Tunneling through a one-dimensional potential barrier

Zafar Ahmed
Nuclear Physics Division, Bhabha Atomic Research Centre, Bombay 400085, India
(Received 29 July 1992)

The exact transmission coefficient of a one-dimensional potential barrier, $V(x)=V_0(1-[\{1-\exp(x/a)\}/\{1+c\exp(x/a)\}]^2)$, of which the Morse and the Eckart barriers are special cases, has been obtained. Comparing the exact and WKB transmission coefficients, the limitation of the semiclassical method has been quantified. Another branch of this potential ($c<0$) has been shown useful in nucleus-nucleus fusion. Using the transmission coefficients, the exact energy eigenvalues of the inverted ($V_0\to -V_0$) potential (oscillator) have been derived.

PACS number(s): 03.65.Nk, 25.70.Jj, 73.40.Gk, 03.65.Ge

I. INTRODUCTION

The potential functions which are amenable to exact analytic quantal solutions are rare. Rarer are the instances where the one-dimensional potentials admit a simple and tractable transmission coefficient [1–4]. In this paper, we present analytic results on the transmission amplitude $\tau(k)$ and the transmission coefficient $T(k)$ for a one-dimensional potential barrier

$$V(x)=V_0(1-[\{1-\exp(x/a)\}/\{1+c\exp(x/a)\}]^2) .$$

From a more general standpoint this potential function interpolates between the symmetric Eckart [2] and the Morse [4] potential. This makes it especially suited to applications in chemical physics. For instance, the investigation of the vibrational spectrum of a bond-stretching diatomic molecule was carried out using this potential function,

$$V(r)=V_0[\{1-\exp(r/a)\}/\{1-c\exp(r/a)\}]^2 .$$

The amenability of the corresponding Schrödinger equation was first suggested by Tietz [5]. Recently, Wei Hua [6] has again utilized this potential to obtain analytic expressions for the energy eigenvalues. It may be remarked that owing to the condition of regularity of the wave function, $\psi(r)$ at $r=0$, the radial Schrödinger equation for this potential in $r\in[0,\infty]$, even for the s wave, does not provide an exact and simple expression for energy eigenvalues of the bound states. However, for this anharmonic oscillator, in Ref. [6] an approximate expression for the s-wave eigenvalues was obtained. Furthermore, an approximate expression for higher partial-wave eigenvalues was proposed.

The transmission coefficient of a potential barrier finds applications in the fields like nuclear fission [23], heavy-ion fusion [6], and the phenomenon of tunneling in solids. On the other hand, it provides a check for the other alternative methods like the path-integral technique [7] and the Wigner function method [8]. It also offers a test of accuracy for the approximate methods such as the WKB [9] and the variable-phase method [10] which are used for intractable potentials. It must be emphasized that when a potential barrier does not converge asymptotically, the calculation of the transmission coefficient through numerical integration of the Schrödinger equation is rendered infeasible. Since the transmission amplitudes and the transmission coefficients [4] contain information about the energy eigenspectrum (bound states, metastable states, and resonances) of the potential, the transmission functions $[\tau(k),T(k)]$ become the very ingredients of the theory.

In this paper we also obtain the energy eigenvalues for the bound states of the inverted potential (oscillator),

$$V(x)=-V_0(1-[\{1-\exp(x/a)\}/\{1+c\exp(x/a)\}]^2)$$

from the transmission functions. These eigenvalues will correspond to the solution of the Schrödinger equation for $x\in[-\infty,\infty]$ subject to the boundary condition $\psi(\pm\infty)=0$.

The Schrödinger equation for the present potential function is transformable to the Gauss hypergeometric equation. This, therefore, suggests that this potential belongs to a broad class of potentials known as Natanzon potentials [11]. Eckart potential [2] and Ginnochio's potential [3] also come in this class of potentials. Some more potentials of this class may be cited in Ref. [12].

The plan of this paper is as follows. In Sec. II we discuss the potential barrier and calculate the exact eigenfunctions, the exact and the WKB-approximated transmission coefficients are presented in Sec. III. An application of a branch of this potential barrier in nucleus-nucleus fusion is suggested in Sec. IV. The exact energy eigenvalues of the corresponding oscillator potential are extracted in Sec. V. Finally, we summarize our findings in Sec. VI.

II. THE POTENTIAL BARRIER AND THE EXACT EIGENFUNCTIONS

The present one-dimensional potential barrier is written as

$$V(x)=V_0(1-[\{1-\exp(x/a)\}/\{1+c\exp(x/a)\}]^2) ,\quad 0\le c\le 1 \qquad (1)$$

and the curvature of the top of the barrier is given by $C_0=2V_0/[(1+c)^2a^2]$. When $c=0$, Eq. (1) gives the Morse barrier, i.e., $V(x)=V_0[2\exp(x/a)-\exp(2x/a)]$ and when $c=1$ one gets the symmetric Eckart barrier, i.e., $V(x)=V_0\text{sech}^2(x/2a)$. These potentials are displayed in Fig. 1. For (1), we write the Schrödinger equation

$$-(\hbar^2/2m)\frac{\partial^2\psi(x,t)}{\partial x^2}+V_0(1-[\{1-\exp(x/a)\}/\{1+c\exp(x/a)\}]^2)\psi(x)=i\hbar\frac{\partial\psi(x,t)}{\partial t} . \tag{2}$$

Using the transformations $z=-c\exp(x/a)$ and $\psi(z,t)=\phi(z)/z^{1/2}\exp(-iEt/\hbar)$ in (2) we obtain

$$\frac{d^2\phi(z)}{dz^2}+[(p^2+q^2+\tfrac{1}{4})/z^2+q^2(b+1)^2/(1-z)^2+2q^2(b+1)/\{z(1-z)\}]\phi(z)=0 , \tag{3}$$

where we have used $p^2=(E-V_0)/\Delta$, $q^2=V_0/\Delta$, $f^2=E/\Delta$, $b^2=1/c^2$, and $\Delta=\hbar^2/(2ma^2)$. Comparing Eq. (3) with one of the standard transformed forms of the Gauss hypergeometric equation [13]

$$\frac{d^2\phi(z)}{dz^2}+[(1-\lambda^2)/4z^2+(1-v^2)/\{4(1-z)^2\}+(1-\lambda^2+\mu^2-v^2)/\{4z(1-z)\}]\phi(z)=0 , \tag{4}$$

we get $\lambda=2if$ $(f=ka)$, $\mu=2is$ $(s=k'a)$, and $v=2ig$, where

$$s=[f^2+(b^2-1)q^2]^{1/2}$$

and

$$g=[q^2(b+1)^2-\tfrac{1}{4}]^{1/2} .$$

Let us express k and k' explicitly as

$$k=[2mE]^{1/2}/\hbar$$

and

$$k'=[2m\{E+(b^2-1)V_0\}]^{1/2}/\hbar .$$

We can write the solutions of (4) as

$$\phi(z)=z^{1/2-\lambda/2}(1-z)^{1/2-v/2}W(z) . \tag{5}$$

$W(z)$ refers to the Gauss hypergeometric function [13], among 24 forms available for $W(z)$; the following will be useful in the sequel:

$$W_1(z)={}_2F_1(\alpha,\beta,\gamma;z) , \tag{6a}$$

$$W_2(z)=z^{1-\gamma}{}_2F_1(1+\alpha-\gamma,1+\beta-\gamma,2-\gamma;z) , \tag{6b}$$

and

$$W_3(z)=(1-z)^{-\alpha}{}_2F_1(\alpha,\gamma-\beta,\alpha+1-\beta;(1-z)^{-1}) . \tag{6c}$$

The parameters of ${}_2F_1$ are $\alpha=\frac{1}{2}-if-ig-is$, $\beta=\frac{1}{2}-if-ig+is$, and $\gamma=1-2if$.

Using (6b) one of the solutions of (2) can be written as

$$\psi(z,t)=(-1)^{-if}(2f)^{-1/2}\exp(-\pi f)z^{if}(1-z)^{1/2-ig}{}_2F_1((\tfrac{1}{2}+if-ig-is),(\tfrac{1}{2}+if-ig+is),(1+2if);z)\exp(-iEt/\hbar) , \tag{7}$$

which behaves as $\exp[i(kx-Et/\hbar)]$ as $x\to-\infty$, denoting a wave (ψ_i) incident on the barrier, assuming the incidence from left to right. Changing k to $-k$ in (7), we obtain the oppositely traveling reflected wave (ψ_ρ)

$$\psi(z,t)=(-1)^{if}(2f)^{-1/2}\exp(\pi f)z^{-if}(1-z)^{1/2-ig}{}_2F_1((\tfrac{1}{2}-if-ig-is),(\tfrac{1}{2}-if-ig+is),(1-2if);z)\exp(-iEt/\hbar) . \tag{8}$$

Using (6c), one more solution of (2) can be written as

$$\psi(z,t)=(-1)^{if}(2s)^{-1/2}\exp(-\pi f)z^{-if}(1-z)^{is+if}$$
$$\times{}_2F_1((\tfrac{1}{2}-if-ig-is),(\tfrac{1}{2}-if+ig+is),(1-2is);(1-z)^{-1})\exp(-iEt/\hbar) , \tag{9}$$

which, in the limit $x\to\infty$, denotes a wave moving from left to right after being transmitted (ψ_τ) through the barrier, i.e., $\exp[i(k'x-Et/\hbar)]$.

III. THE EXACT AND THE WKB TRANSMISSION COEFFICIENTS

In this section we calculate the transmission coefficients exactly as well as using the WKB approximation. Subsequently, we compare the two sets of results to reveal the limitation of the WKB method. Also, we shall be discussing the results for the two interesting limits (the Morse and the Eckart barriers) hence presenting confidence and usefulness of the derived results. It may be noted that ψ_i, ψ_ρ, and ψ_τ obtained in Sec. II satisfy the following the Wronskian relations:

$$[\psi_i^*,\psi_i]=i,\quad [\psi_\rho^*,\psi_\rho]=-i,\quad \text{and } [\psi_\tau^*,\psi_\tau]=i . \tag{10}$$

By virtue of a standard analytic continuation property of the hypergeometric functions [13]

$$(1-\zeta)^{-\alpha}{}_2F_1(\alpha,(\gamma-\beta),(\alpha+1-\beta);(1-\zeta)^{-1})$$
$$=[\{\Gamma(1+\alpha-\beta)\Gamma(1-\gamma)\}/\{\Gamma(1-\beta)\Gamma(1+\alpha-\gamma)\}]\,{}_2F_1(\alpha,\beta,\gamma;\zeta)$$
$$+\exp\{i\pi(\gamma-1)\}[\{\Gamma(1+\alpha-\beta)\Gamma(\gamma-1)\}/\{\Gamma(\alpha)\Gamma(\gamma-\beta)\}]\zeta^{(1-\gamma)}{}_2F_1((1+\alpha-\gamma),(1+\beta-\gamma),(2-\gamma);\zeta)\ , \quad (11)$$

a desirable relation among ψ_i, ψ_ρ, and ψ_τ can be written as

$$\psi_i+\rho(k)\psi_\rho=\tau(k)\psi_\tau\ , \quad (12)$$

where $\tau(k)$ and $\rho(k)$ are the transmission and the reflection amplitudes expressible as

$$\tau(k)=(s/f)^{1/2}[\{\Gamma(\tfrac{1}{2}-if-ig-is)\Gamma(\tfrac{1}{2}-if+ig-is)\}/\{\Gamma(1-2is)\Gamma(-2if)\}] \quad (13a)$$

and

$$\rho(k)=[\{\Gamma(2if)\Gamma(\tfrac{1}{2}-if-ig-is)\Gamma(\tfrac{1}{2}-if+ig+is)\}/\{\Gamma(-2if)(\tfrac{1}{2}+if+ig-is)\Gamma(\tfrac{1}{2}+if-ig-is)\}]\ . \quad (13b)$$

With the help of the Wronskian relations (10), the transmission and reflectance coefficients get defined, respectively, as $T(k)=\tau^*(k)\tau(k)$ and $R(k)=\rho^*(k)\rho(k)$. Employing the properties of the Γ functions, we obtain

$$T(k)=[\sinh(2\pi f)\sinh(2\pi s)]/[\cosh\{\pi(f+s+g)\}\cosh\{\pi(f+s-g)\}] \quad (14a)$$

and

$$R(k)=[\cosh\{\pi(f+g-s)\}\cosh\{\pi(f-g-s)\}]/[\cosh\{\pi(f+g+s)\}\cosh\{\pi(f-g+s)\}]\ . \quad (14b)$$

In a special instance when $c\to 0$, the present potential barrier (1) becomes the Morse barrier [4]. For this case, the eigenfunctions become the confluent hypergeometric functions. Also, we get $(g-s)\to q$ and $(s+g)\to\infty$. In these limits, (14a) degenerates to the transmission coefficient of the Morse barrier [4]

$$T^{\text{Morse}}(E)=\frac{1-\exp(-4\pi f)}{1+\exp\{2\pi(q-f)\}}\ ,\quad f=(E/\Delta)^{1/2} \text{ and } q=(V_0/\Delta)^{1/2}\ . \quad (15)$$

The other special case is of the symmetric Eckart barrier which can be obtained when $c=1$ in (1). For this case ($f=s$), from (14a), we get

$$T^{\text{Eckart}}(E)=\sinh^2(\pi f')/[\sinh^2(\pi f')+\cosh^2(\pi g')]\ , \quad (16)$$

where

$$f'=(E/\Delta')^{1/2}\ ,$$
$$g'=[(V_0/\Delta')-\tfrac{1}{4}]^{1/2}\ ,$$

and

$$\Delta'=\hbar^2/(8ma^2)\ .$$

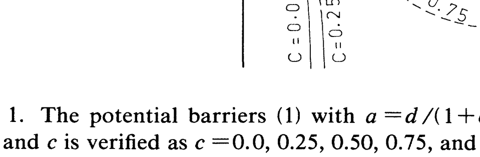

FIG. 1. The potential barriers (1) with $a=d/(1+c)$, when $d=1.0$ and c is verified as $c=0.0$, 0.25, 0.50, 0.75, and 1.0. We get the Morse and the Eckart barrier for $c=0.0$ and 1.0, respectively.

In the limit when $\hbar\to 0$, ($\Delta'\to 0$) (14a) degenerates into the classical limit of $\Theta(E-V_0)$ where Θ is the Heaviside function; another interesting limit is the semiclassical one, i.e., when Δ is very small but finite. For $E>\Delta$ and $V_0>\Delta$ (14a) gives

$$T(E)\approx[1+\exp\{2\pi(g-s-f)\}]^{-1}\ , \quad (17)$$

which interestingly compares well with the expression obtained by the semiclassical WKB approximation, i.e.,

$$T_{\text{WKB}}(E)=[1+\exp(2\pi(g_{\text{WKB}}-s-f)\}]^{-1},\quad g_{\text{WKB}}=q(b+1)\ . \quad (18a)$$

Note the disappearance of $-\frac{1}{4}$ in g_{WKB}.

We observe that Δ sets the lower limit on energy above which the WKB approximation would work well provided $V_0>\Delta$. The condition, $E>\Delta$ interestingly comes to that of the de Broglie wavelength of the incident particle being less than that of the size of the "obstacle" i.e., $\lambda_d<2\pi a$, where $2\pi a$ is a length scale found in the potential. Instances when $\lambda_d>2\pi a$ the tunneling becomes more and more quantal and hence the semiclassical approximation becomes poorer. Equation (18a) yields the WKB transmission of the Morse barrier [4],

$$T^{\text{Morse}}_{\text{WKB}}(E)=[1+\exp\{2\pi(q-f)\}]^{-1}\ . \quad (18b)$$

For the symmetric Eckart barrier, $V(x)=V_0\text{sech}^2(x/2a)$, it yields

$$T_{\rm WKB}^{\rm Eckart}(E)=[1+\exp\{2\pi(q'-f')\}]^{-1}\ ,\quad q'=(V_0/\Delta')^{1/2}\ . \tag{18c}$$

The poor performances of the WKB approximation at low energies ($E>\Delta$) can be assessed from the fact that the exact transmission coefficients, viz., (14a), (15), and (16) vanish at $E=0$, whereas the WKB expressions, viz., (18a), (18b), and (18c) yield a finite value for zero-energy transmission. $E=V_0$ is another interesting point where the WKB transmission coefficient, irrespective of the potential profile, inherently becomes one-half; the exact transmission coefficients show a deviation from this value.

IV. AN APPLICATION TO NUCLEUS-NUCLEUS FUSION

In the fusion of two nuclei the fusion reaction rates are calculated employing the philosophy of barrier penetration [14]. This method consists of obtaining the penetrability (transmission coefficient) of the fusion interaction barrier. The barrier is formed due to the nuclear attraction and Coulomb plus centrifugal repulsion. A generally accepted phenomenological form for this potential is given [15] as

$$V_l(r)=-D[r_1r_2/(r_1+r_2)]\exp[-b(r-(r_1+r_2)]+Z_1Z_2e^2/r+(\hbar^2/2m)l(l+1)/r^2\ , \tag{19}$$

where $b=1.333\ \text{fm}^{-1}$, D is prescribed for different nuclei, for instance for $^{16}\text{O}+^{13}\text{C}$ it is 100.87 MeV/fm. Here, $r_i=R_i-1/R_i$, $R_i=(1.28A_i^{1/3}+0.8/A_i^{1/3}-0.87)$, where $i=1,2$. In this expression, A denotes the atomic weight of a nucleus. At the origin this potential entails a strong singularity; then, there is a well (potential pocket) attached to a barrier. In Fig. 2 we show, by open circles,

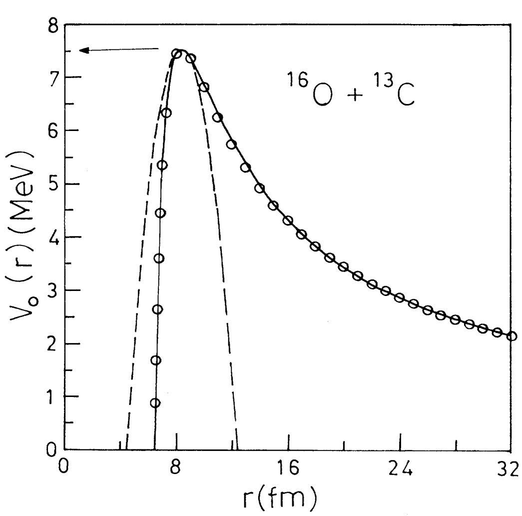

FIG. 2. The solid line represents our barrier model Eq. (24), the open circles denote the actual fusion interaction barrier given by Eq. (19), and the dashed line presents the parabolic fit. The arrow denotes the Coulomb barrier. The fitting parameters are $\hbar\omega_0=2.37$ MeV, $V_0=7.53$ MeV, and $C_0=1.03$.

the typical s-wave interaction barrier for the fusion of ^{16}O and ^{13}C nuclei. The maximum of the s-wave interaction (19), V_0 is called the Coulomb barrier. In the theory of nucleus-nucleus fusion, the height of the Coulomb barrier is a very important parameter, as the physical mechanisms that give rise to fusion are different below and above the Coulomb barrier [16]. Also, we will find that the calculational schemes are different below and above this threshold. The barrier penetration method or model consists in ignoring the pocket and calculating the penetrability through this barrier for each partial wave. The fusion rates are then determined by

$$\sigma^F(E)=(\pi/k^2)\sum_l(2l+1)T_l(E)\ . \tag{20}$$

A convenient and common way [14] is to fit a parabolic barrier

$$V(r)=V_l-m\omega_l^2(r-r_l)^2/2 \tag{21}$$

to the fusion interaction barrier; the parameters V_l and $\hbar\omega_l$ are then used to calculate the penetrability, $T_l(E)$ employing the exact transmission coefficient formula of parabolic barrier (Hill-Wheeler formula [1])

$$T_l(E)=[1+\exp\{2\pi(V_l-E)/\hbar\omega_l\}]^{-1}\ . \tag{22}$$

Note that the parameter r_l is redundant since the transmission through a barrier is independent of its position. For a given partial wave, V_l is the maximum of the function given by (19). The parameter $\hbar\omega_l$ determines the top curvature of a barrier, i.e.,

$$h\omega_l=\left[(h^2/m)\frac{d^2V_l(r)}{dr^2}\right]^{1/2}_{r=r_l}\ . \tag{23}$$

This curvature parameter is also called the Hill-Wheeler frequency. In Fig. 2 we have shown the actual s-wave fusion barrier arising from (19) (open circles) and the parabolic fitting to the same. Excepting the barrier top, the parabolic barrier renders a very poor fit; the tail part of the barrier especially is quite off. Now the question whether the tail part affects the fusion rates arises. Our potential barrier has got the distinction of admitting various tails (see Fig. 1), as the parameter c is varied between 0 and 1, by keeping the height and top curvature fixed (see Fig. 1). To this end, we set in (1) $a_l=d_l/(1+c_l)$, $x=(r-r_l)$. Once again, for positive values of c_l, the fits of these barriers to the actual fusion barriers are bad. Yet, recently, we have shown [17] that if the fitting barriers (parabolic or the present barrier) entail the same height and top curvature, the fusion rates at energies above the Coulomb barrier ($E>V_0$) are hardly affected. Therefore, at these energies, the parabolic, the Morse, the Eckart, and the present barrier work very satisfactorily. However, we must emphasize that at subbarrier energies ($E>V_0$) these barrier models render wrong results.

In the calculation of fusion rates the usage of WKB penetrability was introduced surprisingly late in 1980 [15] whereas this barrier penetration model or method is as old as 1959 [14]. Moreover, the application of this approximation was suggested rather wishfully, without a justification. It must be emphasized here that the coincidence of the exact and the WKB penetrability of the parabolic barrier is a peculiar (nongeneric) feature. The justification of the goodness of the WKB penetrability is still due. The present barrier elicits this justification very neatly. In Sec. IV, as we have seen that the smallness of $\Delta_l=\hbar^2/(2ma_l^2)=[\hbar\omega_l(1-C_l)]^2/V_l(\Delta_l \ll l$ MeV) ensures the goodness of the WKB method at energies, $E>\Delta_l$. Even in the worst case, for instance, for the $^{16}O+^{13}C$ system, $\hbar\omega_0=2.37$ MeV and the Coulomb barrier $V_0=7.53$ MeV gives $\Delta=0.18$ MeV. The experimental fusion rates are measured [18] at energies $E>4$ MeV, that is much larger as compared to Δ. Therefore, the WKB penetrability renders accurate results. For bigger nuclei, the Coulomb barrier (V_0), which roughly goes as $Z_1Z_2/(A_1^{1/3}+A_2^{1/3})$ MeV is much larger and therefore, the values of Δ will be much smaller, consequently, the WKB approximation would work much better. In the light of the present facts the nucleus-nucleus fusion is adjudged to be a semiclassical event. One problem with the WKB penetrability lies in the fact that above the barrier energies, the turning points become complex. Finding the complex roots of a real equation (turning points) is a cumbersome and nonstandard exercise. One therefore, avoids the use of the WKB method energies greater than the Coulomb barrier. In this regard, the parabolic method and the WKB method are practically complementary to each other.

It is here that we invoke another branch of the potential barrier (1) by making c negative, and we suggest a new parametrizing potential barrier which admits a simple WKB penetrability factor. We suggest that the fusion barriers obtained by the aforementioned interaction should be fitted to the three-parameter potential profile (V_l, d_l, and C_l)

$$V(r)=V_l(1-[\{1-\exp(x)\}/\{1-C_l\exp(x)\}]^2)\ , \qquad (24)$$

where $x=(r_l-r)/[d_l(1-C_l)]$. It may be reemphasized that the position of the barrier top r_1 is a redundant parameter for penetrability calculations. By finding the maximum of (19) and the position of the maximum value we get V_l and γ_l, respectively. Next, using (23) we obtain $\hbar\omega_l$. Owing to a good familiarity of the curvature parameter $\hbar\omega_l$ (the Hill-Wheeler frequency of parabolic barrier) in fusion literature, we would like to retain it in our model. We, therefore, connect $\hbar\omega_l$ and d_l by comparing the top curvatures of the parabolic and present barrier (24). Using (23) we get $d_l=[(\hbar^2/2m)(4V_l)/(\hbar\omega_l)^2]^{1/2}$. We then vary C_l to fit the Coulomb tail; for a better fit one may have to change the obtained value of $\hbar\omega_l$ slightly. The variations in C_l are generally from 1.4 to 0.5 as one goes from lower to higher partial waves. The situation of $C_l=1$ is to be avoided and this may require a slight adjustment in the value of $\hbar\omega_l$. It is very important to know the asymptotic distance up to which one has to fit the Coulomb tail. For this, one should see the lowest energy at which the fusion rates are to be obtained. The asymptotic distance should be fairly sufficient to find the turning point at the lowest energy. Our barrier model essentially consists of three parameters, viz., V_l, $\hbar\omega_l$, and C_l. In Fig. 2 we have shown the fitting of the actual s-wave barrier with the barrier given by (24). An overall good reproduction of the geometry of the fusion barrier using the present barrier (24) is remarkable. For the sake of comparison the equivalent parabolic barrier is also plotted in this figure.

The parametrizing barrier (24) entails a singularity ($C_l=1$) at negative energies. However, this singularity being away from the turning point, we can use the WKB approximation legitimately. We get the WKB penetrability

$$T_l(E)=[1+\exp(2\pi\epsilon_l K_l)]^{-1}\ , \qquad (25)$$

where

$$K_l=(E/\Delta_l)^{1/2}-[(E/\Delta_l)+(B_l^2-1)V_l/\Delta_l]^{1/2}+(B_l-1)(V_l/\Delta_l)^{1/2}\ ,$$

$$\epsilon_l=\mathrm{sgn}(C_l-1)\ ,$$

$$\Delta_l=(\hbar\omega_l)^2(1-C_l)^2/(4V_l)\ ,$$

and

$$B_l=1/C_l\ .$$

Note that the value of C_l being between 1.4 to 0.5 the parameter, Δ will get reduced and the WKB approximation will become more accurate here. Our calculations of fusion rates using this barrier (24) and the penetrability factor (25) and their comparison with the experimental data have been reported in Ref. [19]; the detailed results are also to appear elsewhere. Presently, our aim is to demonstrate the accuracy of our parametrizing barrier and the utility of its simple penetrability factor that works at both the energies below and above the Coulomb barrier. To this end, we have calculated the s-wave penetrability for fusion of ^{16}O and ^{13}C nuclei, using the fitted parameters V_0 and $\hbar\omega_0$. In Fig. 3 the dotted line

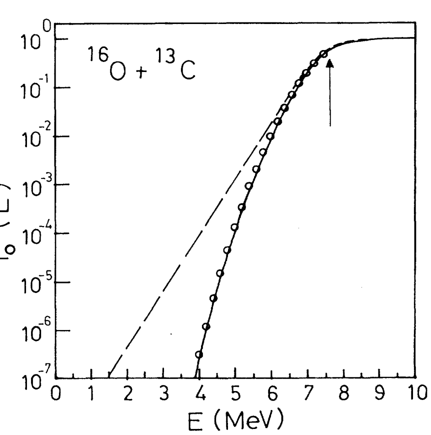

FIG. 3. The solid line is our penetrability formula (25) for an s wave, the open circles denote the WKB-approximated penetrability, calculated numerically at sub-barrier energies. The arrow denotes the Coulomb barrier.

represents the parabolic approximation (the Hill-Wheeler formula) results, the solid line is due to our Eq. (25), and the open circles show the WKB approximated penetrability calculated numerically at subbarrier energies only. The Hill-Wheeler formula overestimates the subbarrier penetrability; to emphasize again, this is due to a poor reduction of the geometry of the fusion barrier by a parabolic barrier. As per our claim, at energies above the Coulomb barrier, all the three penetrabilities coincide. A good match between the numerical WKB penetrability and the one calculated using our analytical WKB penetrability via the barrier (24) testifies to the closeness of our parametrization of the fusion interaction barrier. Thus, our barrier penetration scheme facilitates an improvement over the well-known parabolic barrier penetration method at subbarrier energies. At energies above the barrier energies our penetrability formula bypasses the calculation of complex turning points for the WKB calculations. It may be remarked here that the earlier attempts [20] to improve the parabolic barrier model, at subbarrier energies, have not worked at energies above the Coulomb barrier. This is the reason why on several instances [21] the fusion reaction rates have been calculated avoiding the barrier penetration method. Having adjudged the nucleus-nucleus fusion to be semiclassical, we claim that our barrier penetration method renders accurate estimates of penetrability and hence the fusion rates at both the energies, below and above the Coulomb barrier.

V. THE ENERGY EIGENVALUES OF THE INVERTED BARRIER (OSCILLATOR)

Now our aim is to extract the discrete energy eigenvalues of the present anharmonic oscillator [inverted barrier (1)] using the obtained expression of $\tau(k)$ (13a) and $T(k)$ (14a). To this end we use the simple poles of $\tau(k)$ lying on the upper half of the imaginary line in the complex k plane (physical sheet) which are known to represent the possible bound states of a potential [4]. For the bound states of the inverted potential, we change V_0 to $-V_0$ and define $G=[q^2(b+1)^2+\frac{1}{4}]^{1/2}$, $S=[F_n^2+(b^2-1)q^2]^{1/2}$, where $F_n=if=ik_n a$. We locate the positive poles of F_n in Eq. (14a) and get

$$F_n=G-S-N, \quad N=n+\tfrac{1}{2} . \tag{26}$$

Note that these poles will be positive if and only if $N<(G-S)$. Therefore this oscillator entails only a finite number of bound states. By setting $E=-\hbar^2F_n^2/(2\mu a^2)$, the exact eigenvalues of the present anharmonic oscillator are obtained,

$$E_n=-\Delta[q^2(b^2-1)-(G-N)^2]^2/[4(G-N)^2] . \tag{27}$$

For the case of the Morse oscillator [4], $c=0(b=\infty)$ and also $(G-S)=q$, on using (15) or (26) we get

$$E_n=-\Delta[(V_0/\Delta)^{1/2}-(n+\tfrac{1}{2})]^2 . \tag{28}$$

For the case of the Eckart oscillator [2], $c=1$ and $F=S$, Eqs. (16) or (26) yields

$$E_n=-\Delta'[\{V_0/\Delta'+\tfrac{1}{4}\}^{1/2}-(n+\tfrac{1}{2})]^2 . \tag{29}$$

In an earlier work [4], we have suggested a general method of extraction of the eigenvalues of the bound states of an inverted potential from the transmission coefficient $T(k)$ of the corresponding potential. Since one generally knows the transmission coefficient [$T(k)$ not $\tau(k)$], therefore extraction of eigenvalues from $T(k)$ is more important. It can be checked that Eqs. (27), (28), and (29) can be neatly obtained by an analysis of the simple positive poles in k plane (14a), provided we use the fact that this oscillator can, unlike harmonic oscillator, support only a finite number of bound state [4].

Similar analysis of the semiclassical T_{WKB}, given by Eqs. (18), yield the eigenvalues of these oscillators consistent with the semiclassical Bohr-Sommerfeld quantization law. We notice that for the case of the Morse oscillator the semiclassical and quantal eigenvalues coincide. For the Eckart and the other oscillators $(0<c<1)$ both kinds of eigenvalues differ only slightly, i.e., the semiclassical eigenvalues of the Eckart oscillator are same as (29), excepting a term, "$+\frac{1}{4}$". Similarly the semiclassical eigenvalues for these instances will still be given by (27), wherein $G=q(b+1)$.

VI. SUMMARY

To sum up, in this paper we have presented the exact (14a) and the WKB (18a) transmission coefficients of a general one-dimensional potential barrier which are useful in the fields such as nuclear fission, fusion, and tunneling in solids. We have discussed the limitation of the WKB penetrability. Using our results we have obtained a criterion that enables us to ascertain the validity of the WKB-approximated transmission coefficient. We have argued that the nucleus-nucleus fusion is semiclassical due to which the use of the WKB penetrability in calculation of fusion rates is justified. Hitherto, a potential barrier which parametrizes the full geometry (the Coulomb tail for all partial waves) of the fusion interaction barrier and also admits a simple penetrability factor has been elusive. To this end, a branch of our potential (24) has been shown to be particularly useful in nucleus-nucleus fusion. Notably, our barrier penetration method (25) works at both the energies, below and above the Coulomb barrier. Our barrier can also be useful in parametrizing the charged particle barrier formed in the event of α-decay. The penetrability formula (25) can then be used to calculate the α-decay rates. Currently, in heavy-ion nuclear reactions, new kinds of resonances are being discussed. These are called the barrier-top resonances [22] which entail the complex energy eigenvalues embedded in the positive-energy continuum. We would like to mention here that the present barrier is a suitable candidate for studying such resonances with the help of Eq. (13a). Also, we believe that the exact eigenvalues of the inverted potential (oscillator) presented here (27) shall be useful in the studies on the dynamical aspects of the one-dimensional quantal integrable systems.

ACKNOWLEDGMENTS

It is a pleasure to thank Sudhir Ranjan Jain for discussions. I am grateful to Dr. S. Kataria for suggesting to me the application to nucleus-nucleus fusion. I acknowledge with gratitude an encouraging remark made by Dr. B. K. Jain.

[1] D. Hill and J. Wheeler, Phys. Rev. **89**, 1102 (1953); G. Barton, Ann. Phys. (N.Y.) **116**, 322 (1986).

[2] C. Eckart, Phys. Rev. **35**, 1303 (1930).

[3] J. N. Ginocchio, Ann. Phys. (N.Y.) **152**, 203 (1984).

[4] Z. Ahmed, Phys. Lett. A **157**, 1 (1991).

[5] T. Tietz, J. Chem. Phys. **38**, 3036 (1963).

[6] Wei Hua, Phys. Rev. A **42**, 2524 (1990); G. A. Natanzon, **44**, 3377 (1991).

[7] D. M. Brink and U. Smilansky, Nucl. Phys. A **405**, 301 (1983).

[8] N. L. Balazs and A. Voros, Ann. Phys. (N.Y.) **199**, 123 (1990).

[9] N. Froman and P. O. Froman, *JWKB Approximations: Contributions to the Theory* (North-Holland, Amsterdam, 1965).

[10] L. P. Presnyakov, Phys. Rev. A **44**, 5636 (1991); J. G. Muga, Phys. Lett. A **157**, 325 (1991).

[11] G. A. Natanzon, Theor. Mat. Fiz. **38**, 146 (1979).

[12] G. Levai, J. Phys. A **24**, 131 (1991), and the references therein.

[13] Y. L. Luke, *The Special Functions and Their Approximations* (Academic, New York, 1969); *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1970); A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. I.

[14] D. H. Thomas, Phys. Rev. **116**, 703 (1959); M. Beckerman, Rep. Prog. Phys. **51**, 1047 (1988).

[15] S. G. Steadman, *Fusion Reactions Below the Coulomb Barrier,* Lecture Notes in Physics Vol. 219 (Springer-Verlag, Heidelberg, 1985). In this reference several similar interaction potentials are given, see pp. 43, 131, 291, and 333; also see the references therein.

[16] H. Esbensen, Nucl. Phys. A **352**, 147 (1980).

[17] Z. Ahmed, (unpublished).

[18] B. Dasmahapatra, B. Cujec, and F. Lahlou, Nucl. Phys. A **394**, 301 (1983).

[19] Z. Ahmed, in *Proceedings of the DAE (India) Symposium on Nuclear Physics* (DAE, Bombay, 1991), Vol. 34B, p. 179.

[20] Y. Avishai, Z. Phys. A **286**, 285 (1978); J. L. Dethier and Fl. Stancu, Phys. Rev. C **23**, 1503 (1981).

[21] Q. Haider and B. Cujec, Nucl. Phys. A **429**, 116 (1984).

[22] W. A. Friedman and C. J. Goebel, Ann. Phys. (N.Y.) **104**, 145 (1977); G. R. Satchler, *Direct Nuclear Reactions* (Clarendon, New York, 1983), p. 436.

[23] S. Bjørnholm and J. E. Lynn, Rev. Mod. Phys. **52**, 725 (1980).