Phase measurements

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We address the problem of identifying an operational prescription for quantum phase measurements. As is known, different experiments can lead to different measured phase operators. However, we show that ambiguities of interpretation can arise even if a single experiment, such as that of Noh, Fougères, and Mandel [Phys. Rev. Lett. **67**, 1426 (1991)], is chosen as defining a phase measurement. We show by reference to a simple but fundamental example that it is not possible to deduce a unique phase difference from the measurements.

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I. INTRODUCTION

Recent renewed interest in the theoretical formulation of the quantum optical phase has focused attention on possible phase measurements [1,2]. In designing a suitable experiment, we can follow one of two possible approaches. On the one hand, we can introduce phase as a fundamental mathematical component of the quantum theory of light [3], and only then seek an experiment from which we can infer information about the phase statistics. Alternatively, we could examine classical phase-measuring experiments and define these as measurements of phase in the quantum regime [1,2,4,5]. Unfortunately, in the latter operational approach, the classical description is not sufficient to define a unique quantum phase [2]. Different classical measurement schemes lead to different quantum measured phase operators. While this is disappointing, it might still be possible to choose a particular measurement scheme as the definitive phase measurement. This choice would necessarily represent a *convention* for the definition of phase.

The adoption of an operational definition of quantum phase has been explored by us [4], and more recently advocated by Noh, Fougères, and Mandel [1,2]. The purpose of this paper is to point out that even if one particular measurement scheme is agreed upon, there can still remain an ambiguity in the definition of the phase. In the experiment of Noh, Fougères, and Mandel, this ambiguity leads to inconsistent interpretations of the experiment *as a quantum phase measurement.* Independent methods for extracting the phase from the data lead to equivalent results in the classical regime and therefore to a unique classical phase. However, application of the same methods in the quantum regime can give inconsistent results, precluding the definition of a unique measured phase operator for the chosen experiment.

II. THE EXPERIMENT OF NOH, FOUGÈRES, AND MANDEL

We wish to point out form the outset that the experimental results of Noh, Fougères, and Mandel are in excellent agreement with the predictions of their theoretical analysis, and we do not question the accuracy of either of these. Indeed, the agreement is so good that there is little point in attempting to explain their experimental results in terms of any other theory. We shall, however, question the interpretation of these results as phase measurements. Their approach is to use a classical treatment of the experiment to provide the interpretation of the quantum-mechanical quantities that they measure. That is, their choice of quantum-mechanical operators is based on a direct comparison with the corresponding classical expressions.

A. Classical analysis

Figure 1 shows the experimental scheme of Noh, Fougères, and Mandel, which consists of a similar arrangement to the eight-port homodyne detection scheme of Walker and Carrol [5]. There are four outputs with a photodetector (D_j) at each. There are also four input ports, two of which are used for the fields to be measured. Within the domain of classical optics, the instantaneous amplitudes of the two input fields are

$$V_1=I_1^{1/2}\exp(i\phi_1)\ , \tag{2.1a}$$

$$V_2=I_2^{1/2}\exp(i\phi_2)\ . \tag{2.1b}$$

The phase difference $\phi_2-\phi_1$ is assumed to remain constant during the measurement time T. The integrated light intensities at the four outputs during the measurement time are [1,2]

$$W_3=\tfrac{1}{4}[W_1+W_2-2W_{12}\cos(\phi_2-\phi_1)]\ , \tag{2.2a}$$

$$W_4=\tfrac{1}{4}[W_1+W_2+2W_{12}\cos(\phi_2-\phi_1)]\ , \tag{2.2b}$$

$$W_5=\tfrac{1}{4}[W_1+W_2-2W_{12}\sin(\phi_2-\phi_1)]\ , \tag{2.2c}$$

$$W_6=\tfrac{1}{4}[W_1+W_2+2W_{12}\sin(\phi_2-\phi_1)]\ , \tag{2.2d}$$

where

$$W_l=\alpha\int_t^{t+T}I_l(t')dt'\ , \tag{2.3a}$$

$$W_{12}=\alpha\int_t^{t+T}[I_l(t')I_2(t')]^{1/2}dt' , \tag{2.3b}$$

and α is the quantum efficiency of the detectors. The integrated intensities are clearly dependent on the phase difference $\phi_2-\phi_1$, and we can use the measured values of these intensities to obtain this phase difference. In particular, we find

$$W_4-W_3=W_{12}\cos(\phi_2-\phi_1) , \tag{2.4a}$$

$$W_6-W_5=W_{12}\sin(\phi_2-\phi_1) , \tag{2.4b}$$

$$W_6-W_4=\frac{1}{\sqrt{2}}W_{12}\sin(\phi_2-\phi_1-\pi/4) , \tag{2.4c}$$

$$W_6-W_3=\frac{1}{\sqrt{2}}W_{12}\cos(\phi_2-\phi_1-\pi/4) , \tag{2.4d}$$

$$W_3-W_5=\frac{1}{\sqrt{2}}W_{12}\sin(\phi_2-\phi_1-\pi/4) , \tag{2.4e}$$

$$W_4-W_5=\frac{1}{\sqrt{2}}W_{12}\cos(\phi_2-\phi_1-\pi/4) . \tag{2.4f}$$

These six quantities, together with the trigonometric identity $\cos^2\theta+\sin^2\theta=1$, are more than enough to determine the value of the classical phase difference from the experimental results. For example, we find that

$$\cos(\phi_2-\phi_1)=\frac{W_4-W_3}{[(W_4-W_3)^2+(W_6-W_5)^2]^{1/2}} , \tag{2.5a}$$

$$\sin(\phi_2-\phi_1)=\frac{W_6-W_5}{[(W_4-W_3)^2+(W_6-W_5)^2]^{1/2}} , \tag{2.5b}$$

$$\cos(\phi_2-\phi_1-\pi/4)=\frac{W_6-W_3}{[(W_6-W_3)^2+(W_6-W_4)^2]^{1/2}} \tag{2.5c}$$

$$=\frac{W_4-W_5}{[(W_4-W_5)^2+(W_3-W_5)^2]^{1/2}} \tag{2.5d}$$

$$=\frac{W_6-W_3}{[(W_6-W_3)^2+(W_3-W_5)^2]^{1/2}} \tag{2.5e}$$

$$=\frac{W_4-W_5}{[(W_4-W_5)^2+(W_6-W_4)^2]^{1/2}} , \tag{2.5f}$$

$$\sin(\phi_2-\phi_1-\pi/4)=\frac{W_6-W_4}{[(W_6-W_4)^2+(W_6-W_3)^2]^{1/2}} \tag{2.5g}$$

$$=\frac{W_3-W_5}{[(W_4-W_5)^2+(W_3-W_5)^2]^{1/2}} \tag{2.5h}$$

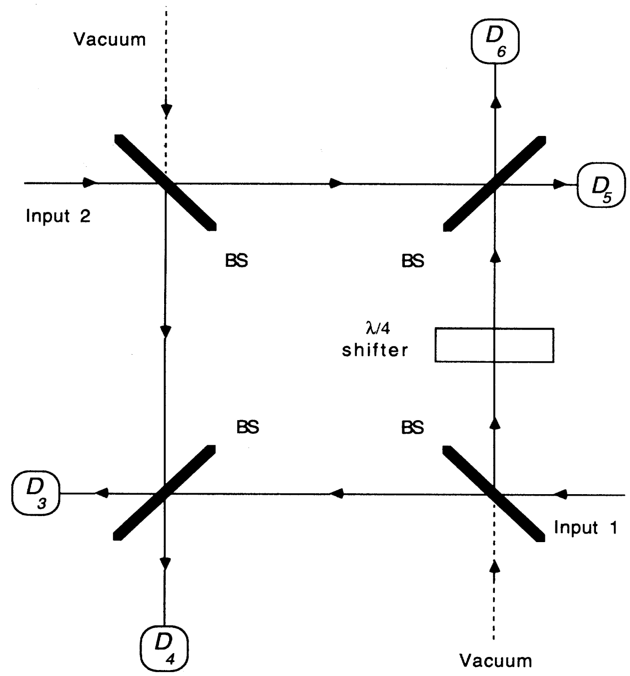

FIG. 1. Outline of the experimental setup used by Noh, Fougères, and Mandel. The thick lines denote 50-50 beam splitters (BS).

$$=\frac{W_3-W_5}{[(W_6-W_3)^2+(W_3-W_5)^2]^{1/2}} \tag{2.5i}$$

$$=\frac{W_6-W_4}{[(W_4-W_5)^2+(W_6-W_4)^2]^{1/2}} . \tag{2.5j}$$

The classical physics of the device ensures the equality between (2.5c), (2.5d), (2.5e), and (2.5f), and between (2.5g), (2.5h), (2.5i), and (2.5j). Any two of the above four trigonometric expressions can be used to determine the phase difference, and the result will be independent of which two are chosen.

B. Quantum analysis

In quantum-mechanical intensity measurements, a discrete number of photoelectron counts are recorded in the counting time. The integrated intensity is proportional to this number and is represented by an operator $\hat{n}_j$, which corresponds to the classical integrated intensity W_j to within a constant of proportionality. From this correspondence, we can define operator functions of $\hat{n}_j$ that relate to the classical trigonometrical expressions (2.5):

$$\hat{C}_M(\phi_2-\phi_1)=\frac{\hat{n}_4-\hat{n}_3}{[(\hat{n}_4-\hat{n}_3)^2+(\hat{n}_6-\hat{n}_5)^2]^{1/2}} , \tag{2.6a}$$

$$\hat{S}_M(\phi_2-\phi_1)=\frac{\hat{n}_6-\hat{n}_5}{[(\hat{n}_4-\hat{n}_3)^2+(\hat{n}_6-\hat{n}_5)^2]^{1/2}} , \tag{2.6b}$$

$$\hat{C}_{M1}(\phi_2-\phi_1-\pi/4)=\frac{\hat{n}_6-\hat{n}_3}{[(\hat{n}_6-\hat{n}_3)^2+(\hat{n}_6-\hat{n}_4)^2]^{1/2}}\,, \tag{2.6c}$$

$$\hat{C}_{M2}(\phi_2-\phi_1-\pi/4)=\frac{\hat{n}_4-\hat{n}_5}{[(\hat{n}_4-\hat{n}_5)^2+(\hat{n}_3-\hat{n}_5)^2]^{1/2}}\,, \tag{2.6d}$$

$$\hat{C}_{M3}(\phi_2-\phi_1-\pi/4)=\frac{\hat{n}_6-\hat{n}_3}{[(\hat{n}_6-\hat{n}_3)^2+(\hat{n}_3-\hat{n}_5)^2]^{1/2}}\,, \tag{2.6e}$$

$$\hat{C}_{M4}(\phi_2-\phi_1-\pi/4)=\frac{\hat{n}_4-\hat{n}_5}{[(\hat{n}_4-\hat{n}_5)^2+(\hat{n}_6-\hat{n}_4)^2]^{1/2}}\,, \tag{2.6f}$$

$$\hat{S}_{M1}(\phi_2-\phi_1-\pi/4)=\frac{\hat{n}_6-\hat{n}_4}{[(\hat{n}_6-\hat{n}_4)^2+(\hat{n}_6-\hat{n}_3)^2]^{1/2}}\,, \tag{2.6g}$$

$$\hat{S}_{M2}(\phi_2-\phi_1-\pi/4)=\frac{\hat{n}_3-\hat{n}_5}{[(\hat{n}_4-\hat{n}_5)^2+(\hat{n}_3-\hat{n}_5)^2]^{1/2}}\,, \tag{2.6h}$$

$$\hat{S}_{M3}(\phi_2-\phi_1-\pi/4)=\frac{\hat{n}_3-\hat{n}_5}{[(\hat{n}_6-\hat{n}_3)^2+(\hat{n}_3-\hat{n}_5)^2]^{1/2}}\,, \tag{2.6i}$$

$$\hat{S}_{M4}(\phi_2-\phi_1-\pi/4)=\frac{\hat{n}_6-\hat{n}_4}{[(\hat{n}_4-\hat{n}_5)^2+(\hat{n}_6-\hat{n}_4)^2]^{1/2}}\,. \tag{2.6j}$$

We note that all the $\hat{n}_j$ operators commute with each other, and therefore no ambiguity of ordering results from the division. Moreover, all of these S_M and C_M operators commute with each other, and therefore represent compatible observables. The normalization ensures that these operators obey the identities

$$\hat{C}_M^2(\phi_2-\phi_1)+\hat{S}_M^2(\phi_2-\phi_1)=\hat{1}\,, \tag{2.7a}$$

$$\hat{C}_{Ml}^2(\phi_2-\phi_1-\pi/4)+\hat{S}_{Ml}^2(\phi_2-\phi_1-\pi/4)=\hat{1},\quad \forall l\,, \tag{2.7b}$$

where $\hat{1}$ is the unit operator. Given these features, it seems eminently reasonable to interpret these operators as representing phase observables. Noh, Fougères, and Mandel chose to use their experimental data to calculate the statistics of $S_M(\phi_2-\phi_1)$ and $C_M(\phi_2-\phi_1)$, which they refer to as their measured sine and cosine of the phase difference. However, there is no *a priori* justification for choosing this particular pair to determine the phase difference. Of course, if all the possible choices yield the same result, as they do classically, then all choices would be equivalent. Unfortunately, this is not the case in the quantum regime, and the different pairs of observables can give results that are inconsistent with a single phase probability distribution. We can illustrate this inconsistency with a simple example.

Consider the case where input field 1 contains precisely one photon and input field 2 is in its vacuum state. A quantum-mechanical analysis shows that only one of the four detectors will register a photocount, and that each detector is equally likely to do so. Following Noh, Fougères, and Mandel, we use the knowledge of which detector was triggered to construct the values of the operators (2.6). If for a particular operator the denominator becomes zero, then that result is not used and the moments for that operator are renormalized accordingly [1,2]. In Table I, we show the four possible outcomes of a single measurement. We emphasize that the appearance of indeterminacies (denoted by question marks in the table) for some operators but not for others is not a fundamental distinction, but rather is dependent on the particular state of the input fields. If we interpret the operators as representing the cosines or sines of their argu-

TABLE I. The four possible outcomes for measurement with only one photon in input field 1. The numbers refer to the measured values of the operators defined in Eq. (2.6). The figures in parentheses are the phase differences in the range $(-\pi,\pi]$, which follow if we interpret these operators as measuring the cosines and sines of their argument. Indeterminate results are denoted by a question mark (?).

	Photodetector registering photocount			
Operator measured	D_3	D_4	D_5	D_6
$\hat{C}_M(\phi_2-\phi_1)$	-1 (π)	1 (0)	0 $(\pm\pi/2)$	0 $(\pm\pi/2)$
$\hat{S}_M(\phi_2-\phi_1)$	0 $(0,\pi)$	0 $(0,\pi)$	-1 $(-\pi/2)$	1 $(\pi/2)$
$\hat{C}_{M1}(\phi_2-\phi_1-\pi/4)$	-1 $(-3\pi/4)$	0 $(-\pi/4, 3\pi/4)$	? (?)	$1/\sqrt{2}$ $(0, \pi/2)$
$\hat{S}_{M1}(\phi_2-\phi_1-\pi/4)$	0 $(\pi/4, -3\pi/4)$	-1 $(-\pi/4)$	? (?)	$1/\sqrt{2}$ $(\pi, \pi/2)$
$\hat{C}_{M2}(\phi_2-\phi_1-\pi/4)$	0 $(-\pi/4, 3\pi/4)$	1 $(\pi/4)$	$-1/\sqrt{2}$ $(\pi, -\pi/2)$	? (?)
$\hat{S}_{M2}(\phi_2-\phi_1-\pi/4)$	1 $(3\pi/4)$	0 $(\pi/4, -3\pi/4)$	$-1/\sqrt{2}$ $(0, -\pi/2)$	? (?)
$\hat{C}_{M3}(\phi_2-\phi_1-\pi/4)$	$-1/\sqrt{2}$ $(\pi, -\pi/2)$	? (?)	0 $(-\pi/4, 3\pi/4)$	1 $(\pi/4)$
$\hat{S}_{M3}(\phi_2-\phi_1-\pi/4)$	$1/\sqrt{2}$ $(\pi, \pi/2)$	? (?)	-1 $(-\pi/4)$	0 $(\pi/4, -3\pi/4)$
$\hat{C}_{M4}(\phi_2-\phi_1-\pi/4)$	? (?)	$1/\sqrt{2}$ $(0, \pi/2)$	-1 $(-3\pi/4)$	0 $(-\pi/4, 3\pi/4)$
$\hat{S}_{M4}(\phi_2-\phi_1-\pi/4)$	? (?)	$-1/\sqrt{2}$ $(0, -\pi/2)$	0 $(\pi/4, -3\pi/4)$	1 $(3\pi/4)$

ments, then we can deduce the phase difference. This is shown in parenthesis in Table I. We note that, although the expectation of the square of each operator is $\frac{1}{2}$, there is no single phase difference that is consistent with the outcome of any *single measurement*. For example, if only D_4 registers a photocount, then assigning a value of $\pi/4$ to the phase difference $\phi_2-\phi_1$ is consistent with S_{M2} and C_{M2} (and possibly with S_{M3} and C_{M3}), but not with any of the other operators.

Comparison with the classical analysis provides no objective reason for preferring any of the operators purporting to represent phase over the others. Indeed, classically there would be no inconsistencies and no reason for choosing between the options discussed above. There are no ambiguities in the interpretation of the classical experiment as a phase-difference measurement. However, this is certainly not the case for the quantum-mechanical experiment. There is no consistent interpretation of the result of a single measurement as a phase-difference measurement.

III. CONCLUSION

The example in Sec. II may appear to be a pathological choice. However, if the experiment of Noh, Fougères, and Mandel had a consistent interpretation as a quantum phase-difference measurement, then this should apply to any choice of input states. Moreover, field states with small numbers of excitations are those very states for which a quantum description of phase is especially important.

It is not our main point to address the physical grounds for accepting a quantum formulation of optical phase, although it is worth mentioning that there are strong reasons for requiring the phase difference between number-state fields to be completely random [6]. Our point is this: It is not even possible to address questions such as phase randomness in the approach favored by Noh, Fougères, and Mandel, because this approach does not lead to a consistent phase-difference probability distribution.

We conclude that the interpretation of the experiments of Noh, Fougères, and Mandel as a phase-difference measurement in the quantum regime is inappropriate.

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