Quantum interference and determination of the traversal time

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The tunneling-time problem is shown to be analogous to the interpretation of the two-slit interference experiment. A measurement assuming an arbitrarily small interaction between a particle and a clock is shown to contradict the uncertainty principle and leads to complex times. A real non-negative traversal time is obtained in a measurement which selects Feynman paths that spend in the barrier a known amount of time; this, however, strongly perturbs the tunneling. The Larmor clock demonstrates both types of behavior.

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I. INTRODUCTION

The question "Can one determine the amount of time τ a quantum particle spends in a specified region of space Ω?" remains open, despite having been discussed for more than five decades (for reviews, see [1]). Quantally the traversal time, like every other quantity, should be described by a wave function, i.e., the amplitude distribution for its possible values τ. In this paper we show that its measurement obeys the usual quantum rules based on the uncertainty principle. We also show that previous attempts to determine the traversal time have contradicted the principle, which have led to the present confusion about its status. Consequently, the solution to the tunneling-time problem is *not* to be found in seeking an "ultimate" candidate for τ [1]; rather the answer lies in the procedure used to quantize classical time parameters.

II. UNCERTAINTY PRINCIPLE

We require Feynman's formulation of the uncertainty principle ([2], p. 9). Suppose a quantum system can reach a known final state Ψ_F via several routes. Let f_i, $i=1,2,\ldots$, be the probability amplitude for taking the ith route. Two situations can arise.

(i) If no observation is made on the system, the routes are *interfering alternatives* ([2], pp. 13 and 14). In this case, the particular route the system has taken is *not* known, while the probability to arrive in the final state is given by $P_F=|\sum_i f_i|^2$.

(ii) If one determines unambiguously how the system reaches the final state, the routes become *exclusive alternatives* and the ith route occurs with probability $|f_i|^2$. However, $P_F=\sum_i |f_i|^2$ is then no longer equal to $|\sum_i f_i|^2$, because observation disturbs the motion ([2], pp. 13 and 14).

The principle applies to any physical measurement. As an example required later, consider the measurement in a two-slit diffraction experiment of a quantity n, which equals 1 if the particle goes through the first slit, and 2 if it goes through the second one. For case (ii), the mean value of n,

$$\langle n\rangle \equiv (1|f_1|^2+2|f_2|^2)/(|f_1|^2+|f_2|^2)\ , \tag{1}$$

is directly related to the probabilities for going through each slit. For case (i), if we evaluate the average

$$\bar{n}\equiv(1f_1+2f_2)/(f_1+f_2) \tag{2}$$

then we find that $\bar{n}$ is in general complex valued, and in accordance with the uncertainty principle, cannot be used to predict which slit the particle will go through.

III. AMPLITUDE DISTRIBUTION FOR THE TRAVERSAL TIME

Consider the traversal time, defined classically as

$$t_{ab}^{\rm cl}[x(t)]=\int_{t_1}^{t_2}\Theta_{ab}(x(t))dt\ ,$$

where $\Theta_{xz}(y)=1$ if $x\le y\le z$ and 0 otherwise, and $x(t)$ is the trajectory. A quantum particle starting at t_1 in some state Ψ_I reaches at $t_2>t_1$ a final state $\Psi_F=|N\rangle$ through a continuum of "routes," each route consisting of all Feynman paths with $t_{ab}^{\rm cl}[x(t)]$ equal to a given value τ. We assign to each route (each τ) an amplitude, by using the traversal time amplitude distribution $\sigma_{ab}(N|\tau|\Psi_I)$ given by [3,4]

$$\sigma_{ab}(N|\tau|\Psi_I)=f(N|\Psi_I)^{-1}\int dx_2\int Dx(\cdot)\int dx_1\Psi_N^*(x_2)\delta(t_{ab}^{\rm cl}[x(\cdot)]-\tau)\exp\{iS[x(\cdot)]/\hbar\}\Psi_I(x_1)\ , \tag{3}$$

where

$$f(N|\Psi_I)\equiv\int dx_2\int Dx(\cdot)\int dx_1\Psi_N^*(x_2)\times\exp\{iS[x(\cdot)]/\hbar\}\times\Psi_I(x_1)\ ,$$

and $\delta(x)$ is the Dirac δ function, $S[x(\cdot)]$ is the classical action, and $\Psi_N(x)\equiv\langle x|N\rangle$. It follows from Eq. (3) that the amplitude to reach Ψ_F and spend in $[a,b]$ a net time between τ and $\tau+\Delta\tau$ is

$$f(N|\Psi_I)\int_\tau^{\tau+\Delta\tau}\sigma_{ab}(N|\tau'|\Psi_I)d\tau'\ .$$

Note that $\sigma_{ab}(N|\tau|\Psi_I)$ is in general a complex-valued function which vanishes for τ outside the interval $[0,t_2-t_1]$, because no Feynman path can stay in $[a,b]$ longer than the total duration of motion, t_2-t_1 [3,4]. We will also assume for simplicity that $\sigma_{ab}(N|\tau|\Psi_I)$ is a smooth function of τ.

IV. INDIRECT MEASUREMENT OF THE TRAVERSAL TIME

The uncertainty principle immediately tells us that nothing can be said about the *actual* amount of time spent in $[a,b]$ unless the particle is restricted to taking paths with a known value of $t_{ab}^{\text{cl}}[x(t)]$. This, however, would perturb its motion (see below). One can try to avoid this difficulty, e.g., by adding to the potential $V(x)$ in which the particle moves an arbitrarily small term $\lambda\Theta_{ab}(x)$, and then extracting information about the traversal time from the perturbed amplitude $f_\lambda(N|\Psi_I)$. For the perturbed action, we have

$$S_\lambda[x(t)]=S[x(t)]-\lambda t_{ab}^{\text{cl}}[x(t)] ,$$

and we find

$$f_\lambda(N|\Psi_I)=f(N|\Psi_I)[1-i\lambda\bar{\tau}/\hbar+O(\lambda^2)] ,$$

where

$$\bar{\tau}\equiv\int_0^{t_2-t_1}\tau\sigma_{ab}(N|\tau|\Psi_I)d\tau \tag{4}$$

is the first moment of $\sigma_{ab}(N|\tau|\Psi_I)$, being analogous to $\bar{n}$ in Eq. (2).

Observables can be calculated from $f_\lambda(N|\Psi_I)$. Comparing their values with and without the perturbation allows one to determine $\bar{\tau}$. The price to be paid for making the interaction between the particle and measuring device arbitrarily small is that the *indirect measurement* result $\bar{\tau}$ is in general complex valued, rather than a real time interval [3–5].

V. DIRECT MEASUREMENT AND PROBABILITY DISTRIBUTION FOR THE TRAVERSAL TIME

We now define a *direct measurement* of the traversal time as follows: If the result of the measurement lies in an arbitrarily narrow interval $[\tau,\tau+\Delta\tau]$, the particle is restricted to taking paths for which $t_{ab}^{\text{cl}}[x(t)]$ lies within the same range. Equivalently, we assume that the measuring device acts as a "slit" on the traversal time scale, excluding all routes except one (for the meaning of an individual Feynman path, see [6]).

Let us divide the whole τ coordinate into arbitrarily narrow intervals $[\tau_k,\tau_k+\Delta\tau]$ where $\tau_k=k\Delta\tau$ and $k=0,\pm1,\pm2\ldots$. Since different intervals are now exclusive alternatives, the *normalized probability distribution* $w(N|\tau|\Psi_I)$ for the result of a direct measurement of the traversal time is given by

$$w(N|\tau|\Psi_I)\equiv\lim_{\Delta\tau\to0}\frac{(\Delta\tau)^{-1}P(N|\tau_k,\tau_k+\Delta\tau|\Psi_I)}{\sum_{k=-\infty}^{\infty}P(N|\tau_k,\tau_k+\Delta\tau|\Psi_I)} , \tag{5}$$

where

$$P(N|\tau,\tau+\Delta\tau|\Psi_I)=\left|f(N|\Psi_I)\int_\tau^{\tau+\Delta\tau}\sigma_{ab}(N|\tau'|\Psi_I)d\tau'\right|^2 .$$

Replacing the sum in Eq. (5) by an integral, we obtain

$$w(N|\tau|\Psi_I)=\frac{|\sigma_{ab}(N|\tau|\Psi_I)|^2}{\int_0^{t_2-t_1}|\sigma_{ab}(N|\tau|\Psi_I)|^2d\tau} . \tag{6}$$

From Eq. (6), the expectation value of the traversal time in a direct measurement $\langle\tau\rangle$ is given by

$$\langle\tau\rangle\equiv\int_0^{t_2-t_1}\tau w(N|\tau|\Psi_I)d\tau , \tag{7}$$

which is analogous to $\langle n\rangle$ in Eq. (1). In accordance with the uncertainty principle, we have obtained a non-negative probability distribution for τ by introducing a measuring device which destroys the interference between paths with different $t_{ab}^{\text{cl}}[x(t)]$.

VI. CONSERVATION OF PROBABILITY IN A DIRECT MEASUREMENT

The probability to find the particle in *some* final state and have *some* measured value of the traversal time must equal unity. More formally, let the final state $|N\rangle$ belong to a set of states $\{|N\rangle\}$ which form a complete orthonormal basis in Hilbert space. Then, we must verify that

$$\mathcal{P}\equiv\lim_{\Delta\tau\to0}\sum_{k,N}P(N|\tau_k,\tau_k+\Delta\tau|\Psi_I)=1 . \tag{8}$$

Proof. It has been shown in [4] that $\sigma_{ab}(N|\tau|\Psi_I)$ can be written as

$$\sigma_{ab}(N|\tau|\Psi_I)=[2\pi f(N|\Psi_I)]^{-1}\times\int_{-\infty}^{\infty}\exp(i\lambda\tau)\langle N|\hat{U}_\lambda(t_2,t_1)|\Psi_I\rangle d\lambda , \tag{9}$$

where $\langle N|\hat{U}_\lambda(t_2,t_1)|\Psi_I\rangle$ is the matrix element of the evolution operator for the potential $V(x)+\hbar\lambda\Theta_{ab}(x)$. Inserting Eq. (9) into $\mathcal{P}$, integrating over the time variables, and using the closure relation $\sum_N|N\rangle\langle N|=1$, we find

$$\mathcal{P}=\lim_{\Delta\tau\to0}(2\pi)^{-2}\int_{-\infty}^{\infty}d\lambda\int_{-\infty}^{\infty}d\lambda'(\lambda'\lambda)^{-1}[\exp(-i\lambda\Delta\tau)-1][\exp(i\lambda'\Delta\tau)-1]\times\langle\Psi_I|\hat{U}_\lambda^{-1}(t_2,t_1)\hat{U}_{\lambda'}(t_2,t_1)|\Psi_I\rangle\sum_{k=-\infty}^{\infty}\exp[i(\lambda'-\lambda)\tau_k] .$$

As $\Delta\tau\rightarrow 0$, the sum on the right-hand side tends to $2\pi(\Delta\tau)^{-1}\delta(\lambda'-\lambda)$ and then we find, as promised,

$$\mathcal{P}=\lim_{\Delta\tau\rightarrow 0}\frac{2}{\pi\Delta\tau}\int_{-\infty}^{\infty}\frac{\sin^2(\lambda\Delta\tau/2)}{\lambda^2}d\lambda=1\ .$$

■

VII. LARMOR CLOCK AS A MEASURING DEVICE

We now show that direct and indirect measurements of the traversal time can be realized in practice by means of a Larmor clock [1,5,7,8]. This consists of a spin with $2j+1$ components, $j=\frac{1}{2},1,\ldots$, which interacts with a uniform magnetic field H, when the particle is inside $[a,b]$. Consider the transition amplitude $f(\beta,N|\alpha,\Psi_I)$ between the states

$$|\alpha,\Psi_I\rangle=\sum_{m=-j}^{j}\alpha_m|m\rangle\int\Psi_I(x)|x\rangle dx$$

at $t=t_1$ and $|\beta,N\rangle$ at $t=t_2$ where $\hat{j}_z|m\rangle=\hbar m|m\rangle$ and $\alpha_m=\langle m|\alpha\rangle$. Since for a spin in state $|m\rangle$, the particle encounters an additional potential [7] $\hbar m\omega\Theta_{ab}(x)$, where ω is the Larmor angular velocity, we obtain using the inverse of Eq. (9)

$$\begin{aligned}f(\beta,N|\alpha,\Psi_I)&=\sum_{m=-j}^{j}\beta_m^*\alpha_m\langle N|\hat{U}_{m\omega}(t_2,t_1)|\Psi_I\rangle\\&=f(N|\Psi_I)\int_0^{t_2-t_1}\sigma_{ab}(N|\tau|\Psi_I)\\&\qquad\times F(\beta|\alpha;j,\omega,\tau)d\tau\ ,\end{aligned}\tag{10}$$

where $\beta_m=\langle m|\beta\rangle$ and

$$F(\beta|\alpha;j,\omega,\tau)=\sum_{m=-j}^{j}\beta_m^*\alpha_m\exp(-im\omega\tau)\ .\tag{11}$$

Equations (10) and (11) define the Larmor clock as a device for measuring the traversal time. Note that $\sigma_{ab}(N|\tau|\Psi_I)$ is the amplitude distribution in the *absence of the clock,* while the weight $F(\beta|\alpha;j,\omega,\tau)$ depends only on the parameters of the clock.

VIII. BAZ' READING OF THE LARMOR CLOCK AS AN INDIRECT MEASUREMENT OF THE TRAVERSAL TIME

In the method originally proposed by Baz', and subsequently used by other authors [1,5,7], the spin at $t=t_1$ is totally polarized along the x axis, so that [7]

$$\alpha_m=\{(2j)!/[2^{2j}(j+m)!(j-m)!]\}^{1/2}\ .$$

In addition, the field directed along the z axis is assumed to be *arbitrarily small,* $\omega\rightarrow 0$.

Let $|\beta\rangle=|n\rangle$. From Eq. (11) we have

$$\lim_{\omega\rightarrow 0}F^{\text{Baz}'}(n|\alpha;j,\omega,\tau)=\alpha_n[1-in\omega\tau+O(\omega^2)]\tag{12}$$

so that, as shown in Fig. 1, all values of τ inside the interval $[0,t_2-t_1]$ contribute to the integral (10). Inserting Eq. (12) into Eq. (10), one finds after standard manipulations [1,5,7] that

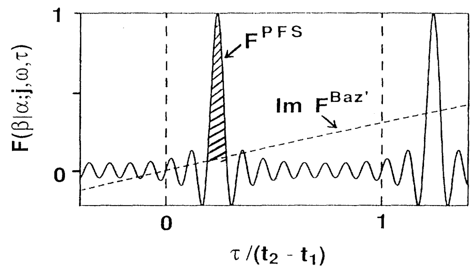

FIG. 1. Imaginary part of $F^{\text{Baz}'}(n|\alpha;j,\omega,\tau)$ for $\omega=0.02\pi/(t_2-t_1)$, $j=10$, $n=-5$ (dashed) and $F^{\text{PFS}}(\gamma^k|\gamma^0;j,\omega,\tau)$ for $\omega=2\pi/(t_2-t_1)$, $j=10$, $k=5$ (solid) vs $\tau/(t_2-t_1)$. The main contribution to $F^{\text{PFS}}(\gamma^k|\gamma^0;j,\omega,\tau)$ comes from the hatched peak.

$$\langle\hat{j}_y\rangle/\hbar j=\omega\,\text{Re}\bar{\tau}+O(\omega^2)$$

and

$$\langle\hat{j}_z\rangle/\hbar j=\omega\,\text{Im}\bar{\tau}+O(\omega^2)\ ,$$

where $\langle\hat{j}_y\rangle$ and $\langle\hat{j}_z\rangle$ are the mean projection of the y and z components of the spin, respectively. (Note that $\omega<0$ when comparing with [1].) Thus the Baz' procedure is an indirect measurement of the complex-valued average time $\bar{\tau}$, leading to a determination of $\text{Re}\bar{\tau}$ and $\text{Im}\bar{\tau}$.

IX. PERES-FODEN-STEVENS (PFS) READING OF THE LARMOR CLOCK AS A DIRECT MEASUREMENT OF THE TRAVERSAL TIME

We now show that a Larmor clock can act as a "slit" for the τ coordinate, projecting [as a measurement requires ([2], p. 106)] $\sigma_{ab}(N|\tau|\Psi_I)$ onto a state localized on this coordinate. Following [8], we use orthonormal states $|\gamma^k\rangle$, $k=0,\ldots,2j$ with

$$\gamma_m^k\equiv\langle m|\gamma^k\rangle=(2j+1)^{-1/2}\exp(-i\phi_k m)\ ,$$

and $\phi_k=2\pi k/(2j+1)$ to describe the clock. From Eq. (11) for $\omega>0$ we have

$$\begin{aligned}F^{\text{PFS}}(\gamma^k|\gamma^0;j,\omega,\tau)&=(2j+1)^{-1}\\&\quad\times\frac{\sin[(2j+1)(\phi_k-\omega\tau)/2]}{\sin[(\phi_k-\omega\tau)/2]}\ .\end{aligned}\tag{13}$$

For large j, the right-hand side of Eq. (13) is sharply peaked around $\tau=\tau_n\equiv(\phi_k+2\pi n)/\omega$, $n=0,\pm 1,\pm 2,\ldots$ with the base width approximately $4\pi/[\omega(2j+1)]$, as shown in Fig. 1. In the limit $j\rightarrow\infty$, for *finite* ω we have

$$\lim_{j\to\infty} F^{\mathrm{PFS}}(\gamma^k|\gamma^0;j,\omega,\tau)=\begin{cases} 2\pi(2j+1)^{-1}\displaystyle\sum_{n=-\infty}^{\infty}\delta(\omega\tau-\phi_k-2\pi n) & \text{if } j=\text{integer} \\ 2\pi(2j+1)^{-1}\displaystyle\sum_{n=-\infty}^{\infty}(-1)^n\delta(\omega\tau-\phi_k-2\pi n) & \text{if } j=\text{half integer}\ . \end{cases} \tag{14}$$

The interference between paths with traversal times differing by an integer number of Larmor periods can be excluded by choosing $\omega=2\pi/(t_2-t_1)$. Then, in the limit $j\to\infty$, the probability

$$P(N|\phi,\phi+d\phi|\Psi_I) \equiv \lim_{j\to\infty}\frac{\displaystyle\sum_{k=0}^{2j}\Theta_{\phi,\phi+d\phi}(\phi_k)|f(\gamma^k,N|\gamma^0,\Psi_I)|^2}{\displaystyle\sum_{k=0}^{2j}|f(\gamma^k,N|\gamma^0,\Psi_I)|^2}$$

for the clock to be in the small interval ϕ/ω to $(\phi+d\phi)/\omega$ becomes, with the help of Eqs. (11) and (14),

$$P(N|\phi,\phi+d\phi|\Psi_I)=\frac{|\sigma_{ab}(N|\phi/\omega|\Psi_I)|^2 d\phi}{\int_0^{2\pi}|\sigma_{ab}(N|\phi/\omega|\Psi_I)|^2 d\phi}\ . \tag{15}$$

Identifying ϕ/ω with τ, we find that Eq. (15) is equivalent to the direct measurement result (6). Thus a direct measurement of the traversal time can be performed with a Larmor clock in its classical limit $j\to\infty, j\omega\to\infty$.

X. CONCLUSIONS

The problem of quantizing the classical traversal time $t_{ab}^{\mathrm{cl}}[x(t)]$ has been solved. We did not have to invoke, as has been suggested in Refs. [9,10], nonstandard interpretations of quantum theory. We have shown that one cannot determine the time spent by a tunneling particle in the barrier region without perturbing its motion, for the same reason that one cannot determine which slit a particle goes through without destroying the interference pattern. Various recent candidates for the duration of tunneling, such as Larmor times, and the closely related Büttiker-Landauer and phase times [1], can be obtained from a complex time $\bar{\tau}$ [11] and are therefore results of indirect measurements. Consequently, although they are real and have a limited physical significance [11], one should not take them as *actual* tunneling times, just as $\mathrm{Re}\bar{n}$, $\mathrm{Im}\bar{n}$, or $|\bar{n}|$ in Eq. (2) cannot be used to label which slit the particle actually goes through. In contrast, a long sought real non-negative traversal time $\langle\tau\rangle$, analogous to the conventional expection value of a dynamical variable, e.g., $\langle x\rangle\equiv\int_{-\infty}^{\infty}x|\Psi(x)|^2dx$, or $\langle n\rangle$ in Eq. (1), is obtained in a direct measurement. However, a direct measurement perturbs the particle's motion to such an extent that an analysis of the tunneling through the unperturbed potential no longer applies. Our approach is also valid for time parameters which are represented classically by functionals other than $t_{ab}^{\mathrm{cl}}[x(t)]$. Thus we conclude that the concept of classical time scales cannot simply be extrapolated to the quantum case, in particular for the case of tunneling. Finally, we emphasize that any search for an "ultimate" tunneling time will encounter the same difficulties that arise in attempts to understand the two-slit diffraction experiment without invoking the uncertainty principle [2].

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