Inversionless amplification in a multilevel system

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We prove that the amplification condition for a multifrequency field in an atomic system with multiple splitting of the operating levels implies a population inversion between the eigenstates of the density matrix. This explains the possibility of amplification without population inversion when a coherent superposition state of sublevels is prepared.

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INTRODUCTION

The amplification of a monochromatic field in a two-level system requires population inversion when the field and the atomic frequencies coincide (resonant operation). Recently, many papers [1–4] dealt with the generalization of this result to the Λ configuration. This configuration can be considered as resulting from the lower-level splitting in a two-level medium. Let the levels be labeled 1, 2, and 3, with atomic transition frequencies $\omega_{21} \ll \omega_{32} < \omega_{31}$. It was shown that the amplification of a bichromatic field with frequencies $\nu_a \simeq \omega_{31}$ and $\nu_b \simeq \omega_{32}$ in a three-level medium with a Λ configuration is possible in the absence of population inversion at both optical transitions ω_{32} and ω_{31}. As demonstrated in [3], this process occurs through the excitation of a coherent superposition of the two lower sublevels when the amplitude of the low-frequency (LF) coherence ρ_{12} is large enough, $|\rho_{12}|^2 > (\rho_{11}-\rho_{33})(\rho_{22}-\rho_{33})$, where ρ_{kk} is the population of level k. Different ways to excite the LF coherence were proposed in [1–4]. In particular, detailed studies were published, which considered the use of either a resonant microwave field or a bichromatic coherent field to pump resonantly an adjacent transition in a double-Λ scheme.

The recent experimental results of Gao *et al.* [5] on amplification without inversion have emphasized the need to address the difficult problem of amplification in multilevel systems more systematically. There are many possible configurations that can be considered, and the experimental scheme of Gao *et al.*, based on Na atoms, is surely not the simplest. As a first step towards a study of multilevel atoms with multimode fields, we propose to generalize the amplification condition for the case of multiple splitting of either or both operating levels. More precisely, we want to formulate a general amplification condition for a multifrequency field in a medium with a multiple-level splitting when each component of the field interacts with its own resonant transition. Although this is a rather particular scheme, it has the advantage of being amenable to an analytic study.

I. MULTIPLE SPLITTING OF THE LOWER LEVEL

A. Characteristic equation

We first consider the propagation of a weak multifrequency field in a medium with multiple splitting of the lower level. The lower levels are labeled $1,2,\ldots,M-1$, in order of increasing energy levels, and the upper state is labeled M. The transition diagram is displayed in Fig. 1. We assume that before the weak field is applied, the medium is in a steady state determined by external sources, which excite a coherent superposition of the lower sublevels. Thus the slowly varying envelope σ_{ij} of the nondiagonal density-matrix elements with $1 \le i \le M-1$ and $1 \le j \le M-1$ are constants fixed by these external sources. The field propagation is described by a set of wave equations for the slowly varying complex amplitudes of the fields α_j with $1 \le j \le M-1$, and by the

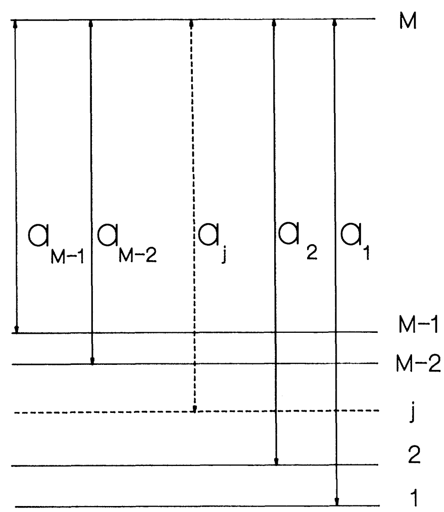

FIG. 1. Transition scheme with lower operating level splitting.

equations for the slowly varying complex amplitudes of the resonant optical polarizations σ_{Mj} with $1 \le j \le M-1$:

$$\frac{\partial \alpha_j}{\partial z} + c^{-1}\frac{\partial \alpha_j}{\partial t} = 2\pi i \nu_j |\mu_{Mj}|^2 N \sigma_{Mj}/c\hbar \ , \tag{1a}$$

$$\frac{\partial \sigma_{Mj}}{\partial t} = -\sigma_{Mj}(\gamma_{Mj} + i\Delta_{Mj}) - i\alpha_j \sigma_{MM} + i\sum_{k=1}^{M-1}\alpha_k \sigma_{kj} \ . \tag{1b}$$

Here ν_j is the carrier frequency of the j component of the field and N is the atomic density. The optical transition $M-j$ is characterized by a dipole matrix element μ_{Mj}, a polarization relaxation rate γ_{Mj}, and an atomic frequency difference ω_{Mj}. The detuning is $\Delta_{Mj} = \omega_{Mj} - \nu_j$. All summations in this section are from 1 to $M-1$. Therefore, as a convention, we shall omit the lower and upper bounds in all sums. Similarly, all products are from 1 to $M-1$ and therefore their limits will also be omitted.

The assumption that the σ_{ij} with i and $j \le M-1$ are constants implies that the set of Eqs. (1) is linear in α_j and σ_{Mj}. Therefore it has solutions of the form $\alpha_j(z,t) = \alpha_j \exp(-i\omega t + ikz)$ and $\sigma_{Mj}(z,t) = \sigma_{Mj}\exp(-i\omega t + ikz)$. This idea is considered in more detail in the Appendix. Substituting these expressions in Eq. (1b) leads, for the optical polarizations, to the expression

$$\sigma_{Mj} = i\frac{\sum_k \alpha_k \sigma_{kj} - \alpha_j \sigma_{MM}}{\gamma_{Mj} + i(\Delta_{Mj} - \omega)} \ .$$

With this result, Eq. (1a) yields a set of algebraic equations:

$$\sum_k A_{kj}\alpha_k = G\alpha_j \ , \quad G = i(k - \omega/c) \ ,$$

$$A_{kj} = g_j(-\sigma^*_{jk} + \sigma_{MM}\delta_{jk}) \ , \tag{2a}$$

$$g_j = \frac{2\pi\omega_j |\mu_{Mj}|^2 N}{c\hbar[\gamma_{Mj} + i(\Delta_{Mj} - \omega)]} \ , \tag{2b}$$

where $\delta_{jj} = 1$ and $\delta_{jk} = 0$ if $k \ne j$.

The normal waves and their gains coincide with the $M-1$ eigenvectors $\boldsymbol{\alpha}^{(p)}$ and eigenvalues G_p of the $(M-1)\times(M-1)$ matrix A whose elements are the A_{kj}. Gains are determined through the characteristic equation

$$\det(A - GI) = 0 \ , \tag{3}$$

which is the equation that determines the dispersion relation: It gives the complex wave number of the normal modes as a function of the real frequency ω, i.e., $k_n = \omega/c - iG_n(\omega)$. The amplification threshold for each normal wave, $G_n = 0$, is therefore defined by the condition $\det(A) = 0$. In order to calculate this determinant, we introduce the basis of eigenvectors of the density submatrix $\tilde{\sigma}$ corresponding to the set of LF transitions between sublevels:

$$\tilde{\sigma}\mathbf{s}^{(k)} = \lambda_k \mathbf{s}^{(k)} \ , \quad \tilde{\sigma} = \{\sigma_{kj}; 1 \le k, j \le M-1\} \ . \tag{4}$$

The eigenvectors of the matrix σ will be

$$\begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \mathbf{s}^{(1)} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \mathbf{s}^{(2)} \end{bmatrix}, \dots, \quad \begin{bmatrix} 0 \\ \mathbf{s}^{(M-1)} \end{bmatrix},$$

since there was no excited optical polarization in the medium before the action of the multifrequency probe field. Using this basis, we obtain

$$\det(A) = \prod_k g_k(\sigma_{MM} - \lambda_k) \ . \tag{5}$$

Hence the amplification threshold of the normal wave $\boldsymbol{\alpha}^{(i)} = \{\alpha_1^{(i)}, \alpha_2^{(i)}, \dots, \alpha_{M-1}^{(i)}\}$ corresponds to the equality between the upper-level population σ_{MM} and one of the eigenvalues λ_i.

B. Amplification condition

From Eq. (5), it follows that the amplification condition is

$$\sigma_{MM} > \min(\lambda_k) \ , \quad 1 \le k \le M-1 \ . \tag{6}$$

In other words, the amplification condition implies an inversion between the upper-level population and the least populated of the eigenstates $\mathbf{s}^{(k)}$. To show this point rigorously, we write Eq. (2a) as a matrix equation

$$(\sigma^* + G_k g^{-1})\boldsymbol{\alpha}^{(k)} = \sigma_{MM}\boldsymbol{\alpha}^{(k)} \ , \quad g = \mathrm{diag}(g_j) \ , \quad g^{-1}g = I \ . \tag{7}$$

Given the structure of Eq. (7), it is natural to introduce the expansion of the vectors $\boldsymbol{\alpha}^{(k)}$ on the basis $\{\mathbf{s}^{(m)}\}$ and, conversely, the expansion of the vectors $\mathbf{s}^{(k)}$ on the basis $\{\boldsymbol{\alpha}^{(m)}\}$:

$$\boldsymbol{\alpha}^{(k)} = \sum_m C_m^k \mathbf{s}^{(m)*} \ , \quad \mathbf{s}^{(k)} = \sum_m D_m^k \boldsymbol{\alpha}^{(m)} \ .$$

Inserting the expansion of $\boldsymbol{\alpha}^{(k)}$ in Eq. (7) and multiplying it by $\boldsymbol{\alpha}^{(k)*}$ leads to the relation

$$\sum_n C_n^{k*}\mathbf{s}^{(n)} \sum_m (\lambda_m C_m^k \mathbf{s}^{(m)*} + G_k g^{-1}\boldsymbol{\alpha}^{(k)}) = \sigma_{MM}|\boldsymbol{\alpha}^{(k)}|^2 \ .$$

Solving this equation for the gain G_k yields the expression

$$G_k = \sum_m \frac{(\sigma_{MM} - \lambda_m)|C_m^k|^2}{\sum_n g_j^{-1}|\alpha_j^{(k)}|^2} \ . \tag{8}$$

For the sake of simplicity, we shall consider the case of line-center operation $\omega = \Delta_{nj}$, bearing in mind that usually gain is maximum on resonance.

Let us first show that the eigenvalue G in Eq. (2a) is real. In the case of equal coupling constants ($g_j \equiv g$, $\forall j$), this property is obvious because A_{kj} is a Hermitian matrix. When $g_j \ne g$, the reality of G follows from the fact that the matrix $B = (\sqrt{\Gamma})^{-1}A\sqrt{\Gamma}$, where $\sqrt{\Gamma} = \mathrm{diag}(\sqrt{g_k})$ and $(\sqrt{\Gamma})^2 = \Gamma$, is obviously Hermitian and has the same eigenvalues as the matrix A. This also means that the eigenvectors of the matrix A are not mutually orthogonal if $g_j \ne g$, because the eigenvectors of B

form an orthogonal basis. However, the relation $\boldsymbol{\beta}^{(m)}=(\sqrt{\Gamma})^{-1}\boldsymbol{\alpha}^{(m)}$ gives the property $(\boldsymbol{\beta}^{(m)},\boldsymbol{\beta}^{(n)})=(\boldsymbol{\alpha}^{(m)},\Gamma^{-1}\boldsymbol{\alpha}^{(n)})=\delta_{mn}$.

From Eq. (8), we conclude that a positive value for G_k implies $\sigma_{MM}>\min(\lambda_m)$, $m=1,2,\ldots,M-1$. In order to prove that this inequality is also sufficient, we multiply Eq. (7) by D_m^k, sum over the index m, and multiply the resulting expression by the vector $\mathbf{s}^{(k)*}$; we obtain

$$\sigma_{MM}-\lambda_k=\sum_{m,p} G_m D_p^{k*} D_m^k (\boldsymbol{\alpha}^{(p)},\Gamma^{-1}\boldsymbol{\alpha}^{(m)})$$
$$=\sum_m G_m |D_m^k|^2 \ . \tag{9}$$

If there is inversion, then $\sigma_{MM}-\lambda_k>0$ and there is obviously at least one positive G_m that provides amplification.

Thus we conclude that the necessary and sufficient condition for amplification is population inversion between the upper level and the least populated of the eigenstates $\mathbf{s}^{(i)}$. This is due to the fact that the density submatrix $\tilde{\sigma}$ is of course diagonal in the basis $\{\mathbf{s}^{(i)}\}$. Hence, there is no interference in this basis and only population inversion is responsible for the amplification.

C. Gain

We calculate the gain near the threshold $\sigma_{MM}=\lambda_k$, assuming that all other eigenvalues λ_i are far enough from σ_{MM}. First, we rewrite the characteristic Eq. (3) in the basis $\{\mathbf{s}^{(k)}\}$ as

$$\left[\prod_i g_i\right]\det(\tilde{n}_{Mi}\delta_{ik}-GF_k^i)=0 \ , \quad \tilde{n}_{Mi}=\sigma_{MM}-\lambda_i \ , \tag{10}$$

where $F_k^i=\sum_j s_j^{(i)} g_j^{-1} s_j^{(k)*}$. In the vicinity of the threshold $G=0$, we only retain the terms that are linear in G, and obtain

$$G_k=\frac{\tilde{n}_{Mk}}{\sum_j |s_j^{(k)}|^2/g_j} \ . \tag{11}$$

It follows from this result that even a weak population of the upper level σ_{MM} can lead to amplification, i.e., $G_k>0$, provided it exceeds the threshold value λ_k. This result is consistent with Eq. (8). Indeed, from Eq. (7) near threshold ($G_k\simeq 0$), it follows that the normal waves coincide with the eigenvectors of the matrix A, i.e., $\boldsymbol{\alpha}^{(k)}=\mathbf{s}^{(k)*}$. Then $C_m^k=\delta_{m,k}$ and Eq. (8) becomes Eq. (11). In the case of equal coupling constants ($g_i=g$), the equality $\boldsymbol{\alpha}^{(k)}=\mathbf{s}^{(k)*}$ remains true even away from threshold, and from Eq. (8) the corresponding gain is

$$G_k=g\tilde{n}_{Mk}=g(\sigma_{MM}-\lambda_i) \ . \tag{12}$$

Another way to interpret this result is as follows. Consider the incident field

$$\boldsymbol{\alpha}=\mathrm{col}(\alpha_1,\alpha_2,\ldots,\alpha_{M-1})=\sum_m \alpha_m \boldsymbol{\Psi}_m \ ,$$

where the $\boldsymbol{\Psi}_m$ are the atomic states (or energy eigenstates). The $\boldsymbol{\Psi}_m$ are represented by a column vector of dimension $M-1$ whose elements are zeros, except for the mth element, which equals 1. In that basis, we also have the decomposition $\mathbf{s}^{(k)}=\sum_m H_m^k \boldsymbol{\Psi}_m$. The atomic medium will amplify the field $\boldsymbol{\alpha}$ that coincides with $\boldsymbol{\alpha}^{(k)}$ and therefore with $\mathbf{s}^{(k)*}$. In other terms, the amplified field will have components such that $\alpha_p/\alpha_q=(H_p^k/H_q^k)^*$, and will be amplified with gain G_k provided $\tilde{n}_{Mk}>0$.

D. The Λ scheme

In the particular case of simple splitting ($M-1=2$), we can easily find the eigenvalues and eigenvectors of the density submatrix:

$$\lambda_{1,2}=\frac{\sigma_{11}+\sigma_{22}}{2}\pm\left[\left[\frac{\sigma_{11}-\sigma_{22}}{2}\right]^2+|\sigma_{21}|^2\right]^{1/2} \ ,$$
$$\mathbf{s}^{(1,2)}=\begin{bmatrix}\lambda_{1,2}-\sigma_{11}\\ \sigma_{21}\end{bmatrix} \ . \tag{13}$$

As the LF coherence σ_{21} increases, the smaller eigenvalue λ_1 tends to zero, while the larger eigenvalue λ_2 tends to the sum of energy-level populations, as displayed in Fig. 2. The characteristic equation $\det(A-GI)=0$ yields

$$(g_1g_2)^{-1}G^2-G(F_2^2\tilde{n}_{31}+F_1^1\tilde{n}_{32})+\tilde{n}_{31}\tilde{n}_{32}=0 \ . \tag{14a}$$

It coincides with the characteristic equation that was obtained for the Λ scheme in [3]. Because $F_2^2\tilde{n}_{31}+F_1^1\tilde{n}_{32}$ is obviously positive in the absence of population inversion between energy states at the optical transitions ($n_{13}>0$, $n_{23}>0$), the necessary condition for amplification takes the form $\tilde{n}_{31}\tilde{n}_{32}<0$. It means that there must be population inversion between the upper level and the least populated of the eigenstates ($\tilde{n}_{32}>0$ and $\tilde{n}_{31}<0$).

In the basis of energy eigenstates, the characteristic equation becomes

$$(g_1g_2)^{-1}G^2-G(g_1n_{13}+g_2n_{23})/g_1g_2$$
$$+n_{13}n_{23}-|\sigma_{21}|^2=0 \ . \tag{14b}$$

The gain condition is $|\sigma_{21}|^2>n_{13}n_{23}$. If we solve that in-

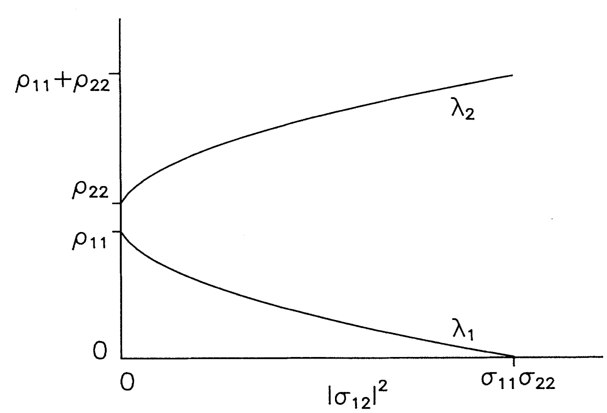

FIG. 2. Eigenvalues of the density submatrix corresponding to the low-frequency transition as a function of the magnitude of the low-frequency coherence.

equality with respect to the upper-level population, we obtain again $\tilde{n}_{32} > 0$. However, we also have $\tilde{n}_{31} < 0$ and $|\tilde{n}_{31}| > \tilde{n}_{32}$. Because of this last relation, it seems at first sight that absorption at the noninverted transition ($\tilde{n}_{31} < 0$) should prevail over amplification at the inverted transition ($\tilde{n}_{32} > 0$), and hence prevent amplification. However, this is not correct. To clarify this point, let us consider the interaction Hamiltonian $V = -\hbar\alpha_1|\psi_3\rangle\langle\psi_1| - \hbar\alpha_2|\psi_3\rangle\langle\psi_2| + \text{c.c.}$, which describes the interaction between the medium and the bichromatic field $\boldsymbol{\alpha} = \text{col}(\alpha_1, \alpha_2)$. In the representation of the density-submatrix eigenstates, V becomes

$$V = \frac{\hbar}{\sigma_{12}(\lambda_2 - \lambda_1)} \times \{[\alpha_1\sigma_{12} + \alpha_2(\sigma_{11} - \lambda_2)]|\psi_3\rangle\langle s^{(1)}| - [\alpha_1\sigma_{12} + \alpha_2(\sigma_{11} - \lambda_1)]|\psi_3\rangle\langle s^{(2)}|\} + \text{c.c.} ,$$

where we used the relations

$$|\psi_1\rangle = (|s^{(1)}\rangle - |s^{(2)}\rangle)/(\lambda_1 - \lambda_2) ,$$

$$|\psi_2\rangle = [(\sigma_{11} - \lambda_2)|s^{(1)}\rangle - (\sigma_{11} - \lambda_1)|s^{(2)}\rangle]/[\sigma_{21}(\lambda_1 - \lambda_2)] .$$

Decomposing the bichromatic field in normal waves $\boldsymbol{\alpha} = a_1\boldsymbol{\alpha}^{(1)} + a_2\boldsymbol{\alpha}^{(2)}$ and taking into account that in the vicinity of the threshold the normal waves coincide with the eigenstates of the density submatrix, $\boldsymbol{\alpha}^{(i)} = \mathbf{s}^{(i)*}$, we can express the interaction Hamiltonian in the form $V = -\hbar[a_1|\psi_3\rangle\langle s^{(1)}| + a_2|\psi_3\rangle\langle s^{(2)}|] + \text{c.c.}$ It follows from this result that each normal wave interacts only with its own effective transition. As a result, the absorption of the normal wave $\boldsymbol{\alpha}^{(1)}$, interacting with the noninverted transition $\tilde{n}_{31} < 0$, does not prevent the amplification of the normal wave $\boldsymbol{\alpha}^{(2)}$ interacting with the inverted transition $\tilde{n}_{32} > 0$, since both normal waves propagate independently of each other.

Let us consider the basis consisting of the absorbing $\boldsymbol{\Psi}_a = [{\alpha_1 \atop \alpha_2}]$ and nonabsorbing $\boldsymbol{\Psi}_n = [{\alpha_2 \atop -\alpha_1}]$ states, where $\boldsymbol{\Psi}_n$ does not interact with the field. The condition $|\sigma_{21}|^2 > n_{13}n_{23}$ is equivalent to the population inversion between the upper level and the absorbing state. The basis $\{\boldsymbol{\Psi}_a, \boldsymbol{\Psi}_n\}$ is parameterized by the ratio of the two wave amplitudes α_1/α_2. If we choose this ratio to be the same as in the amplified normal wave $\boldsymbol{\alpha}^{(2)}$ [3,6,7], then this basis coincides with the basis of the eigenstates of the density submatrix $\mathbf{s}^{(1,2)}$ at threshold ($|\sigma_{21}|^2 = n_{13}n_{23}$) and hence both interpretations are equivalent.

E. Nonlinear propagation

Let us analyze now the nonlinear propagation of the multifrequency field. For simplicity, we consider only the case of equal coupling and relaxation constants for all the transitions and exact resonance $\Delta_{Mi} = 0$. This propagation is described by Eqs. (1) plus an equation for the density submatrix corresponding to the LF transitions:

$$\frac{\partial\sigma_{ij}}{\partial t} = i(\alpha_i^*\sigma_{Mj} - \alpha_j\sigma_{iM}) , \quad i,j \le M-1 . \tag{15}$$

We neglected here the longitudinal and transverse relaxation processes between the LF transitions, as well as the longitudinal relaxation processes at optical transitions, assuming that the multifrequency pulse duration is much shorter than the corresponding relaxation times. At the same time, we suppose that the transverse relaxation times for the optical transitions are essentially smaller than the pulse duration and we eliminate the optical polarizations adiabatically. As a result, we obtain the set of equations

$$\frac{\partial\alpha_i}{\partial z} + c^{-1}\frac{\partial\alpha_i}{\partial t} = -g\sum_j(\sigma_{ji} - \sigma_{MM}\delta_{ij})\alpha_j , \tag{16a}$$

$$\gamma\frac{\partial\sigma_{ij}}{\partial t} = -\alpha_i^*\sum_p(\sigma_{pj} - \sigma_{MM}\delta_{pj})\alpha_p + \alpha_j\sum_p(\sigma_{ip} - \sigma_{MM}\delta_{ip})\alpha_p^* . \tag{16b}$$

In the vicinity of the threshold ($\sigma_{MM} \gtrsim \min\lambda_k$), only one normal wave corresponding to this threshold can propagate with amplification. Because $g_k = g$, we have $\boldsymbol{\alpha}^{(k)} = \mathbf{s}^{(k)*}$. If just before the pulse arrives, the medium was prepared in the eigenstate of the density submatrix, i.e., $\mathbf{s}^{(m)} = \sum_k s_k^{(m)}\boldsymbol{\Psi}_k$, with $(\boldsymbol{\Psi}_k, \mathbf{s}^{(m)}) = s_k^{(m)}$, we have $\sigma_{pq}(t = -\infty) = \sum_m \lambda_m s_p^{(m)} s_q^{(m)*}$, and after the action of this normal wave only one eigenvalue λ_m will be changed. Therefore we can seek a solution of Eqs. (16) in the form

$$\boldsymbol{\alpha} = \alpha(t,z)\mathbf{s}^{(m)*} , \tag{17a}$$

$$\sigma_{ij}(t) = \sigma_{ij}(t = -\infty) + [\lambda_m(t) - \lambda_m(t = -\infty)]s_i^{(m)}s_j^{(m)*} . \tag{17b}$$

After substitution of the ansatz (17) into Eqs. (16), we obtain

$$\frac{\partial\alpha}{\partial z} + c^{-1}\frac{\partial\alpha}{\partial t} = \alpha(\lambda_m - \sigma_{MM}) , \tag{18a}$$

$$\frac{\partial\lambda_m}{\partial t} = -2(\lambda_m - \sigma_{MM})|\alpha|^2/\gamma . \tag{18b}$$

Multiplying the first equation by α^*, and taking into account that

$$\sigma_{MM}(t) = 1 - \lambda_m(t) - \sum_l(1 - \delta_{lm})\lambda_l(t = -\infty) ,$$

we obtain finally the set of equations

$$\frac{\partial I}{\partial z} + c^{-1}\frac{\partial I}{\partial t} = 2g\tilde{n}I , \tag{19a}$$

$$\frac{\partial\tilde{n}}{\partial t} = -4I\tilde{n}/\gamma , \tag{19b}$$

where $I = |\alpha|^2$ and $\tilde{n} = \sigma_{MM} - \lambda_m$. These equations coincide with the equations describing the resonant propagation of the ultrashort pulse in a two-level inversionless medium [9] with a population difference between the energy levels equal to $\tilde{n}$. The formal solution of Eq. (19b) can be written as $\tilde{n}(t) = \tilde{n}_0\exp[-4\int_{-\infty}^{t} I(t')dt'/\gamma]$. Substituting this solution into Eq. (19a), we find the law of nonlinear propagation:

$$I(t,z)=\frac{I_0(t-z/c)}{1-\left[1-\exp\left[g\int_0^z \tilde{n}_0 dz\right]\right]\exp\left[-4\int_{-\infty}^{t} I(t')dt'/\gamma\right]} . \tag{20}$$

Thus a normal wave pulse propagates in a multilevel system like a "monochromatic" pulse in an effective two-level medium with a population difference $\tilde{n}=\sigma_{MM}-\lambda_m$. The interaction of this pulse with the multilevel medium is reduced to the saturation of the effective two-level transition.

II. MULTIPLE SPLITTING OF BOTH OPERATING LEVELS

All the results obtained in Sec. I are directly generalized for the case when there is also an upper-level splitting. In particular, the instability condition for the multifrequency field consisting of $N(M-1)$ components in a medium with $M-1$ sublevels of the lower level and N sublevels of the upper level, when each field component interacts only with its own resonant transition and LF coherencies are excited by the external sources, takes the form

$$\max(\lambda_m^u)>\min(\lambda_n^l) . \tag{21}$$

The superscripts u and l refer to the upper and lower levels, respectively. Two simple consequences follow from Eq. (21). First, in the case when only the upper level is split, the sum of the upper-level populations must exceed the population of the lower level in order to get amplification. Indeed, from (21), we have

$$\sum_{i=2}^{2+N}\sigma_{ii}=\sum_{m=2}^{2+N}\lambda_m^u \ge \max\lambda_m^u > \min\lambda_n^l=\sigma_{11} . \tag{22}$$

This does not mean that amplification without population inversion due to the LF coherence excitation is impossible in the schemes with upper-level splitting. It is possible in the sense that, in the absence of LF coherence, the population of each energy level could be smaller than the ground-state population and hence there would not be amplification. The excitation of the LF coherence allows one to collect effectively all the atoms in one eigenstate and hence to achieve amplification when the sum of the upper-sublevel populations exceeds the sum of the lower-sublevel populations.

However, the case of lower-level splitting is more advantageous in the sense that any amount of atoms in the upper level is sufficient, in principle, to get amplification, because $\min(\lambda_m^l)=0$. Thus an empty eigenstate can result from the excitation by the LF coherence.

Another consequence of Eq. (21) is the following. It is possible to prepare the medium in a coherent state such that inversionless amplification will occur under the action of an ultrashort microwave pulse only if, before the pulse action, there was population inversion at least for one optical transition. This follows from the fact that a unitary transformation (which describes ultrashort pulse action) does not change the eigenvalues of the matrix. Let us remark that, for the particular case of the three-level problem, the same result has been obtained in [8] in a different way.

CONCLUSIONS

If there is a multiple splitting of both operating levels, and LF coherencies are excited between the sublevels, a multifrequency field will be amplified under the condition of population inversion between the most populated and the lowest populated eigenstates of density submatrices, corresponding to the upper and lower operating levels, respectively. In other words, in general the amplification condition can be considered as a population inversion, but not in the basis of the energy atomic levels.

The introduction of LF coherencies between sublevels is equivalent to the redistribution of populations in a new basis of the density-matrix eigenstates. As a result, there can appear a state with the sum of energy-level populations as well as an empty state. The first state plays the key role for inversionless amplification in systems with the upper operating level splitting; the second one is especially important in systems with the lower-level splitting. From the point of view of the inversionless amplification mechanism that we considered here, the systems with lower-level splitting are more advantageous because, in this case, amplification can occur with any amount of atoms in the upper level, while in the first case, the sum of the upper-level populations should exceed at least the population of the lower level. Note, however, that in [10–12], it was shown that there is another mechanism of amplification in systems with a splitting of the operating levels when at least one component of the optical field interacts simultaneously with different transitions. In systems with an upper-level splitting, it does not require that the sum of upper-level populations exceeds the population of the lower level.

It is worth stressing also that the interaction of a multifrequency normal wave with the multilevel system near the threshold is equivalent to the interaction of a monochromatic wave with an effective two-level system, not only in the linear stage of amplification, but also in the full nonlinear case.

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APPENDIX: SOLUTIONS OF THE LINEARIZED EQUATIONS

Let us recall the principles of the linear analysis. The assumption that the submatrix σ_{ij} is independent of the weak field is tantamount to a linearization assumption. It has been removed in Sec. I E, when we considered the full nonlinear propagation problem. As long as we only consider the *linearized* problem, the σ_{ij} are field independent and therefore time independent: they are determined by the initial conditions up to corrections of $O(\alpha^2)$.

Since the resulting equations [Eqs. (1)] are linear in the variables α_j and σ_{Mj}, it follows from the general theorems on linear differential equations that the general solution will be a sum of exponentials of the form $\alpha_j=\sum_n^{M-1}a_j(n)\exp(-i\omega_n t+ik_n z)$ and $\sigma_{Mj}=\sum_n^{M-1}\sigma_{Mj}(n)\exp(-i\omega_n t+ik_n z)$, with real ω_n and therefore complex k_n. This is the obvious generalization of the well-known result that the finite-dimensional set of ordinary differential equations

$$dx_k/dt=f_k(x_1,x_2,\ldots,x_N)\ ,\quad k=1,\ldots,N\ , \tag{A1}$$

where f_k is a linear function of the $\{x_j\}$, has solutions of the form $x_k=\sum_p^N c(p,k)\exp(\lambda_k t)$. The $\{\lambda_k\}$ are the N roots of the characteristic equation obtained by introducing into (A1) the formal solution $x_k=c(k)\exp(\lambda_k t)$. In a completely analogous way, the $M-1$ solutions $\{k_n\}$ are determined by inserting in Eqs. (1) the formal solution $\alpha_j(z,t)=\alpha_j\exp(-i\omega t+ikz)$ and $\sigma_{Mj}(z,t)=\sigma_{Mj}\exp(-i\omega t+ikz)$. This leads, via the rules of matrix calculus, to an algebraic equation of degree $M-1$ whose roots are the $\{k_n\}$. The coefficients $a_j(n)$ and $\sigma_{Mj}(n)$ can then be computed for each k_n.

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