How to reset an atom after a photon detection: Applications to photon-counting processes

Gerhard C. Hegerfeldt
Institut für Theoretische Physik, Universität Göttingen, Germany
(Received 13 April 1992)

The question is studied as to which state a radiating atom has to be reset to if a photon is detected and absorbed. This is important for applications of a recently proposed approach to photon counts from a radiating atom which is based upon repeated gedanken measurements and associated reductions as a technical tool. The general reset density matrix is determined. It may depend on the atomic state shortly before the photon detection, and may thus be time dependent. It may also have off-diagonal terms which may lead to interesting coherence effects. It is shown that the repeated measurements together with the reset matrix lead to the optical Bloch equations for a general N-level atom. The general photon-counting distribution for an N-level radiating atom is derived and the axiomatic continuous-measurement theory of Srinivas and Davies [Opt. Acta **28**, 981 (1981); **29**, 235 (1982)] for quantum-counting processes is shown to follow in the present framework from ordinary quantum mechanics.

PACS number(s): 32.90.+a, 42.50.−p, 03.65.Bz

I. INTRODUCTION

Experiments on single ions have become feasible in the last years, mainly due to advances for laser cooling in Paul traps [1]. New effects have been discovered which are washed out if one deals with an ensemble of atoms instead of a single one. Particularly spectacular are the macroscopic dark periods of a fluorescing system with two excited states, one rapidly decaying and the other metastable, as originally predicted by Dehmelt [2] and later verified experimentally [3] and analyzed theoretically [4–6]. In this way new and interesting theoretical challenges have arisen in connection with photon counting from single fluorescing atoms.

With such applications in mind, Wilser and the present author [7–9] proposed an approach to photon counting from a radiating atom which is based on repeated gedanken measurements of the spontaneous photons, at instances a short time Δt apart, and with ensuing reductions according to the von Neumann–Lüders rule [10]. A very similar, if not identical, approach was then also proposed independently by Dalibard, Castin, and Mølmer [11] who emphasized applications with Monte Carlo simulations, and also by Carmichael [12]. Applications are given by Dum, Zoller, and Ritsch [13]. The reductions serve as a technical tool and allow the determination of the time development in a simple way. The above approach to photon counting differs from the one of Glauber [14] which is based on correlation functions. Our approach is particularly adapted to the description of single radiating atoms. It lends itself to an analytic treatment as well as to simulations; the latter point was particularly stressed in Ref. [11].

The optical Bloch equations are not used as an input but rather follow as a consequence. For systems with a single ground state this was shown in Refs. [8, 11]; the general case is treated at the end of Sec. III. In Ref. [9] it was pointed out that the no-photon probability obtained by the approach was also compatible with results of Porrati and Putterman [15] who calculated a no-photon probability at time t with no measurements and no reductions in between; this problem was then studied in great generality by Reibold [16] using the projector formalism. In Refs. [7–9] it was shown that the approach leads to a simple quantum-mechanical understanding of Dehmelt's dark periods and to quantitative agreement with results of Cohen-Tannoudji and Dalibard [5]. It was further shown in Refs. [8, 9] that the axiomatic continuous-measurement theory of Davies and Srinivas [17] can be derived for systems with a single ground state by the above approach. Hegerfeldt and Plenio [18] have used the approach to show that through a coherence effect one can have macroscopic dark periods also without a metastable state provided the separation of the two upper levels is very small and their dipole moments for the transition to the ground state are parallel. This explains, through the behavior of individual atoms, the nonabsorption resonances in gases found with the help of optical Bloch equations by Cardimona, Raymer, and Stroud [19]. The approach can also be applied to the Λ system, as pointed out in Ref. [18], and explains in a similar way the nonabsorption resonances found by Orriols [20].

The term "photon emission" will be replaced by the more precise term "photon detection" in the following. The applications in Refs. [7–9, 18] use atomic systems with transitions to the same ground state (except for the Λ system in Ref. [18]). For these simple systems it seems intuitively clear that after detection of a photon the atom should be in the ground state; in Ref. [7] it was pointed out that this can be verified in second-order perturbation theory. Once in the ground state it takes some time

for the atom to be reexcited and to emit the next photon. The familiar antibunching [21] can be understood in this way. For a general N-level atom with transitions between various levels and for broadband detection of photons, without frequency resolution, one might expect a diagonal mixture of the different final atomic states. If the levels are far apart this is approximately so, but in general there may arise off-diagonal terms which may lead to coherence effects. This is explicitly discussed for the Λ system, a three-level system with allowed transitions from the highest level to two lower levels [22]. After a photon detection one has a mixture of both lower levels with nonvanishing off-diagonal terms if the transition dipole moments are not orthogonal. If the two lower levels are far apart the off-diagonal terms will, for most questions, play no role and may be omitted, but they may lead to coherence effects for small level separation [18]. The determination of the reset matrix is an important question in applications since one has to know how to continue the analysis or simulation after detection of a photon.

In Sec. II of this paper we study to which state or density matrix an atom has to be reset after a broadband photon detection, provided that shortly before no photon was found. This condition is adapted to the above approach of repeated gedanken measurements and it allows the determination of the general reset matrix in a simple way. For an atom with transitions to a single ground state the atom is reset to it, as expected. In general the reset matrix contains nonzero off-diagonal elements. It may also depend on the atomic state shortly before the photon detection and may thus be time dependent. For the Λ system the reset matrix is always the same, independent of the prior atomic state, but it may have nonzero off-diagonal terms which may lead to interference effects.

In Sec. III the reset matrix is applied, in the context of the above approach of repeated gedanken measurements, to broadband photon counting. With its help the general counting distributions are determined. It is shown that the repeated measurements together with the general reset matrix lead to the optical Bloch equations and to the photon correlation function $g^{(2)}(\tau)$. A connection is pointed out with the continuous-measurement theory of Davies and Srinivas [17] who had axiomatically introduced postulates for operators of a so-called quantum-counting process, operators which then had to be determined semiphenomenologically. On a coarse-grained time scale, coarse compared to the above Δt, it is shown—again generalizing results of Refs. [8, 9]—that these operators not only arise in a natural way in our approach but can be explicitly determined.

In Sec. IV, finally, the results are discussed, especially the importance of the reset matrix for simulations. Generalizing the results of Ref. [9] it is pointed out that the photon counts from a *single* atom can be considered as a sample path of a stochastic process constructed in Sec. III and that ergodic properties allow the calculation of time averages for such a sample path (i.e., for a single atom) by ensemble averages. By simulation of such sample paths these quantities as well as a solution of the Bloch equations can be determined numerically, just as simulations of sample paths for the Langevin equation may be used for a numerical solution of the Fokker-Planck equation.

II. THE RESET MATRIX

The von Neumann–Lüders rule [10] for the reduction after a measurement states the following. Let $\mathcal{E}$ be an ensemble of systems described by a state vector $|\,\psi\rangle$ or, more generally, by a density matrix ρ, let $\mathcal{O}$ be an observable with discrete eigenvalues $\{\alpha\}$, and let P_α be the projector onto the eigenspace belonging to the eigenvalue α. Measuring $\mathcal{O}$ on the systems one will in general find different values, unless $|\,\psi\rangle$ is an eigenstate of $\mathcal{O}$. Denoting by $\mathcal{E}_\alpha$ the subensemble of systems for which a particular value α has been found, then right after the measurement $\mathcal{E}_\alpha$ is described by $P_\alpha \rho P_\alpha \,/\, \mathrm{tr}(\,)$. According to the statistical interpretation of quantum mechanics the norm squared or the trace give the relative magnitude of $\mathcal{E}_\alpha$, i.e., the probability of finding the value α.

As a system we consider an atom, possibly driven, plus its radiation field and measure whether there are photons or not. As observable one may take either

$$P_0 \equiv |\,0_{\mathrm{ph}}\rangle \mathbb{1}_A \langle 0_{\mathrm{ph}}\,| \tag{1}$$

or

$$P_1 \equiv \mathbb{1} \;-\; P_0\;. \tag{2}$$

The subensemble with observed photons belongs to the eigenvalue 1 of P_1 and is described by

$$P_1\;\rho\;P_1\;/\;\mathrm{tr}(\,)\;, \tag{3}$$

and analogously with P_0 for no photons.

The von Neumann–Lüders rule applies, by its very formulation, to nondemolition measurements only. Therefore the reduced state in Eq. (3) still contains the photons. But photon detection is usually by absorption [23], and the above rule does not state how to describe, for a coupled system, the absorption of a subsystem. In particular, in a resonator or with mirrors present a photon absorption might have a reaction on the atomic system.

For infinite space, however, and without mirrors one may argue physically that for the atomic description alone it should make no difference whether or not the photons are absorbed, as long as they are sufficiently far away from the atom and are no longer interacting with it [24]. After a *nondemolition* photon measurement the density matrix for the atom alone would be given by the partial (photonic) trace of the expression in Eq. (3),

$$\mathrm{tr}_{\mathrm{ph}}\; P_1 \rho P_1 \;/\; \mathrm{tr}(\,)\;. \tag{4}$$

By the above argument this should also be the atomic density matrix after a demolition measurement. But after absorption no photons are present any longer and the complete system would then be described by

$$|\,0_{\mathrm{ph}}\rangle\;(\mathrm{tr}_{\mathrm{ph}}\;P_1\rho P_1)\;\langle 0_{\mathrm{ph}}\,|\;/\;\mathrm{tr}(\,)\;. \tag{5}$$

Equation (5) extends the von Neumann–Lüders reduction rule to demolition measurements, and it has been used before as the obvious procedure [25]. As an example, for a two-level system it is usually taken for granted that after a photon detection the system is in the ground state. This indeed results from Eq. (5) as follows. For a nonabsorption photon measurement the usual von Neumann–Lüders reduction rule and ordinary perturbation theory yields the state $\mid 1_{\rm ph}\rangle \mid 1\rangle$, and absorption of the photon gives $\mid 0_{\rm ph}\rangle \mid 1\rangle$ by Eq. (5). We take Eq. (5) as a starting point for the following determination of the reset matrix.

Let us now consider an ensemble where no photons are present at time t, thus described by $\mid 0_{\rm ph}\rangle\rho_A(t)\langle 0_{\rm ph}\mid$ where ρ_A is the density matrix of the atoms. By time $t+\Delta t$ this has developed into

$$U(t+\Delta t,t)\mid 0_{\rm ph}\rangle\rho_A(t)\langle 0_{\rm ph}\mid [U(t+\Delta t,t)]^*,$$

where U is the complete unitary time-development operator. Performing a broadband photon measurement [26] at time $t+\Delta t$, the subensemble where photons are found is described by Eq. (5), with atomic part

$$\begin{aligned}\rho_A^{(1)} &= {\rm tr}_{\rm ph}\ P_1 U(t+\Delta t,t)\mid 0_{\rm ph}\rangle \\ &\quad\times\rho_A\ \langle 0_{\rm ph}\mid [U(t+\Delta t,t)]^*\ /\ {\rm tr}(\,) \\ &\equiv \hat{J}\rho_A\ \Delta t\ /\ {\rm tr}(\,),\end{aligned} \tag{6}$$

which serves as a definition of the linear superoperator $\hat{J}$ acting on ρ_A. The size of the subensemble where photons were found relative to that described by ρ_A is given by ${\rm tr}\hat{J}\rho_A\Delta t$. Equation (6) can be evaluated by simple perturbation theory. With an external field $\mathbf{E}_e(t)$, the standard Hamiltonian in dipole form and in the limit of long wavelengths and in the rotating-wave approximation is [27]

$$\begin{aligned}H = &\sum_i \hbar\omega_i \mid i\rangle\langle i\mid\ + H_F^0 + e\mathbf{D}^{(-)}\cdot(\mathbf{E}_e^{(+)}+\mathbf{E}^{(+)}) \\ &+ {\rm H.c.},\end{aligned} \tag{7}$$

where $\mathbf{D}^{(-)} = \sum_{i>j}\mathbf{D}_{ij}\mid i\rangle\langle j\mid$ and $\mathbf{D}_{ij} = \langle i\mid\mathbf{X}\mid j\rangle$. Going over to the interaction picture with $H_A^0+H_F^0$ and retaining only terms which become proportional to Δt the external field drops out in Eq. (6), and one obtains for the numerator of Eq. (6), with $\omega_{ij} = \omega_i - \omega_j$ and $\omega = c\mid\mathbf{k}\mid$,

$$\begin{aligned}&\hbar^{-2}{\rm tr}_{\rm ph}\ P_1\ e^{-iH_0\Delta t/\hbar}\int_0^{\Delta t} dt'\ \mathbf{D}^{(+)}\cdot\mathbf{E}^{(-)}\mid 0_{\rm ph}\rangle\rho_A\langle 0_{\rm ph}\mid\int_0^{\Delta t} dt'\mathbf{D}^{(-)}\cdot\mathbf{E}^{(+)}e^{iH_0\Delta t/\hbar} \\ &= \sum_{\substack{i,j,\ell,m\\ i>j\\ \ell>m}}\int_0^{\Delta t}dt'\int_0^{\Delta t}dt''\left\{\sum_{\mathbf{k},\lambda}\frac{e^2\omega}{2\epsilon_0\hbar V}(\mathbf{D}_{ji}\cdot\boldsymbol{\epsilon}_{\mathbf{k}\lambda})(\boldsymbol{\epsilon}_{\mathbf{k}\lambda}\cdot\mathbf{D}_{\ell m})\ e^{i\omega\ (t'-t'')}\right\} \\ &\quad\times e^{-i\omega_{ij}t'}e^{i\omega_{\ell m}t''}e^{-i(\omega_j-\omega_m)\Delta t}\mid j\rangle\langle i\mid\rho_A\mid\ell\rangle\langle m\mid .\end{aligned} \tag{8}$$

We denote the expression in curly brackets, a correlation function, by $\kappa_{ji\ell m}(\tau)$, $\tau = t'-t''$. It vanishes rapidly for $\mid\tau\mid$ larger than a correlation time τ_c which is short compared to the inverse optical transition frequencies [28]. To exploit this, we decompose the rectangular integration domain in Eq. (8) into two triangles, $\int_0^{\Delta t}dt'\int_0^{t'}dt''$ and $\int_0^{\Delta t}dt''\int_0^{t''}dt'$. The first can be written as

$$\int_0^{\Delta t}dt'\ e^{-i(\omega_{ij}-\omega_{\ell m})t'}\int_0^{t'}d\tau\ \kappa_{ji\ell m}(\tau)\ e^{-i\omega_{\ell m}\tau}\ . \tag{9}$$

By a standard textbook argument [28] the inner integral can be extended to infinity if $\Delta t \gg \tau_c$ and gives $\Gamma_{ji\ell m}$, with

$$\Gamma_{ji\ell m} \equiv e^2\ \mathbf{D}_{ji}\cdot\mathbf{D}_{\ell m}\mid\omega_{\ell m}\mid^3\ /\ 6\pi\epsilon_0\hbar c^3, \tag{10}$$

where, as usual, a term corresponding to a line shift has been neglected. Note that $\Gamma_{ijji} \equiv A_{ij}$ is the Einstein coefficient for the i-j transition.

Now two different ranges of Δt are considered. First, let Δt be smaller than the inverse optical frequencies, $\omega_{ij}^{-1} \gg \Delta t \gg \tau_c$. Then the remaining integral over t' is Δt. For the second triangle one only has to interchange (ji) and (ℓm). Thus we obtain

$$\hat{J}\rho_A = \sum_{\substack{i,j,\ell,m\\ i>j\\ \ell>m}}\{\Gamma_{ji\ell m}+\Gamma_{\ell mji}\}\mid j\rangle\langle i\mid\rho_A\mid\ell\rangle\langle m\mid, \tag{11}$$

$$\rho_A^{(1)} = \hat{J}\rho_A\ /\ {\rm tr}(\hat{J}\rho_A), \tag{12}$$

which generalizes Eq. (16) of Ref. [9].

To this density matrix one has to reset the subensemble of atoms for which photons were found, provided that a broadband photon measurement at a time Δt earlier had yielded zero [29] and the atomic density matrix after this earlier measurement had been ρ_A. Typically Δt is here of the order of 10^{-16} sec.

Secondly, we now consider $\Delta t \sim 10^{-12}$s, i.e., Δt much larger than the inverse optical frequencies but much smaller than the level lifetimes. One now obtains a slightly different but essentially equivalent result. If two optical transition frequencies are far apart, i.e., $\mid\omega_{ij}-\omega_{\ell m}\mid\Delta t \gg 1$, then

$$\int_0^{\Delta t'}dt'\ e^{-i(\omega_{ij}-\omega_{\ell m})t'} \approx 0$$

and the expression in Eq. (9) becomes for these transitions

$$\Gamma'_{ji\ell m} = \delta_{i\ell}\,\delta_{jm}\,\Gamma_{ji\ell m}\,. \tag{13}$$

On the other hand, if two optical transition frequencies are very close, i.e., if $|\omega_{ij}-\omega_{\ell m}|\,\Delta t \ll 1$, then one again obtains Eq. (10). Instead of the above reset matrix $\rho_A^{(1)}$ one thus gets

$$\rho_A^{(1)'} = \hat{J}'\rho_A \,/\, \mathrm{tr}(), \tag{14}$$

where $\hat{J}'$ is obtained from $\hat{J}$ by replacing $\Gamma_{ji\ell m}$ by $\Gamma_{ji\ell m}\delta_{i\ell}\delta_{jm}$ whenever ω_{ij} and $\omega_{\ell m}$ are far apart.

What then is the correct reset matrix? At the end of Sec. III it will be shown that $\rho_A^{(1)}$ of Eq. (12) leads to the usual optical Bloch equations while $\rho_A^{(1)'}$ in general does not. However, the only difference consists of some rapidly oscillating terms which play no role in most applications. One can therefore use either expression. In Secs. III it is shown that the reset matrix can also be obtained directly from the Bloch equations. In the following examples (i) and (ii) $\rho_A^{(1)}$ does not depend on the atomic state ρ_A preceding the measurement, but in example (iii) it does.

(i) *Single ground state.* $\mathbf{D}_{ij} = \mathbf{0}$ unless i or j equals 1, the ground state. Then $\rho_A^{(1)} = |\,1\rangle\langle 1\,|$, by Eq. (11). This means that after a photon detection the atom is in the ground state, as expected.

(ii) Λ *system.* Two ground states $|\,1\rangle$ and $|\,2\rangle$, excited state $|\,3\rangle$, i.e., only $\mathbf{D}_{13}$ and $\mathbf{D}_{23}$ are nonzero. Then one has, with the Einstein coefficients A_{31} and A_{32},

$$\rho_A^{(1)} = \frac{1}{A_{31}+A_{32}} \begin{pmatrix} A_{31} & \Gamma_{1332}+\Gamma_{3213} & 0 \\ \Gamma_{2331}+\Gamma_{3123} & A_{32} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

The off-diagonal terms vanish only if $\mathbf{D}_{31}\cdot\mathbf{D}_{23}=0$. The same matrix appears in Ref. [22] as an initial condition. For levels 1 and 2 sufficiently far apart the off-diagonal terms can be neglected for most questions so that the atom can be taken to be either in state $|\,1\rangle$ or $|\,2\rangle$ after a photon detection. On the other hand, for levels 1 and 2 close together the off-diagonal terms lead to interesting coherence effects such as dark periods [18] and quantum beats in the correlation function $g^{(2)}(\tau)$ under illumination with *incoherent* light [30].

(iii) *Cascade three-level system.* If $\mathbf{D}_{21},\mathbf{D}_{32} \neq \mathbf{0}$ then $\rho_A^{(1)}$ depends on $\rho_A(t)$ and thus on t. One has

$$\hat{J}\rho_A = \begin{pmatrix} A_{21}\rho_{22} & (\Gamma_{1232}+\Gamma_{3212})\rho_{23} & 0 \\ (\Gamma_{2321}+\Gamma_{2123})\rho_{32} & A_{32}\rho_{33} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{16}$$

The dependence on the state prior to detection, and thus on time, is easy to understand for large level separation. If, for example, the atom were in state $|\,2\rangle$ there would only be a transition to $|\,1\rangle$; if it were in a superposition of $|\,2\rangle$ and $|\,3\rangle$ it could go to $|\,1\rangle$ or $|\,2\rangle$ or a mixture thereof.

III. PHOTON-COUNTING PROCESSES

The approach of repeated gedanken measurements proposed in Refs. [7–9, 11] and the reset matrix of the preceding section will now be used to determine the statistics of broadband photon counting for a general N-level atom. This generalizes the results of Refs. [7–9] for an N-level atom with transitions to a single ground state. Starting at time t_0 with a state $|\,0_{\mathrm{ph}}\rangle\,|\,\psi_A(t_0)\rangle$ the subensemble with no photons until time $t = n\,\Delta t + t_0$ is described by

$$P_0\,U(t,t-\Delta t)P_0\cdots P_0\,U(\Delta t+t_0,t_0)\,|\,0_{ph}\rangle\,|\,\psi_A(t_0)\rangle$$
$$\equiv |\,0_{\mathrm{ph}}\rangle\,|\,\psi_A^{(0)}(t)\rangle\,. \tag{17}$$

The norm squared of this is the probability of finding no photons for the measurements between t_0 and t [31]. For an initial density matrix $|\,0_{\mathrm{ph}}\rangle\rho_A(t_0)\langle 0_{\mathrm{ph}}\,|$ an analogous expression holds if $|\,0_{\mathrm{ph}}\rangle\,|\,\psi_A^{(0)}(t)\rangle$ is replaced by $|\,0_{\mathrm{ph}}\rangle\,\rho_A^{(0)}(t)\,\langle 0_{\mathrm{ph}}\,|$ on the right-hand side of Eq. (17). The no-photon probability is then given by the trace.

It was shown in Ref. [9] by straightforward perturbation theory that the time development of $|\,\psi_A(t)\rangle$, on a *coarse-grained* time scale on which Δt is very small, is given by a "reduced" or "effective" non-Hermitian Hamiltonian H_{red}. Explicitly one obtains

$$H_{\mathrm{red}} = H_A(t) - i\hbar\,\Gamma, \tag{18}$$

$$\Gamma = \sum_{\substack{\alpha,i,j \\ \alpha<i \\ \alpha<j}} \Gamma_{i\alpha\alpha j}\,|\,i\rangle\langle j\,|, \tag{19}$$

where $H_A(t)$ is the atomic Hamiltonian including external driving fields, typically given by lasers. The matrix Γ consists of generalized damping terms, determined from Eq. (10). We define the nonunitary time-development operator U_{red} by $\dot{U}_{\mathrm{red}} = -i/\hbar\,H_{\mathrm{red}}U_{\mathrm{red}}$ and define, as in Eq. (17) of Ref. [9], the superoperator $\hat{S}_{t,t_0}$ by

$$\hat{S}_{t,t_0}\rho_A \equiv U_{\mathrm{red}}(t,t_0)\rho_A\,U_{\mathrm{red}}(t,t_0)^* \tag{20}$$

so that $\rho_A^{(0)}(t) = \hat{S}_{t,t_0}\rho_A(t_0)$ describes the subensemble with no photons found between t_0 and t. Its trace gives, if $\rho_A(t_0)$ is normalized, the probability $P_0(t;\rho_A(t_0))$ of finding no photon between t_0 and t.

The subensemble with (i) no photons found between t_0 and t and (ii) a photon at $t+\Delta t$, is described by $\hat{J}\hat{S}_{t,t_0}\rho_A(t_0)$ since $\hat{J}$ performs the resetting. The size of the subensemble relative to the original ensemble, and thus the probability $w_1(t,t_0;\rho_A(t_0))\Delta t$ to find the first photon after t_0 at $t+\Delta t$, is the trace of this times Δt. From Eqs. (11) and (14) one finds

$$w_1(t,t_0;\rho_A(t_0)) = \mathrm{tr}[\hat{J}\hat{S}_{t,t_0}\rho_A(t_0)]$$
$$= -\frac{d}{dt}\,P_0(t,t_0;\rho_A(t_0)) \tag{21}$$

as expected intuitively since the decrease of the no-photon probability in Δt must equal the probability of finding a photon in Δt.

Analogously, the subensemble for which photons are found at $t_1 + \Delta t, \ldots, t_n + \Delta t$ but none in between and none between $t_n + \Delta t$ and t is described by

$$\hat{S}_{t,t_n} \hat{J} \hat{S}_{t_n,t_{n-1}} \cdots \hat{J} \hat{S}_{t_1,t_0} \rho_A(t_0), \tag{22}$$

where $\hat{S}_{t_i,t_{i-1}+\Delta t}$ has been approximated by $\hat{S}_{t_i,t_{i-1}}$. The size of this subensemble relative to the original one is the trace times $(\Delta t)^n$, and it gives the probability for this event. Going over to the coarse-grained time scale we have that the probability density $w(t_1, \ldots, t_n; [t_0, t])$ for finding exactly n photons at times $t_1 < t_2 \cdots < t_n$ in the interval $[t_0, t]$ is given by

$$w(t_1, \cdots t_n; [t_0, t]) = \mathrm{tr}[\hat{S}_{t,t_n} \hat{J} \hat{S}_{t_n,t_{n-1}} \cdots \hat{J} \hat{S}_{t_1,t_0} \rho_A(t_0)]. \tag{23}$$

Putting $t = t_n$ the operator $\hat{S}_{t,t_n}$ drops out and one obtains the probability density $w(t_1, \ldots, t_n)$ for finding n photons at $t_1 < \cdots < t_n$ and none in between. Equations (20)–(23) are extensions to the general case of Eqs. (14)–(17) in Ref. [9].

If the reset matrix is always the same, as in examples (i) and (ii) of Sec. II, i.e., if after a photon detection an atom is always reset to the same state or density matrix $\rho_A^{(1)}$, then the memory is lost after each detection and one has a renewal process [32]. Then one has from Eq. (12) $\hat{J}\rho_A = \rho_A^{(1)} \mathrm{tr}(\hat{J}\rho_A)$ with the same $\rho_A^{(1)}$ for any ρ_A, and Eq. (23) factorizes into single-photon probabilities,

$$\begin{aligned} w(t_1, \ldots, t_n; [t_0, t]) &= \mathrm{tr}\left(\hat{S}_{t,t_n} \rho_A^{(1)}\right) \mathrm{tr}\left(\hat{J} \hat{S}_{t_n,t_{n-1}} \rho_A^{(1)}\right) \cdots \mathrm{tr}\left(\hat{J} \hat{S}_{t_2,t_1} \rho_A^{(1)}\right) \mathrm{tr}\left[\hat{J} \hat{S}_{t_1,t_0} \rho_A(t_0)\right] \\ &= P_0(t, t_n; \rho_A^{(1)})\, w_1(t_n, t_{n-1}; \rho_A^{(1)}) \cdots w_1(t_1, t_0; \rho_A(t_0)). \end{aligned} \tag{24}$$

In this case the analysis becomes particularly easy and many quantities can be calculated by Laplace transform as in Ref. [8] for the case of a single ground state. The form of the correlation functions obtained by Agarwal and Jha [22] for a Λ system is consistent with Eq. (24).

The above probability densities determine a classical stochastic process whose sample paths are given by the photon-detection times of a single radiating atom. Without external pumping these paths terminate. Ergodicity allows one to replace time averages by ensemble averages which in many cases can be computed analytically. In some situations it may be advantageous to simulate the paths. If $P_0(t)$ can be calculated, analytically or numerically, one can proceed by simulating the time of the first photon, reset the atom to the state determined by reset matrix, simulate the next photon, and so on. In more complex situations one may have to simulate individual measurements, with the size of Δt adapted to the problem.

Between photon detections an atom may be described by the reduced density matrix

$$U_{\mathrm{red}}(t, t_i)\, \rho_A^{(1)}(t_i) [U_{\mathrm{red}}(t, t_i)]^*, \tag{25}$$

where t_i is the time of a photon detection and $\rho_A^{(1)}(t_i)$ is the corresponding reset matrix.

Connection with continuous measurements. In order to describe continuous measurements Davies and Srinivas [17] have extended the axiomatics of quantum mechanics by postulates for "homogeneous quantum-counting processes." In particular, their postulates imply the existence of two superoperators J and S_t which map trace class operators to trace class operators and satisfy certain properties. For an individual system of an ensemble described by a density matrix ρ their meaning is as follows. $\mathrm{tr}(S_t \rho)$ is the probability of finding no counting event in $[0, t]$, and the probability density $w(t_1, \ldots, t_n; [0, t])$ for finding a counting event exactly at the times $t_1, \ldots, t_n$ in $[0, t]$ is given by

$$w(t_1, \ldots, t_n; [0, t]) = \mathrm{tr}\left(S_{t-t_n} J S_{t_n - t_{n-1}} J \cdots J S_{t_2 - t_1} J S_{t_1} \rho\right). \tag{26}$$

For a particular system J and S_t have to be determined phenomenologically or by intuition.

A comparison with Eq. (23) shows that just this structure has been obtained by our approach of repeated measurements and reductions and that the unknown superoperators J and S_t have to coincide with our $\hat{J}$ and $\hat{S}_t$ constructed above. In contrast to the axiomatic theory of Davies and Srinivas, the superoperators are explicitly known in our case. In this way we have derived their continuous-measurement theory for the general N-level atom within the framework of standard quantum mechanics and the extension of the reduction rule of von Neumann and Lüders to demolition measurements, except that we have introduced a temporal coarse graining. This generalizes the analogous result of Refs. [8, 7] for atoms with transitions to a single ground state.

Connection with optical Bloch equations. For the case of a single ground state it was pointed out in Refs. [7, 8] that the reduced atomic density matrix $\rho_A(t)$ for the *complete* ensemble, including the measurements and associated reductions, satisfy the optical Bloch equations if an atom is reset to the ground state after each photon detection. The same result was also derived independently in Ref. [11]. This will now be shown to hold also for the general case with the reset matrix $\rho_A^{(1)}$ of Eq. (12). This is proved just as in Ref. [8]. Let $I(t)$ be the—yet to be determined—probability density for finding a photon at time t for the complete ensemble. This ensemble decomposes into a subensemble with no photons until t and

subensembles with the last photon detection, before t, at time t'. Thus, with the reset matrix $\rho_A^{(1)}$ of Eq. (12),

$$\rho_A(t) = \hat{S}_{t,t_0}\rho_A(t_0) + \int_{t_0}^{t} dt' I(t')\hat{S}_{t,t'}\rho_A^{(1)}(t') \ .$$

Differentiation yields, by Eq. (20),

$$\dot{\rho}_A = -\frac{i}{\hbar}\left[H_{\rm red}\rho_A - \rho_A H_{\rm red}^*\right] + I(t)\rho_A^{(1)}(t) \ .$$

Since $\mathrm{tr}\rho_A = \mathrm{tr}\rho_A^{(1)} \equiv 1$ and $\mathrm{tr}\dot{\rho}_A = 0$ one obtains with Eq. (18)

$$I(t) = \mathrm{tr}\left\{(\Gamma + \Gamma^*)\rho_A(t)\right\}. \tag{27}$$

From Eqs. (11) and (27) one finds by an elementary calculation that

$$\mathrm{tr}\hat{J}\rho_A = \mathrm{tr}\left\{(\Gamma + \Gamma^*)\rho_A\right\}$$

[33]. Thus one has

$$\dot{\rho}_A = -\frac{i}{\hbar}[H_{\rm red}\rho_A - \rho_A\, H_{\rm red}^*] + \hat{J}\rho_A. \tag{28}$$

This coincides with the usual optical Bloch equations [34]. Conversely, starting from the latter one could read off $H_{\rm red}$ or the reset superoperator $\hat{J}$ through Eq. (28). If the levels are far apart the off-diagonal terms in the reset matrix will lead to rapidly oscillating terms in the interaction picture and thus can be neglected for most purposes. This would amount to using $\rho_A^{(1)'}$ of Eq. (14) as a reset matrix.

The photon probability density $I(t)$ in Eq. (27) depends on the initial condition, i.e., $I(t) = I(t;\rho_A(t_0))$. For a free gas of atoms, represented as an ensemble, the initial state would usually be the ground state or a mixture with Boltzmann weights. On the other hand, the correlation function $g^{(2)}(\tau)$ for a single atom in broadband detection involves the reset matrix $\rho_A^{(1)}$ since $g^{(2)}(\tau)$ is proportional to the probability density for finding a photon at time τ provided a photon was found at time 0. If the reset matrix is always the same, then $\rho_A(0) = \rho_A^{(1)}$ in this case and

$$g^{(2)}(\tau) = I(\tau;\rho_A^{(1)}) \ / \ \bar{I}. \tag{29}$$

In cases where the Bloch equations are too complicated to solve one may use simulations of the above-mentioned sample paths to obtain a numerical solution or directly relevant properties. This aspect was stressed in Ref. [11].

The relation between the Bloch equation for an ensemble of atoms and the description of the history of a single atom in terms of a random path of reduced density matrices, as in Eq. (25), is very similar to that between the Langevin equation and its associated Fokker-Planck equation.

IV. DISCUSSION

For a single atom, the photon-detection times form a sample path of the classical stochastic process of the preceding section, a process which is governed by quantum mechanics. On a coarse-grained time scale the process can be regarded as practically continuous. Without external pumping these paths terminate, and one can make no definite statements about an individual system. With external pumping, however, this is possible due to ergodic properties of the process. Ergodicity allows one to replace time averages over a single sample path by ensemble averages. For a renewal process this is more or less evident. Indeed, by resetting the atom always to the same state after a photon detection—although at stochastic times—one may imagine to have created an ensemble by repetition. In the general case, with a time-dependent reset matrix, it is not so obvious that one can speak about an ensemble created by repetition. One uses ergodicity to explain that although the statistical interpretation of quantum mechanics deals with ensembles, one can make certain predictions in the form of time averages for a single atom.

As an alternative to an analytic treatment a simulation of sample paths suggests itself, as in Refs. [18, 11]. For situations with not too many degrees of freedom it is in general not necessary to simulate the outcome of each gedanken measurement, at times Δt apart. Instead it is usually sufficient to simulate just the detection times using Eq. (21). This reduces the computing considerably. With a general reset matrix one may, however, have to work with density matrices, instead of wave functions as in Ref. [7, 9, 18, 11]. But the time development is still given by the effective Hamiltonian $H_{\rm red}$, an $N \times N$ matrix, and is therefore computationally much easier to handle than that in the corresponding optical Bloch equation which involves an $(N^2-1)\times(N^2-1)$ matrix. For a large number of degrees of freedom it may be numerically advantageous to simulate the outcome of many gedanken measurements separately, in particular if atomic momenta in a complicated external field are included [35], and this may be an efficient way to solve the Bloch equation numerically by simulation.

In order to describe continuous measurements Davies and Srinivas [17] have extended the axiomatics of quantum mechanics by postulates for "homogeneous quantum-counting processes." Wilser [8] and Hegerfeldt and Wilser [9] have used the above approach of repeated gedanken measurements, with ensuing reductions, to derive superoperators $\hat{S}_t$ and $\hat{J}$ that satisfy the postulates of Davies and Srinivas for the special case that after a photon detection the atom is reset to the ground state. Equations (12), (19), and (23) carry this over to the general case. It should be noted, however, that here we deal with a coarse-grained time. We thus arrive at the conclusion that the axiomatic continuous-measurement theory of Davies and Srinivas can be derived from standard quantum mechanics, taken together with the extension of the von Neumann–Lüders reduction rule to demolition measurements, if one relaxes the "continuous" and goes over to a coarse-grained time scale.

ACKNOWLEDGMENTS

I would like to thank R. Reibold for stimulating discussions. This research was supported in part by the Deutsche Forschungsgemeinschaft.

[1] W. Paul, O. Osberghaus, and E. Fischer, Forschungsber. Wirtsch. Verkehrsminist. Nordrhein-Westfalen **415**, 1 (1958); W. Neuhauser, M. Hohenstatt, P.E. Toschek, and H.G. Dehmelt, Phys. Rev. Lett. **41**, 233 (1978); D.J. Wineland, R.E. Drullinger, and F.L. Walls, *ibid.* **40**, 1639 (1978); M. Degenais and L. Mandel, Phys. Rev. A **18**, 2217 (1978); D.J. Wineland and W. Itano, *ibid.* **20**, 1521 (1979).

[2] H.G. Dehmelt, Bull. Am. Phys. Soc. **20**, 60 (1975).

[3] Th. Sauter, R. Blatt, W. Neuhauser, and P.E. Toschek, Opt. Commun. **60**, 287 (1986); W. Nagourney, J. Sandberg, and H. Dehmelt, Phys. Rev. Lett. **56**, 2797 (1986); J.C. Bergquist, W.M. Itano, R.G. Hulet, and D.J. Wineland, Phys. Scr. **T22**, 79 (1988).

[4] R.J. Cook and H.J. Kimble, Phys. Rev. Lett. **54**, 1023 (1985).

[5] C. Cohen-Tannoudji and J. Dalibard, Europhys. Lett. **1**, 441 (1986).

[6] A. Schenzle, R.G. DeVoe, and R.G. Brewer, Phys. Rev. A **33**, 2127 (1986); J. Javanainen, *ibid.* **33**, 2121 (1986); D.T. Pegg, R. Loudon, and P.L. Knight, *ibid.* **33**, 4085 (1986), G. Nienhuis, *ibid.* **35**, 4639 (1987); P. Zoller, M. Marte, and D.J. Walls, *ibid.* **35**, 198 (1987); M. Porrati and S. Putterman, *ibid.* **36**, 925 (1987); S. Reynaud, J. Dalibard, and C. Cohen-Tannoudji, IEEE J. Quantum Electron. **24**, 1395 (1988).

[7] G.C. Hegerfeldt and T.S. Wilser (unpublished).

[8] T. Wilser, doctoral dissertation, Göttingen, 1991.

[9] G.C. Hegerfeldt and T.S. Wilser, in *Proceedings of the Second International Wigner Symposium, Goslar, 1991*, edited by H.D. Doebner, W. Scherer, and F. Schroeck (World Scientific, Singapore, 1992).

[10] G. Lüders, Ann. Phys. (Leipzig) **8**, 323 (1951) generalizes von Neumann's prescription for observables with nondegenerate eigenvalues to the degenerate case, but differs from von Neumann's proposal for the latter [cf. J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955) Chap. V.1].

[11] J. Dalibard, Y. Castin, and K. Mølmer, Phys. Rev. Lett. **68**, 580 (1992).

[12] H.J. Carmichael (unpublished).

[13] R. Dum, P. Zoller, and H. Ritsch, Phys. Rev. A **45**, 4879 (1992).

[14] R.J. Glauber, Phys. Rev. **130**, 2529 (1963); *ibid.* **131**, 2766 (1963); in *Les Houches Lectures 1964*, edited by C. DeWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach, New York, 1965); P.L. Kelley and W.H. Kleiner, Phys. Rev. **136**, A316 (1964).

[15] M. Porrati and S. Putterman, Phys. Rev. A **39**, 3010 (1989).

[16] R. Reibold, Physica (to be published); J. Phys. A (to be published).

[17] E.B. Davies, Commun. Math. Phys. **15**, 277 (1969); **19**, 83 (1970); **22**, 5 (1971); M.D. Srinivas, J. Math. Phys. **18**, 2138 (1977); M.D. Srinivas and E.B. Davies, Opt. Acta **28**, 981 (1981); **29**, 235 (1982).

[18] G.C. Hegerfeldt and M.B. Plenio, Phys. Rev. A **46**, 373 (1992).

[19] D.A. Cardimona, M.G. Raymer, and C.R. Stroud Jr., J. Phys. B **15**, 55 (1982).

[20] G. Orriols, Nuovo Cimento B **53**, 1 (1979); E. Arimondo and G. Orriols, Lett. Nuovo Cimento **17**, 333 (1976).

[21] H.J. Carmichael and D.F. Walls, J. Phys. B **9**, L43 (1975); H. Kimble and L. Mandel, Phys. Rev. A **13**, 2123 (1976).

[22] For the Λ system the photon correlation functions were analyzed by G.S. Agarwal and S.S. Jha, Z. Phys. B **35**, 391 (1979).

[23] For a type of nondemolition photon measurement cf. M. Brune, S. Haroche, J.M. Raimond, and N. Zagury, Phys. Rev. Lett. **65**, 976 (1990); H. Paul, Quantum Opt. **3**, 169 (1991).

[24] This argument neglects a part of the radiation field in the vicinity of the atom.

[25] H. Paul, Ann. Phys. (Leipzig) **14**, 166 (1964) uses a similar point of view to determine the reset matrix for a nuclear cascade involving two γ's.

[26] Photon frequencies are not observed, and therefore this detection does not imply restrictions on Δt. If frequencies were observed also, the time-frequency uncertainty relation would apply.

[27] R. Loudon, *The Quantum Theory of Light*, 2nd ed. (Clarendon, London, 1983).

[28] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Processus d'Interaction Entre Photons et Atomes* (InterEditions, Paris, 1988).

[29] For the determination of the reset matrix it has been essential that the radiation field is known to have been in the vacuum state a time Δt before the photon detection. This is due to the gedanken measurements. Without this knowledge it would be much more complicated to determine the state of an atom after the detection because then $U(t,0)$ would enter for larger times. Moreover, since the photon detection times are random a single gedanken measurement before a photon detection would not do.

[30] G.C. Hegerfeldt and M.B. Plenio (unpublished).

[31] Equation (17) should be compared with the expression $P_0 U(t,t_0) \mid 0_{\rm ph}\rangle \mid \psi_A(t_0)\rangle$ which was calculated in Ref. [15] and whose norm squared gives the probability for a single measurement to find no photon at t, with no measurements, and no knowledge thereof, in between. In principle both expressions are different, e.g., the former decays exponentially while the latter can do this only approximately in some finite interval, not truly asymptotically. This is due to positivity of the energy and ensuing analyticity. As regards the calculation of $P_0(t)$, for the external fields considered in Ref. [15], both results are compatible within the approximations employed. The underlying reason for this agreement has been investigated and clarified in a very general context by means of the projector formalism in Ref. [16]. However, to find a reset matrix within the approach of Ref. [15] and without gedanken measurements would involve the difficulties mentioned in Ref. [29].

[32] Cf., e.g., D.R. Cox, *Renewal Theory* (Methuen, London, 1962).

[33] One can also show directly from Eq. (6) that $\mathrm{tr}\hat{J}\rho_A = \mathrm{tr}(H_{\rm red}\rho_A - \rho_A H_{\rm red}^*)$.

[34] Cf. Ref. [15] and P.W. Milonni, Phys. Rep. **25C**, 1 (1976).

[35] P.D. Lett (private communication).