\rangle \to |2\rangle$, and γ_p is the rate of collisionally induced phase decay of the atoms (Fig. 1 shows the level scheme, with the transition rates ω_{12} and ω_{21} marked).

$\gamma_\parallel$ is the "longitudinal" decay rate of the population inversion D, $\gamma_\perp$ represents the "transverse" decay rate of the dipole moment v, and $\gamma_{12} = \omega_{12} - \omega_{21}$ is the rate at which the population inversion is built up by the pumping process. The pumping rate ω_{12} must be larger than the atomic decay rate ω_{21} for lasing operation. The detuning Δ between the cavity mode and the lasing transition is defined by

$$\Delta \equiv \omega - \omega_L \ . \quad (2.20)$$

We now carry out the adiabatic elimination. We suppose that the thermal photon number $\bar{n} \simeq 0$, and that

$$\gamma_\perp, \gamma_\parallel \gg \gamma \ . \quad (2.21)$$

This allows us to set $(\dot{v},\dot{D}) = 0$ in Eqs. (2.10) and (2.11) and solve for the adiabatic values of D and v, obtaining

$$D = \frac{\gamma_{12}N + \Gamma_D - \frac{2g}{\gamma_\perp}(\alpha\Gamma_{v^\dagger} + \alpha^\dagger\Gamma_v)}{\gamma_\parallel(1 + \alpha^\dagger\alpha/n_s)} \ , \quad (2.22)$$

$$v = \frac{gD\alpha + \Gamma_v}{\gamma_\perp} \ , \quad (2.23)$$

where the saturation photon number n_s is defined by

$$n_s \equiv \frac{\gamma_\perp\gamma_\parallel}{4g^2} \ . \quad (2.24)$$

We may then substitute the adiabatic values (2.22) and (2.23) back into the equation for the field (2.9) to get

$$\dot{\alpha} = -\gamma\left[1 - \frac{C}{1 + \frac{\alpha^\dagger\alpha}{n_s}}\right]\alpha - i\Delta\alpha + \Gamma'_\alpha \ , \quad (2.25)$$

where the new stochastic force Γ'_α is related to the old forces by

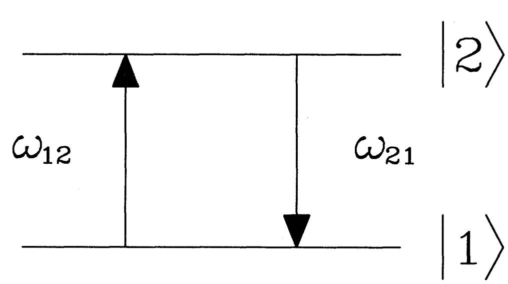

FIG. 1. The two-level atom, showing the transition rates ω_{12} and ω_{21}.

$$\Gamma'_\alpha=\Gamma_\alpha+\left[\frac{g}{\gamma_\perp}\right]\left[\frac{1+\alpha^\dagger\alpha/2n_s}{1+\frac{\alpha^\dagger\alpha}{n_s}}\right]\Gamma_v-\left[\frac{g}{\gamma_\perp}\right]\frac{\alpha^2/2n_s}{1+\frac{\alpha^\dagger\alpha}{n_s}}\Gamma_{v^\dagger}+\left[\frac{g^2}{\gamma_\perp\gamma_\parallel}\right]\frac{\alpha}{1+\frac{\alpha^\dagger\alpha}{n_s}}\Gamma_D\ . \tag{2.26}$$

The scaled laser pump parameter C is defined by

$$C\equiv\frac{Ng^2\gamma_{12}}{\gamma\gamma_\perp\gamma_\parallel}\ . \tag{2.27}$$

We now calculate the correlation functions of the stochastic force Γ'_α. To do this, we evaluate the correlation functions (2.12)–(2.16) at the semiclassical steady state given by solving $(\dot\alpha,\dot v,\dot D)=0$ and ignoring Γ_α, Γ_v and Γ_D; that is, we ignore the noise in the calculation of the noise coefficients. We finally arrive at

$$\begin{aligned}\langle\Gamma'_\alpha(t)\Gamma'_\alpha(t')\rangle&=\langle\Gamma'_{\alpha^\dagger}(t)\Gamma'_{\alpha^\dagger}(t')\rangle^\dagger\\&=-\frac{\gamma C\frac{\alpha^2}{n_s}}{4\sigma_0 p(1+i^s)^3}\{\sigma_0(2+i^s)^2+\sigma_0(i^s)^2+2(1+i^s)(1-\sigma_0)\\&\quad-2(2+i^s)[(\sigma_0+1)(1+i^s)+(p-1)(1+i^s+\sigma_0)]\\&\quad-2\sigma_0(2+i^s)(1+\sigma_0)+2\sigma_0 i^s(1+\sigma_0)\}\delta(t-t')\ ,\end{aligned} \tag{2.28}$$

$$\begin{aligned}\langle\Gamma'_\alpha(t)\Gamma'_{\alpha^\dagger}(t')\rangle&=\langle\Gamma'_{\alpha^\dagger}(t)\Gamma'_\alpha(t')\rangle\\&=\frac{\gamma C}{4\sigma_0 p(1+i^s)^3}\{-2(2+i^s)(i^s)^2\sigma_0+2p(1-\sigma_0)i^s(1+i^s)\\&\quad+\sigma_0[(2+i^s)^2+(i^s)^2][(\sigma_0+1)(1+i^s)+(p-1)(1+i^s+\sigma_0)]\\&\quad-4\sigma_0(1+\sigma_0)i^s\}\delta(t-t')\ ,\end{aligned} \tag{2.29}$$

where

$$i\equiv\frac{\alpha^\dagger\alpha}{n_s} \tag{2.30}$$

is a scaled intensity,

$$\sigma_0\equiv\frac{\gamma_{12}}{\gamma_\parallel} \tag{2.31}$$

is the inversion per atom at the lasing threshold, and

$$p\equiv\frac{2\gamma_\perp}{\gamma_\parallel}=\frac{\gamma_\parallel+\gamma_p}{\gamma_\parallel} \tag{2.32}$$

is a measure of phase damping. p is greater than or equal to 1; the equality occurs when $\gamma_p=0$.

To simplify Eqs. (2.28) and (2.29), we take $\omega_{12}\gg\omega_{21}$, so that

$$\sigma_0=1\ , \tag{2.33}$$

and we consider the limit of large phase damping,

$$p\gg1\ . \tag{2.34}$$

This will turn out (Sec. IV below) to be the logical limit in which to compare the two-level system with Louisell's three-level system. In this limit the Langevin equation (2.25) may be written

$$\frac{d}{dt}\begin{bmatrix}\alpha\\\alpha^\dagger\end{bmatrix}=\mathbf{A}_{\alpha\alpha^\dagger}+\tilde{\mathbf{D}}^{1/2}_{\alpha\alpha^\dagger}\boldsymbol{\eta}(t)\ , \tag{2.35}$$

where $\boldsymbol{\eta}(t)$ is a 2×1 δ-correlated stochastic force vector and the drift and diffusion coefficients are

$$A_\alpha=\gamma\left[\frac{C}{1+\frac{\alpha^\dagger\alpha}{n_s}}-1\right]\alpha-i\Delta\alpha\ , \tag{2.36}$$

$$D_{\alpha\alpha}=D^\dagger_{\alpha^\dagger\alpha^\dagger}=-\frac{\gamma C\alpha^2/n_s(2+i)^2}{2(1+i)^3}\ , \tag{2.37}$$

$$D_{\alpha^\dagger\alpha}=D_{\alpha\alpha^\dagger}=\frac{\gamma C(2+i)(2+2i+i^2)}{2(1+i)^3}\ . \tag{2.38}$$

We are interested in the fluctuations in the intensity of the laser, so we change to amplitude and phase variables (r,ϕ), defined by

$$\alpha=\sqrt{n_s}re^{-i\phi}\ , \tag{2.39}$$

and obtain a new Langevin equation

$$\frac{d}{dt}\begin{bmatrix}r\\\phi\end{bmatrix}=\mathbf{A}_{r\phi}+\tilde{\mathbf{D}}^{1/2}_{r\phi}\boldsymbol{\eta}(t)\ , \tag{2.40}$$

where $\boldsymbol{\eta}(t)$ is the same δ-correlated stochastic force and the new drift and diffusion coefficients are (assuming

$|\alpha|^2 \gg 1$)

$$A_r = \gamma \left[\frac{C}{1+r^2} - 1 \right] r \ , \tag{2.41}$$

$$A_\phi = \Delta \ , \tag{2.42}$$

$$D_{rr} = \frac{\gamma C (1 + r^2/2)}{n_s (1+r^2)^3} \ , \tag{2.43}$$

$$D_{\phi\phi} = \frac{\gamma C (1+r^2/2)}{r^2 n_s (1+r^2)} \ , \tag{2.44}$$

$$D_{r\phi} = 0 \ . \tag{2.45}$$

We note that the amplitude and phase equations decouple; that is, we may write

$$\frac{dr}{dt} = A_r + D_{rr}^{1/2} \eta_1(t) \ , \tag{2.46}$$

$$\frac{d\phi}{dt} = A_\phi + D_{\phi\phi}^{1/2} \eta_2(t) \ . \tag{2.47}$$

We now solve for the deterministic steady state by setting the drift coefficients equal to zero. We find that the steady-state intensity is simply

$$(r^s)^2 = C - 1 \ , \tag{2.48}$$

but the phase ϕ has no steady state. When the detuning $\Delta = 0$, the phase diffuses freely; for nonzero detuning there is a drift motion (a rotation at constant frequency Δ) superimposed on the phase diffusion. This means that the zero-frequency component of the phase spectrum will diverge. We may, however, calculate the amplitude spectrum of the output light:

$$S_{XX}^{\text{out}}(\omega) \equiv \int_{-\infty}^{\infty} dt \, e^{-i\omega t} \langle X^{\text{out}}(t) X^{\text{out}}(0) \rangle^s \ , \tag{2.49}$$

where X^{out} is the usual output amplitude quadrature, and the spectrum is normalized so that zero corresponds to no fluctuations, and one to the vacuum level. Following Gardiner [8] we can calculate $S_{XX}^{\text{out}}(\omega)$ by linearizing Eq. (2.46) about the steady-state amplitude r^s. We find that

$$S_{XX}^{\text{out}}(\omega) = 1 + 8\gamma n_s {:}S_{rr}(\omega){:} \ , \tag{2.50}$$

where

$${:}S_{rr}(\omega){:} \equiv \int_{\infty}^{\infty} dt \, e^{-i\omega t} \langle {:}r(t) r(0){:} \rangle^s \tag{2.51}$$

$$= \left. \frac{D_{rr}}{(\partial A_r / \partial r)^2 + \omega^2} \right|_{r=r^s} . \tag{2.52}$$

Equation (2.52) comes from Gardiner [8]. We note that because the Langevin equations are derived via the P function, the spectra obtained are normally ordered, as indicated by the :: symbols in Eq. (2.51). To get the symmetrically ordered spectrum, we have added one in Eq. (2.50). We finally obtain

$$S_{XX}^{\text{out}}(\omega) = 1 + \frac{8(1+i^s/2)}{4(i^s)^2 + \left[\dfrac{\omega}{\gamma} \right]^2 (1+i^s)^2} \ . \tag{2.53}$$

Figure 2 plots the spectrum in (2.53) for different values of the laser pump parameter C. The spectrum is a simple Lorentzian, with full width at half maximum $4\gamma i^s/(1+i^s)$. We note from (2.53) that the zero-frequency fluctuations are divergent at threshold ($C=1$, $i^s=0$). Above threshold the fluctuations tend towards Poissonian ($S_{XX}^{\text{out}}=1$) at all frequencies.

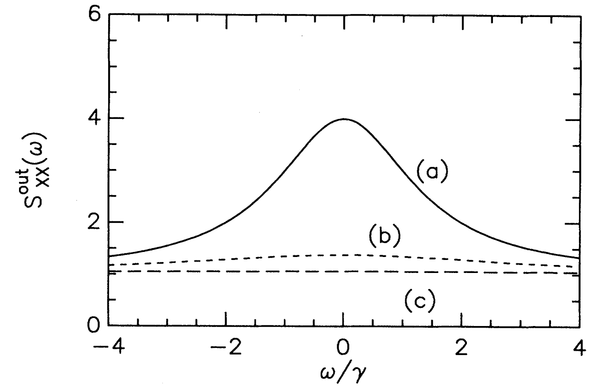

FIG. 2. Output amplitude spectrum for Haken's model, in the adiabatic limit: a plot of Eq. (3.34), for (a) $C=2$ (solid line), (b) $C=5$ (dots), and (c) $C=20$ (dashes).

III. THE THREE-LEVEL MODEL

The results of this section have all been derived by Louisell in Chap. 10 of Ref. [2], so we merely summarize the results contained therein.

Louisell assumes the active laser medium to consist of N homogeneously broadened three-level atoms (see energy-level diagram in Fig. 3). The lasing transition is between the upper two levels. The Hamiltonian for the system is exactly as in Eqs. (2.1)–(2.3), but with the extra atomic level $|0\rangle$, and extra heat baths to model the $|0\rangle \leftrightarrow |1\rangle$ and $|0\rangle \leftrightarrow |1\rangle$ transitions. As in Haken's model, each atom is assumed to couple to its own independent heat baths (the assumption of homogeneous broadening).

The system operators for the three-level system are a, $a^\dagger$, S_-, S_+, N_1, and N_2, where a and $a^\dagger$ are the boson operators for the cavity mode,

$$S_- \equiv \sum_{j=1}^{N} |1\rangle_j \langle 2|_j \ , \tag{3.1}$$

$$S_+ \equiv \sum_{j=1}^{N} |2\rangle_j \langle 1|_j \tag{3.2}$$

are the collective dipole moments for the lasing transition $|1\rangle \leftrightarrow |2\rangle$, and

$$N_1 \equiv \sum_{j=1}^{N} |1\rangle_j \langle 1|_j \ , \tag{3.3}$$

$$N_2 \equiv \sum_{j=1}^{N} |2\rangle_j \langle 2|_j \tag{3.4}$$

are the populations in levels $|1\rangle$ and $|2\rangle$, respectively. Comparing the definitions (3.1) and (3.2) of the dipole moments with Eq. (2.5), we note that there is no factor $e^{\pm i\mathbf{k}\cdot\mathbf{x}_j}$, dependent on the position of each atom. However, given the assumption of homogeneous broadening,

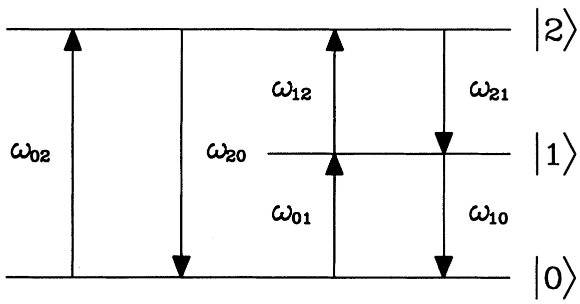

FIG. 3. The three-level atom, showing the various transition rates.

this factor makes no difference to the final equations.

It is assumed that there is a very large number of atoms available in the ground state to be pumped to other levels, so that the depletion of the ground state is negligible, i.e.,

$$N \gg N_1, N_2 \ . \tag{3.5}$$

With the following correspondence between operators and c numbers

$$a \leftrightarrow \alpha \ , \tag{3.6}$$

$$a^\dagger \leftrightarrow \alpha^\dagger \ , \tag{3.7}$$

$$S_- \leftrightarrow v \ , \tag{3.8}$$

$$S_+ \leftrightarrow v^\dagger \ , \tag{3.9}$$

$$N_1 \leftrightarrow \mathcal{N}_1 \ , \tag{3.10}$$

$$N_2 \leftrightarrow \mathcal{N}_2 \ , \tag{3.11}$$

and with a truncation of the Fokker-Planck equation to derivatives of first and second order only, valid when Eq. (3.5) holds, the Langevin equation is

$$\dot{\alpha} = -(\gamma + i\Delta)\alpha + gv + \Gamma_\alpha \ , \tag{3.12}$$

$$\dot{v} = -\gamma_\perp v + g(\mathcal{N}_2 - \mathcal{N}_1)\alpha + \Gamma_v \ , \tag{3.13}$$

$$\dot{\mathcal{N}}_1 = R_1 - \Gamma_1 \mathcal{N}_1 + \omega_{21}\mathcal{N}_2 + g(v^\dagger \alpha + v\alpha^\dagger) + \Gamma_{\mathcal{N}_1} \ , \tag{3.14}$$

$$\dot{\mathcal{N}}_2 = R_2 - \Gamma_2 \mathcal{N}_2 + \omega_{12}\mathcal{N}_1 - g(v^\dagger \alpha + v\alpha^\dagger) + \Gamma_{\mathcal{N}_2} \ , \tag{3.15}$$

where

$$\langle \Gamma_v(t)\Gamma_v(t')\rangle = \langle \Gamma_{v^\dagger}(t)\Gamma_{v^\dagger}(t')\rangle^\dagger = 2gv\alpha\delta(t-t') \ , \tag{3.16}$$

$$\langle \Gamma_{v^\dagger}(t)\Gamma_v(t')\rangle = \langle \Gamma_v(t)\Gamma_{v^\dagger}(t')\rangle = 2[R_2 + (\Gamma_1 + 2\Gamma_{12}^{\rm ph})\mathcal{N}_2 + \omega_{12}\mathcal{N}_1]\delta(t-t') \ , \tag{3.17}$$

$$\langle \Gamma_{\mathcal{N}_1}(t)\Gamma_{\mathcal{N}_1}(t')\rangle = [R_1 + \omega_{21}\mathcal{N}_2 + \Gamma_1\mathcal{N}_1 - g(v^\dagger\alpha + v\alpha^\dagger)]\delta(t-t') \ , \tag{3.18}$$

$$\langle \Gamma_{\mathcal{N}_2}(t)\Gamma_{\mathcal{N}_2}(t')\rangle = [R_2 + \omega_{12}\mathcal{N}_1 + \Gamma_2\mathcal{N}_2 - g(v^\dagger\alpha + v\alpha^\dagger)]\delta(t-t') \ , \tag{3.19}$$

$$\langle \Gamma_{\mathcal{N}_1}(t)\Gamma_{\mathcal{N}_2}(t')\rangle = 2[-\omega_{21}\mathcal{N}_2 - \omega_{12}\mathcal{N}_1 + g(v^\dagger\alpha + v\alpha^\dagger)]\delta(t-t') \ , \tag{3.20}$$

$$\langle \Gamma_{\mathcal{N}_1}(t)\Gamma_v(t')\rangle = \langle \Gamma_{\mathcal{N}_1}(t)\Gamma_{v^\dagger}(t')\rangle^\dagger = 2\Gamma_1 v\delta(t-t') \ , \tag{3.21}$$

$$\langle \Gamma_{\mathcal{N}_2}(t)\Gamma_v(t')\rangle = \langle \Gamma_{\mathcal{N}_2}(t)\Gamma_{v^\dagger}(t')\rangle^\dagger = -2\omega_{12}v\delta(t-t') \ , \tag{3.22}$$

and the various rates are defined as follows:

$$R_1 \equiv N\omega_{01} \ , \tag{3.23}$$

$$R_2 \equiv N\omega_{02} \ , \tag{3.24}$$

$$\Gamma_1 \equiv \omega_{10} + \omega_{12} \ , \tag{3.25}$$

$$\Gamma_2 \equiv \omega_{20} + \omega_{21} \ , \tag{3.26}$$

$$\Gamma_{12} = \Gamma_{12}^{\rm ph} + \tfrac{1}{2}(\Gamma_1 + \Gamma_2) \ . \tag{3.27}$$

$\Gamma_{12}^{\rm ph}$ is an atomic dephasing rate due to collisions and other phase-destroying processes, γ, Δ, and g are as defined in Sec. II, and the transition rates ω_{jk} are as defined in the level picture of Fig. 3.

To simplify the model, Louisell assumes that Γ_1 is much larger than all other decay rates. The atoms in level $|1\rangle$ then decay to the ground state $|0\rangle$ so rapidly that we may set

$$\mathcal{N}_1 = 0 \tag{3.28}$$

and ignore the $\mathcal{N}_1$ equation entirely. This is a sensible limit to take for a normal laser, as the population inversion $\mathcal{N}_2 - \mathcal{N}_1 > 0$ becomes easier to achieve with a very small population in level $|1\rangle$. To complete the adiabatic assumptions, we assume that Γ_2 is much larger than the cavity loss rate γ so that we have

$$\Gamma_2 \gg \Gamma_1 \gg \gamma \ . \tag{3.29}$$

Under these conditions the atomic variables may be adiabatically eliminated to give equations for the field of the form of (2.35), where

$$A_\alpha = \gamma \left[\frac{C}{1 + \dfrac{\alpha^\dagger \alpha}{n_s}} - 1 \right] \alpha - i\Delta\alpha \ , \tag{3.30}$$

$$D_{\alpha\alpha} = D^\dagger_{\alpha^\dagger\alpha^\dagger} = \frac{\gamma C \alpha^2 / n_s}{\left[1 + \dfrac{\alpha^\dagger\alpha}{n_s} \right]^2} \ , \tag{3.31}$$

$$D_{\alpha^\dagger\alpha} = D_{\alpha\alpha^\dagger} = 2\gamma C \frac{1 + \dfrac{\alpha^\dagger\alpha}{2n_s}}{\left[1 + \dfrac{\alpha^\dagger\alpha}{n_s} \right]^2} \ , \tag{3.32}$$

and

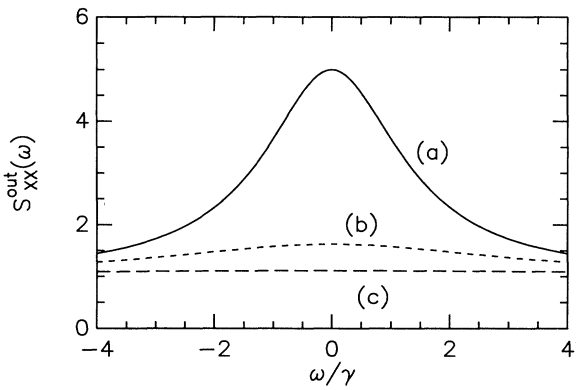

FIG. 4. Output amplitude spectrum from Louisell's model, in the adiabatic limit: a plot of Eq. (3.74), for (a) $C=2$ (solid line), (b) $C=5$ (dots), and (c) $C=20$ (dashes).

$$n_s \equiv \frac{\Gamma_2 \Gamma_{12}}{2g^2} , \tag{3.33}$$

$$C \equiv \frac{R_2}{2\gamma n_s} . \tag{3.34}$$

The saturation photon number n_s and scaled pump rate C are the three-level model equivalents of those defined in Eqs. (2.24) and (2.27).

Changing to the same amplitude and phase variables defined in Eq. (2.39), we find that the amplitude and phase decouple, with the phase rotating at angular frequency Δ and freely diffusing as well. The Langevin equations are of the form of (2.46) and (2.47), with

$$A_r = \gamma \left[\frac{C}{1+r^2} - 1 \right] r , \tag{3.35}$$

$$A_\phi = \Delta \quad , \tag{3.36}$$

$$D_{rr} = \frac{\gamma C}{n_s(1+r^2)^2} , \tag{3.37}$$

$$D_{\phi\phi} = \frac{1}{r^2 n_s} \frac{\gamma C}{1+r^2} . \tag{3.38}$$

Linearizing these equations, we find the output amplitude fluctuation spectrum:

$$S_{XX}^{out}(\omega) = 1 + 8\gamma n_s :S_{rr}(\omega): \tag{3.39}$$

$$= 1 + \frac{8(1+i^s)}{4(i^s)^2 + \left[\frac{\omega}{\gamma}\right]^2 (1+i^s)^2} , \tag{3.40}$$

which is plotted in Fig. 4 for the same values of C as in Fig. 2.

IV. COMPARING THE TWO MODELS IN THE ADIABATIC LIMIT

Louisell's first step in deriving Eqs. (3.30)–(3.32) was to assume that the decay rate Γ_1 from the lower lasing level was so great that the population $\mathcal{N}_1$ of that level could be set to zero, and the $\mathcal{N}_1$ equations ignored entirely. In that case we can compare Eqs. (2.10) and (2.11) of Haken with Eqs. (3.13) and (3.15) of Louisell, identifying Haken's population inversion D with Louisell's upper level population $\mathcal{N}_2$. The deterministic parts of the equations are the same if we make the following identifications:

Haken	Louisell
D	$\mathcal{N}_2$
$\gamma_{12}N$	R_2
$\gamma_\perp$	$\Gamma_{12} = \frac{\Gamma_1+\Gamma_2}{2} + \Gamma_{12}^{ph}$
$\gamma_\parallel$	Γ_2

It is clear that to approximate Louisell's model using Haken's, in the adiabatic limit, we must make two assumptions.

(i) Louisell has taken Γ_1 much larger than all other decay rates, so that

$$\Gamma_{12} \simeq \Gamma_1/2 \gg \Gamma_2 . \tag{4.1}$$

To simulate this Haken's model we must therefore take $\gamma_\perp \gg \gamma_\parallel$, or in other words,

$$p = \frac{2\gamma_\perp}{\gamma_\parallel} = \frac{\gamma_\parallel + \gamma_p}{\gamma_\parallel} \gg 1 . \tag{4.2}$$

We are thus modeling the fast decay from the lower lasing level as a strong phase-destroying process, $\gamma_p \gg \gamma_\parallel$.

(ii) In Louisell's adiabatic approximation the atoms in the lower lasing level $|1\rangle$ decay so quickly to level $|0\rangle$ that there is little probability of reexcitation to level $|2\rangle$. In Haken's model therefore we take

$$\omega_{12} \ll \omega_{21} . \tag{4.3}$$

The unsaturated inversion per atom is then

$$\sigma_0 = \frac{\omega_{21} - \omega_{12}}{\omega_{21} + \omega_{12}} \simeq 1 . \tag{4.4}$$

The best possible agreement between the models of Haken and Louisell then should be in the limit

$$p \gg 1 , \tag{4.5}$$

$$\sigma_0 = 1 . \tag{4.6}$$

This is just the limit in which Eqs. (2.41)–(2.44) are valid, leading to the spectrum (2.53). We note that the amplitude diffusion coefficient D_{rr} is smaller for Haken's model than for Louisell's by a factor

$$\frac{D_{rr}^H}{D_{rr}^L} = \frac{1+i^s/2}{1+i^s} , \tag{4.7}$$

where the superscript H denotes Haken's model, and the superscript L Louisell's model. The drift terms are equal, so the normally ordered spectra will differ only by the factor in Eq. (4.7). We can see from (4.7) that in the

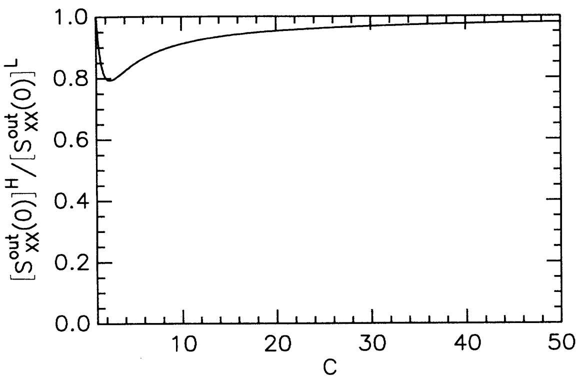

FIG. 5. Comparison between Haken's and Louisell's models in the adiabatic limit: we plot the ratio of zero-frequency symmetrically ordered spectra for the two models.

high-noise region just above threshold ($i^s \ll 1$) the two models agree. Well above threshold ($i^s \gg 1$), the normally ordered spectrum will be smaller by a factor of 2 for the two-level case; however, in both cases the symmetrically ordered spectra (which measures the actual noise present) tend to 1. In Fig. 5 we plot the ratio of the zero-frequency spectra for the two models. There is good agreement just above threshold and well above threshold; the worst disagreement occurs at

$$C = 1 + \sqrt{2} \simeq 2.4 \ , \tag{4.8}$$

when $[S^{\text{out}}_{XX}(0)]^H$ is smaller than $[S^{\text{out}}_{XX}(0)]^L$ by a factor of $(3-\sqrt{2})/2 \simeq 0.79$.

The phase diffusion term $D_{\phi\phi}$ is larger for Haken's than for Louisell's model by a factor

$$\frac{D^H_{\phi\phi}}{D^L_{\phi\phi}} = 1 + i^s/2 \ . \tag{4.9}$$

We have been forced to model the fast transition $|1\rangle \rightarrow |0\rangle$ of Louisell as a phase-destroying process in Haken's model, which has inflated the phase noise. Provided we are primarily interested in the amplitude noise, and the amplitude and phase noise are decoupled (which is certainly the case for the bare laser system considered above, and is often the case even with nonlinear elements inside the laser cavity), we are not concerned by this.

V. CONCLUSION

We were able to make a useful comparison between two often used laser models, those of Haken and Louisell, in the adiabatic limit. To make a connection between the two models in this limit, we modeled the fast transition to the ground state in the three-level model of Louisell, by a fast atomic dephasing process in the two-level model of Haken. This seems reasonable, as in the three-level model we expect the transition to the ground state to destroy the phase relationship between the lasing levels.

We obtained fairly good agreement in the output intensity spectra for the two models, from near threshold to well above threshold. However, the effect of introducing the fast atomic dephasing in two-level model was to inflate the phase noise.

Our overall conclusion then is that Haken's simpler model is an acceptable approximation to that of Louisell, provided that we are primarily interested in intensity fluctuations, and that the intensity and phase fluctuations decouple. However, if the intensity and phase fluctuations are coupled, by introducing a nonlinear element inside the laser cavity (for example, intravity second-harmonic generation, in the bistable region [5,9,10]), then the more realistic model of Louisell should be used.

ACKNOWLEDGMENTS

This research was supported by the New Zealand Vice Chancellors Committee, the University of Auckland Research Committee, the New Zealand Lottery Grants Board, and IBM New Zealand.

---

*Present address: Department of Physics, Princeton University, Princeton, NJ 08544.

[1] H. Haken, *Laser Theory* (Springer-Verlag, Berlin, 1984).
[2] W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).
[3] M. Lax and W. H. Louisell, Phys. Rev. **185**, 568 (1969).
[4] M. Sargent, M. O. Scully, and W. E. Lamb, *Laser Physics* (Addision-Wesley, Reading, MA, 1974).
[5] R. Schack, A. Sizmann, and A. Schenzle, Phys. Rev. A **43**, 6303 (1991).
[6] M. A. M. Marte and D. F. Walls, Phys. Rev. A **37**, 1235 (1988).
[7] M. A. M. Marte, H. Ritsch, and D. F. Walls, Phys. Rev. A **38**, 3577 (1988).
[8] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1983).
[9] D. F. Walls, M. J. Collett, and A. S. Lane, Phys. Rev. A **42**, 4366 (1990).
[10] R. B. Levien, M. Sc. thesis, University of Auckland, 1991.