Squeezed states with thermal noise. II. Damping and photon counting

Paulina Marian
Laboratory of Physics, Department of Chemistry, University of Bucharest, Boulevard Carol I 13, R-70346 Bucharest, Romania

Tudor A. Marian
Department of Physics, University of Bucharest, P.O. Box MG-11, R-76900 Bucharest-Măgurele, Romania
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We consider a single-mode radiation field initially in a displaced squeezed thermal state. The weak interaction of such a field with a heat bath of arbitrary temperature is shown to preserve the Gaussian form of the characteristic function. Accordingly, the study of the time development of the density operator reduces to our previous description [P. Marian and T. A. Marian, preceding paper, Phys. Rev. A **47**, 4474 (1993)] of the initial quantum state. As examples, photon statistics and squeezing properties of the damped field are analyzed. Based on the close relation between field dissipation and photon detection, we derive simple analytic formulas for the counting distribution and its factorial moments. Nonclassical features of a displaced squeezed thermal state, such as oscillations of the photon-number distribution, survive in the counting process, provided that the quantum efficiency of the detector is high enough.

I. INTRODUCTION

The effect of damping on a squeezed state has been largely investigated in recent years [1-4]. In the case of a harmonic oscillator *weakly* coupled to a heat bath, Milburn and Walls [1] have derived and solved the Fokker-Planck equation for the Q function of an initial displaced squeezed vacuum state (DSVS). Then they have discussed the influence of damping on squeezing. The photon statistics of damped squeezed light was investigated by Peřinova, Křepelka, and Peřina [2] also using a Fokker-Planck-equation approach. A more general treatment of damping based on a functional-integral method was given by Schramm and Grabert [3]. They studied the time evolution of a DSVS when the field is coupled to a dissipative environment of given temperature, for arbitrary strength of the damping. In the case of a weak coupling to a zero-temperature heat bath, Milburn and Walls [4] analyzed the attenuation of the oscillations in the photon-number distribution of a DSVS.

On the other hand, the theory of photodetection is a central problem in quantum optics. The earlier full quantum-mechanical derivations of the counting formula for a single-mode free field [5,6] are valid only to the first order of perturbation theory. The attenuation of the field by the detection process was considered first by Mollow [7] and, independently, by Scully and Lamb [8]. Their results have been confirmed in further developments of photodetection theory [9–11], which yielded the same functional equation for the photon-counting distribution. Essentially, in Refs. [7] and [8] a master-equation-type technique is used to obtain the time evolution of the field-detector system. The counting probability obeys the same equation as that describing the photon-number distribution of a field mode weakly coupled to a zero-temperature reservoir. This is by no means surprising because detection with nonunit quantum efficiency and damping of the field amplitude are related phenomena.

Consequently, the present paper is devoted to both damping and photon-counting statistics for the broad class of quantum states investigated in our preceding paper [12], hereafter denoted as I. The weak damping of a single-mode radiation field initially in a displaced squeezed thermal state (DSTS) is studied in Secs. II–IV. In Sec. II we derive the characteristic function (CF) of the damped field as a solution of the partial differential equation describing its time evolution for any temperature of the reservoir. The time development of the correlation functions is considered in Sec. III. We also examine various representations of the density operator, in the Schrödinger and interaction pictures, as well as the squeezing effect. Section IV focuses on the mode dissipation in the presence of a heat bath at zero temperature. Starting from a formal solution of the master equation in the Fock basis, we develop here a second method of finding the corresponding density matrix, besides the more general one presented in Secs. II and III. This method is connected with a direct derivation, in Sec. V, of the photon-counting distribution and its factorial moments. Section VI summarizes the principal results. In Appendix A we solve formally the master equation in the Fock representation for the case of a zero-temperature heat bath, by employing the Laplace-transform method. Appendix B deals with the summation of a series involving Laguerre polynomials.

II. CHARACTERISTIC FUNCTION OF THE DAMPED FIELD

A standard treatment of the weak damping in the Markoff approximation leads to the following master equation for the reduced density operator of a single-mode field, in the interaction picture [13]:

$$\frac{\partial\rho_I}{\partial t}=\frac{\gamma}{2}(2a_I\rho_I a_I^\dagger-a_I^\dagger a_I\rho_I-\rho_I a_I^\dagger a_I)+\gamma\bar{n}_b(a_I^\dagger\rho_I a_I+a_I\rho_I a_I^\dagger-a_I^\dagger a_I\rho_I-\rho_I a_I a_I^\dagger)\ . \quad (2.1)$$

In Eq. (2.1), γ is the damping constant and $\bar{n}_b$ is the mean photon number of the thermal state (TS) at the temperature T_b of the heat bath,

$$\bar{n}_b=\left[\exp\left[\frac{\hbar\omega}{k_B T_b}\right]-1\right]^{-1}, \quad (2.2)$$

while a_I and $a_I^\dagger$ are the photon annihilation and creation operators, respectively, in the interaction picture,

$$a_I(t)=e^{-i\omega t}a\ ,\quad a_I^\dagger(t)=e^{i\omega t}a^\dagger\ . \quad (2.3)$$

The nonvanishing terms for $\bar{n}_b=0$ describe the energy loss from the mode to the reservoir, whereas the terms proportional to $\bar{n}_b$ account for the transfer of the fluctuation energy from the reservoir into the mode.

Our aim is to evaluate the normally ordered CF

$$\chi_N(\lambda,t)\equiv\mathrm{Tr}[\rho_I(t)e^{\lambda a_I^\dagger(t)}e^{-\lambda^* a_I(t)}]\ , \quad (2.4)$$

which is independent of the picture. Starting from Eqs. (2.1) and (2.3), and making use of the commutation relations [14]

$$[a,f(a,a^\dagger)]=\frac{\partial f}{\partial a^\dagger} \quad (2.5a)$$

and

$$[a^\dagger,f(a,a^\dagger)]=-\frac{\partial f}{\partial a}\ , \quad (2.5b)$$

where $f(a,a^\dagger)$ is an arbitrary analytic function, one readily finds the linear first-order partial differential equation

$$\frac{\partial\chi_N}{\partial t}=-\gamma\bar{n}_b|\lambda|^2\chi_N-\left[\frac{\gamma}{2}-i\omega\right]\lambda\frac{\partial\chi_N}{\partial\lambda}-\left[\frac{\gamma}{2}+i\omega\right]\lambda^*\frac{\partial\chi_N}{\partial\lambda^*}\ . \quad (2.6)$$

We require a CF (2.4) subjected to the initial condition

$$\chi_N(\lambda,0)=\exp[-A|\lambda|^2-\tfrac{1}{2}B^*\lambda^2-\tfrac{1}{2}B(\lambda^*)^2+C^*\lambda-C\lambda^*]\ , \quad (2.7)$$

which describes a DSTS, as shown by Eqs. (2.8)–(2.11) of I. Integration of the subsidiary equations affords the characteristic curves [15,16]. In turn, they provide a unique solution that fulfills the initial condition (2.7),

$$\chi_N(\lambda,t)=\exp[-A(t)|\lambda|^2-\tfrac{1}{2}B^*(t)\lambda^2-\tfrac{1}{2}B(t)(\lambda^*)^2+C^*(t)\lambda-C(t)\lambda^*]\ , \quad (2.8)$$

with the time-depending coefficients

$$A(t)=Ae^{-\gamma t}+\bar{n}_b(1-e^{-\gamma t})\ , \quad (2.9a)$$

$$B(t)=Be^{-(\gamma+2i\omega)t}\ , \quad (2.9b)$$

$$C(t)=Ce^{-[(\gamma/2)+i\omega]t}\ . \quad (2.9c)$$

It is remarkable that the Gaussian form of the CF (2.7) is not altered by the damping (2.1) of the radiation field. According to Eqs. (2.9), for $t\to\infty$, the CF (2.8) takes the form

$$\chi_N(\lambda,t)\to\exp(-\bar{n}_b|\lambda|^2)\ , \quad (2.10)$$

describing the TS with the mean photon number (2.2).

However, it is useful to define a hybrid normally ordered CF:

$$(\chi_I)_N(\lambda,t)\equiv\mathrm{Tr}[\rho_I(t)e^{\lambda a^\dagger}e^{-\lambda^* a}]\ , \quad (2.11)$$

including the density operator in the interaction picture $\rho_I(t)$, just as the CF (2.4) is expressed in terms of the Schrödinger density operator $\rho(t)$:

$$\chi_N(\lambda,t)=\mathrm{Tr}[\rho(t)e^{\lambda a^\dagger}e^{-\lambda^* a}]\ . \quad (2.12)$$

We recall the relation

$$\rho_I(t)=\exp(i\omega t a^\dagger a)\rho(t)\exp(-i\omega t a^\dagger a)\ . \quad (2.13)$$

Substitution of Eq. (2.13) into Eq. (2.11), followed by use of Eqs. (2.3) and (2.8), gives the exponential function

$$(\chi_I)_N(\lambda,t)=\exp[-A_I(t)|\lambda|^2-\tfrac{1}{2}B_I^*(t)\lambda^2-\tfrac{1}{2}B_I(t)(\lambda^*)^2+C_I^*(t)\lambda-C_I(t)\lambda^*]\ , \quad (2.14)$$

with the nonoscillating coefficients

$$A_I(t)=A(t)\ , \quad (2.15a)$$

$$B_I(t)=e^{2i\omega t}B(t)\ , \quad (2.15b)$$

$$C_I(t)=e^{i\omega t}C(t)\ . \quad (2.15c)$$

The outstanding form (2.8) of the CF for a damped DSTS enables us to write, in analogy with Eq. (2.20) of I,

$$\mathrm{Tr}\{[\rho(t)]^2\}=\tfrac{1}{2}\{[A(t)+\tfrac{1}{2}]^2-|B(t)|^2\}^{-1/2}\ . \quad (2.16)$$

Insertion of Eqs. (2.9) and (2.10) of I into Eqs. (2.9a) and (2.9b), respectively, yields an explicit expression of the degree of purity of the transient state:

$$\mathrm{Tr}\{[\rho(t)]^2\}=\tfrac{1}{2}\{[\bar{n}e^{-\gamma t}+\bar{n}_b(1-e^{-\gamma t})+\tfrac{1}{2}]^2+(2\bar{n}+1)(2\bar{n}_b+1)(\sinh r)^2\times e^{-\gamma t}(1-e^{-\gamma t})\}^{-1/2}\ . \quad (2.17)$$

In contrast with the initial situation, specified by Eq. (2.21) of I, the degree of purity (2.17) depends on the squeeze parameter r. We notice a symmetry property of the function (2.17): it is not modified by the simultaneous interchanges

$$\bar{n}\leftrightarrow\bar{n}_b \quad (2.18a)$$

and

$$e^{-\gamma t}\leftrightarrow 1-e^{-\gamma t}\ . \quad (2.18b)$$

The transformation (2.18b) has the fixed point

$$t_1=\frac{1}{\gamma}\ln 2 \tag{2.19}$$

for which

$$\mathrm{Tr}\{[\rho(t_1)]^2\} =[(\bar{n}-\bar{n}_b)^2+(2\bar{n}+1)(2\bar{n}_b+1)(\cosh r)^2]^{-1/2} . \tag{2.20}$$

Note that the degree of purity (2.17) evolves from the initial value $(2\bar{n}+1)^{-1}$, at $t=0$, to the limit $(2n_b+1)^{-1}$ when $t\rightarrow\infty$. If the squeeze parameter r satisfies the condition

$$(\sinh r)^2>\max\left\{\frac{\bar{n}-\bar{n}_b}{2\bar{n}+1},\frac{\bar{n}_b-\bar{n}}{2\bar{n}_b+1}\right\}, \tag{2.21}$$

then the degree of purity of a damped DSTS decreases first to the absolute minimum

$$\mathrm{Tr}\{[\rho(t_m)]^2\}=\frac{[(2\bar{n}+1)(2\bar{n}_b+1)(\sinh r)^2-(\bar{n}-\bar{n}_b)^2]^{1/2}}{(2\bar{n}+1)(2\bar{n}_b+1)\sinh r\cosh r} \tag{2.22}$$

achieved at the time

$$t_m=\frac{1}{\gamma}\ln\left\{\frac{2[(2\bar{n}+1)(2\bar{n}_b+1)(\sinh r)^2-(\bar{n}-\bar{n}_b)^2]}{(2\bar{n}_b+1)[\bar{n}-\bar{n}_b+(2\bar{n}+1)(\sinh r)^2]}\right\} \tag{2.23}$$

and then increases. Otherwise, it varies monotonically between the values imposed by the heat baths with temperatures T and T_b.

III. PHOTON STATISTICS, DENSITY OPERATOR, AND SQUEEZING

Our starting point in investigating the photon-number statistics for a damped DSTS consists of the CF's (2.8) and (2.14). We shall take advantage of the preservation of their initial Gaussian form (2.7), with the time development concentrated in the coefficients (2.9) and (2.15), respectively. Accordingly, all the results, both in the Schrödinger and interaction pictures, parallel the initial ones, at $t=0$, already obtained in Secs. III–VI of I. In the corresponding formulas, one should merely replace the initial coefficients A, B, and C, introduced in Eqs. (2.9)–(2.11) of I, by their current values (2.9) or (2.15).

For instance, any correlation function of the damped field is analogous to its initial expression (3.2) of I:

$$\langle (a^\dagger)^l a^m\rangle|_t=\sum_{k=0}^{\min\{l,m\}} k!\binom{l}{k}\binom{m}{k}[A(t)]^k[\tfrac{1}{2}B^*(t)]^{(l-k)/2} \times[\tfrac{1}{2}B(t)]^{(m-k)/2}H_{l-k}([2B^*(t)]^{-1/2}C^*(t))H_{m-k}([2B(t)]^{-1/2}C(t)) . \tag{3.1}$$

From Eqs. (2.9) it follows that the correlation function (3.1) has no periodic dependence on time except for the factor $\exp[i(l-m)\omega t]$. Obviously, one can write special correlation functions, as in Eqs. (3.5), (B9), and (3.11) from I, or correlation functions for particular damped states, as displaced thermal states (DTS's), Eq. (3.12) of I, and squeezed thermal states (STS's), Eqs. (3.13) of I.

From Eq. (2.8), by analogy with Eqs. (4.3)–(4.7) and (4.23) of I, there follows the R function in the Schrödinger picture,

$$R(\beta^*,\beta',t)=\pi Q(0,t)\exp[\tilde{A}(t)\beta^*\beta'-\tfrac{1}{2}\tilde{B}(t)(\beta^*)^2-\tfrac{1}{2}\tilde{B}^*(t)(\beta')^2+\tilde{C}(t)\beta^*+\tilde{C}^*(t)\beta'] , \tag{3.2}$$

$$\tilde{A}(t)=\frac{A(t)[A(t)+1]-|B(t)|^2}{[A(t)+1]^2-|B(t)|^2} , \tag{3.3a}$$

$$\tilde{B}(t)=\frac{B(t)}{[A(t)+1]^2-|B(t)|^2} , \tag{3.3b}$$

$$\tilde{C}(t)=\frac{[A(t)+1]C(t)+B(t)C^*(t)}{[A(t)+1]^2-|B(t)|^2} , \tag{3.3c}$$

and

$$\pi Q(0,t)=\{[A(t)+1]^2-|B(t)|^2\}^{-1/2} \times\exp\left[-\frac{[A(t)+1]|C(t)|^2+\frac{1}{2}\{B(t)[C^*(t)]^2+B^*(t)[C(t)]^2\}}{[A(t)+1]^2-|B(t)|^2}\right] . \tag{3.4}$$

In a similar way, using the CF (2.14), we find by analogy with the above-mentioned equations of I, Glauber's R function in the interaction picture,

$$R_I(\beta^*,\beta',t)=\pi Q(0,t)\exp[\tilde{A}_I(t)\beta^*\beta'-\tfrac{1}{2}\tilde{B}_I(t)(\beta^*)^2 -\tfrac{1}{2}\tilde{B}_I^*(t)(\beta')^2+\tilde{C}_I(t)\beta^* +\tilde{C}_I^*(t)\beta'] . \quad (3.5)$$

The coefficients in Eq. (3.5) are connected to the functions (3.3) by relations identical with Eqs. (2.15):

$$\tilde{A}_I(t)=\tilde{A}(t) , \quad (3.6a)$$

$$\tilde{B}_I(t)=e^{2i\omega t}\tilde{B}(t) , \quad (3.6b)$$

$$\tilde{C}_I(t)=e^{i\omega t}\tilde{C}(t) . \quad (3.6c)$$

Therefore, they no longer contain the oscillatory factors displayed by the coefficients (3.3) in the Schrödinger picture.

The quasiprobability functions $W(\beta,t,s)$ $(s=-1,0,1)$ have the expression (4.13) of I, where the coefficients A, B, and C must be replaced by the functions (2.9) in the Schrödinger picture, or (2.15) in the interaction picture. Exactly like the CF (2.11), all the CF's $\chi_I(\lambda,t,s)$ and the associated quasiprobability densities $(1/\pi)W_I(\beta,t,s)$ are hybrid quantities. However, they are determined by the density operator $\rho_I(t)$ in the interaction picture in the same way as the functions $\chi(\lambda,t,s)$ and $(1/\pi)W(\beta,t,s)$ $(s=-1,0,1)$ are determined by the Schrödinger density operator $\rho(t)$.

According to a condition analogous to Eq. (4.14) of I, written for $s=1$, Glauber's P representation for a damped DSTS exists, provided that

$$A(t)>|B(t)| . \quad (3.7)$$

It is now adequate to recall the threshold (3.17) of I,

$$r_s\equiv\tfrac{1}{2}\ln(2\bar{n}+1) . \quad (3.8)$$

The condition (3.7) is fulfilled either if

$$r\le r_s \quad (3.9)$$

or if

$$r>r_s , \quad (3.10a)$$

but only at times

$$t>t_s \quad (3.10b)$$

subsequent to the moment

$$t_s\equiv\frac{1}{\gamma}\ln\left[1+\frac{1-\exp[2(r_s-r)]}{2\bar{n}_b}\right] . \quad (3.11)$$

Notice that the critical time t_s increases with the squeeze parameter r and decreases as the reservoir temperatures T and/or T_b increase.

In view of Eq. (5.2) of I, the density-matrix elements in the Schrödinger picture and Fock representation are

$$\rho_{lm}(t)=\frac{\pi Q(0,t)}{(l!m!)^{1/2}}\sum_{k=0}^{\min\{l,m\}}k!\binom{l}{k}\binom{m}{k}[\tilde{A}(t)]^k[\tfrac{1}{2}\tilde{B}(t)]^{(l-k)/2} \times[\tfrac{1}{2}\tilde{B}^*(t)]^{(m-k)/2}H_{l-k}([2\tilde{B}(t)]^{-1/2}\tilde{C}(t))H_{m-k}([2\tilde{B}^*(t)]^{-1/2}C^*(t)) . \quad (3.12)$$

Due to Eq. (3.5), in the interaction picture a similar formula holds and affords the relation

$$(\rho_I)_{lm}(t)=e^{i(l-m)\omega t}\rho_{lm}(t) , \quad (3.13)$$

in agreement with Eq. (2.13). Accordingly, the photon-number distribution $\rho_{ll}(t)$ and its generating function

$$G(s,t)=\sum_{l=0}^{\infty}\rho_{ll}(t)s^l \quad (|s|\le 1) , \quad (3.14)$$

written explicitly by analogy with Eqs. (5.10) or (5.11) of I, are picture independent. We mention that our results concerning the lth-order correlation function $\langle(a^\dagger)^l a^l\rangle|_t$ and the l-photon probability $\rho_{ll}(t)$, when set in the equivalent forms (B9) and (5.6) of I, coincide, in the special case of a DSVS, with the previous ones reported by Peřinova, Křepelka, and Peřina [17].

Since $\bar{n}=0$ in the case of a damped DSVS or squeezed coherent state (SCS), there follows $A=0$, but $\tilde{A}(t)\neq 0$ for $t>0$. Consequently, the initial expression, at $t=0$, of the density-matrix elements for a SCS, given by Eq. (5.15) of I, is no longer valid for $t>0$, when the general formula (3.12) holds, with the coefficients (3.3) written explicitly with $\bar{n}=0$. In contrast, the coefficients $\tilde{B}(t)$ vanishes at the same time with B, and $\tilde{C}(t)$ vanishes simultaneously with C. It follows that the initial functional form of the matrix elements $\rho_{lm}(t)$, specific for a DTS, Eq. (5.17) of I, or for a STS, Eqs. (5.18) of I, subsists. The time development of the density matrix reduces to that of the corresponding coefficients (3.3) or (3.6).

As stated by Eqs. (2.3), the quadrature operators in the interaction picture,

$$\hat{X}_{1I}(t)=\tfrac{1}{2}[a_I(t)+a_I^\dagger(t)] , \qquad \hat{X}_{2I}(t)=\frac{1}{2i}[a_I(t)-a_I^\dagger(t)] , \quad (3.15)$$

arise from the corresponding Schrödinger operators by means of a rotation,

$$\hat{X}_{1I}(t)=\cos(\omega t)\hat{X}_1+\sin(\omega t)\hat{X}_2 , \qquad \hat{X}_{2I}(t)=-\sin(\omega t)\hat{X}_1+\cos(\omega t)\hat{X}_2 . \quad (3.16)$$

By analogy with Eqs. (6.6) and (6.7) of I, we write the distribution functions of the quadratures in the interaction picture:

$$[P(X_{1I},t)=\left[\frac{2\hbar}{M\omega}\right]^{1/2}\langle q|\rho_I(t)|q\rangle$$
$$=\{\pi[\tfrac{1}{2}+A_I(t)-\mathrm{Re}B_I(t)]\}^{-1/2}\times\exp\left\{-\frac{[X_{1I}-\mathrm{Re}C_I(t)]^2}{\tfrac{1}{2}+A_I(t)-\mathrm{Re}B_I(t)}\right\}, \qquad (3.17a)$$

and

$$P(X_{2I},t)=(2M\hbar\omega)^{1/2}\langle p|\rho_I(t)|p\rangle$$
$$=\{\pi[\tfrac{1}{2}+A_I(t)+\mathrm{Re}B_I(t)]\}^{-1/2}\times\exp\left\{-\frac{[X_{2I}-\mathrm{Im}C_I(t)]^2}{\tfrac{1}{2}+A_I(t)+\mathrm{Re}B_I(t)}\right\}. \qquad (3.17b)$$

Employing Eqs. (2.15) and (2.9), after inserting the coefficients (2.9) and (2.10) of I, the variances of the quadratures (3.15) read

$$[\Delta X_{1I}(t)]^2=\tfrac{1}{4}\{(2\bar{n}+1)e^{-\gamma t}[\cosh(2r)+\cos\varphi\sinh(2r)]+(2\bar{n}_b+1)(1-e^{-\gamma t})\} \qquad (3.18a)$$

and

$$[\Delta X_{2I}(t)]^2=\tfrac{1}{4}\{(2\bar{n}+1)e^{-\gamma t}[\cosh(2r)-\cos\varphi\sinh(2r)]+(2\bar{n}_b+1)(1-e^{-\gamma t})\}. \qquad (3.18b)$$

In the particular case of a DSVS and for $\varphi=\pi$, the expressions (3.18) reduce to the result of Milburn and Walls [18].

Just as in I, squeezing in the quadrature X_1 becomes effective to any even order N, when, for $\varphi=\pi$,

$$[\Delta X_{1I}(t)]^2<\tfrac{1}{4}. \qquad (3.19)$$

The condition (3.19) for Nth-order squeezing is satisfied if the squeeze parameter exceeds the threshold (3.8),

$$r>r_s, \qquad (3.20a)$$

and for a time prior to the critical moment (3.11),

$$t<t_s. \qquad (3.20b)$$

The fact that the restrictions (3.20) for squeezing are complementary to the conditions (3.9) and (3.10) for the existence of Glauber's P representation of the density operator attests once more the nonclassical nature of the squeezing effect.

IV. DISSIPATION

The damping of a displaced squeezed thermal mode in contact with a reservoir at zero temperature ($\bar{n}_b=0$) is characterized by a *time scaling* of the coefficients A, B, and C, defined by Eqs. (2.9)–(2.11) of I:

$$A_I(t)=Ae^{-\gamma t},\quad B_I(t)=Be^{-\gamma t},\quad C_I(t)=Ce^{-(\gamma/2)t}. \qquad (4.1)$$

Because of Eqs. (4.1), the evolution of the correlation function (3.1) reads

$$\langle(a^\dagger)^l a^m\rangle|_t=\exp[i(l-m)\omega t]\exp[-\tfrac{1}{2}(l+m)\gamma t]\times\langle(a^\dagger)^l a^m\rangle|_{t=0}. \qquad (4.2)$$

As proved by Eq. (A13), the scaling (4.2) is a *general* property of the modes which dissipate according to the master equation (A1).

In what follows we develop another method to evaluate the density matrix of a damped DSTS determined by the coefficients (4.1). This time, our main tool is the relation (A11) between the density-matrix elements in the Fock basis and their initial values. We start by inserting Eq. (B1) of I into Eq. (5.2) of I to obtain a double-integral representation of the density-matrix elements at $t=0$,

$$\rho_{lm}(0)=\left[\frac{l!}{m!}\right]^{1/2}Q(0)\tilde{A}^l\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}du\,dv\,e^{-(u^2+v^2)}[\tilde{f}^*(v)]^{m-l}\times L_l^{(m-l)}\left[-\frac{1}{\tilde{A}}\tilde{f}(u)\tilde{f}^*(v)\right]\quad(m\ge l), \qquad (4.3)$$

with

$$\tilde{f}(v)\equiv\tilde{C}-i(2\tilde{B})^{1/2}v \qquad (4.4)$$

and $L_l^{(m-l)}(z)$ a Laguerre polynomial, Eq. (B4) of I. In Eqs. (4.3) and (4.4) we have employed the quantity Q (0), given by Eqs. (4.23) and (4.4) of I, as well as the coefficients (4.5)–(4.7) of I, whose explicit expressions are written as Eqs. (4.9)–(4.11) of I. Note that, in the diagonal case, the integral representation (4.3) is similar to Eq. (B2) of I. When substituting Eq. (4.3) into Eq. (A11), after interchanging the order of operations, we find a series of the type (B1). Making use of our result (B8), we get the formula

$$(\rho_I)_{lm}(t)=\left[\frac{l!}{m!}\right]^{1/2}Q(0)\frac{\tilde{A}^l(e^{-\gamma t})^{(l+m)/2}}{[1-\tilde{A}(1-e^{-\gamma t})]^{m+1}}\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}du\,dv\,\exp\left[-(u^2+v^2)+\frac{1-e^{-\gamma t}}{1-\tilde{A}(1-e^{-\gamma t})}\tilde{f}(u)\tilde{f}^*(v)\right]\times[\tilde{f}^*(v)]^{m-l}L_l^{(m-l)}\left[-\frac{1}{1-\tilde{A}(1-e^{-\gamma t})}\frac{1}{\tilde{A}}\tilde{f}(u)\tilde{f}^*(v)\right]. \qquad (4.5)$$

Notice that, due to the symmetry property [19,20]

$$\frac{z^l}{l!}L_m^{(l-m)}(-z)=\frac{z^m}{m!}L_l^{(m-l)}(-z)\ , \tag{4.6}$$

the expression (4.5) is general.

The next step is to evaluate Glauber's R function at the moment t, in the interaction picture,

$$R_I(\beta^*,\beta',t)=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(l!m!)^{1/2}}(\rho_I)_{lm}(t)(\beta^*)^l(\beta')^m\ . \tag{4.7}$$

We carry out first the summation on l, employing a Taylor expansion due to Erdélyi [21]. After performing the simpler resulting summation on m, we are left with a double integral of the form (A6) of I. We apply Eq. (A8) of I, and, after some elementary algebra, we recover our result (3.5) written with the particular coefficients (4.1). Accordingly, the evolution of the density matrix of the field mode in a dissipating DSTS reads

$$(\rho_I)_{lm}(t;A,B,C)=(\rho_I)_{lm}(0;Ae^{-\gamma t},Be^{-\gamma t},Ce^{-(\gamma/2)t})\ . \tag{4.8}$$

To resume, the above-presented method of evaluating the density matrix in the Fock representation is more complicated and less general than the method of finding out, primarily, the time development of the CF. However, being based on the relation (A11), it is rewarding especially when one is concerned about the process of photodetection.

V. PHOTON COUNTING

In Mollow's model of photodetection [7], the detector consists of a large number of harmonic oscillators weakly coupled to the radiation field. The diagonal elements of the reduced density matrix of the field are shown to satisfy the system of differential equations [22]:

$$\frac{d}{dt}\rho_{nn}=\gamma[(n+1)\rho_{n+1,n+1}-n\rho_{nn}]\ , \tag{5.1}$$

where γ is a coupling constant. The system (5.1) coincides with the diagonal case $q=0$ of Eq. (A3). Consequently, its solution has the structure (A11) written for $q=0$:

$$\rho_{nn}(t)=e^{-n\gamma t}\sum_{N=n}^{\infty}\begin{bmatrix}N\\ n\end{bmatrix}(1-e^{-\gamma t})^{N-n}\rho_{NN}(0)\ . \tag{5.2}$$

The probability $p_n(t)$ of counting n photons in the time interval $[0,t]$ is connected to the n-photon probability of the attenuated field at the moment t by the substitution (2.18b) [23]:

$$p_n(t)=\mu^n\sum_{N=n}^{\infty}\begin{bmatrix}N\\ n\end{bmatrix}(1-\mu)^{N-n}\rho_{NN}(0)\ . \tag{5.3}$$

In Eq. (5.3) we have denoted by

$$u\equiv 1-e^{-\gamma t} \tag{5.4}$$

the quantum efficiency of the detector.

A brief history of Eq. (5.3) would be valuable at this point. Scully and Lamb [8] consider a detector consisting of many independent atoms placed in a cavity. The probability of recording n photocounts is proved to have the same form (5.3) [24]. A direct, probabilistic derivation of the counting formula (5.3) is due to Rousseau [9]. After treating the simpler case of a photodetector immersed in a one-photon field, she develops the generalization to an arbitrary field state. In the work [10] of Selloni *et al.*, the detector consists of many identical two-level atoms. Not only the field attenuation, as in Refs. [7–9], but also

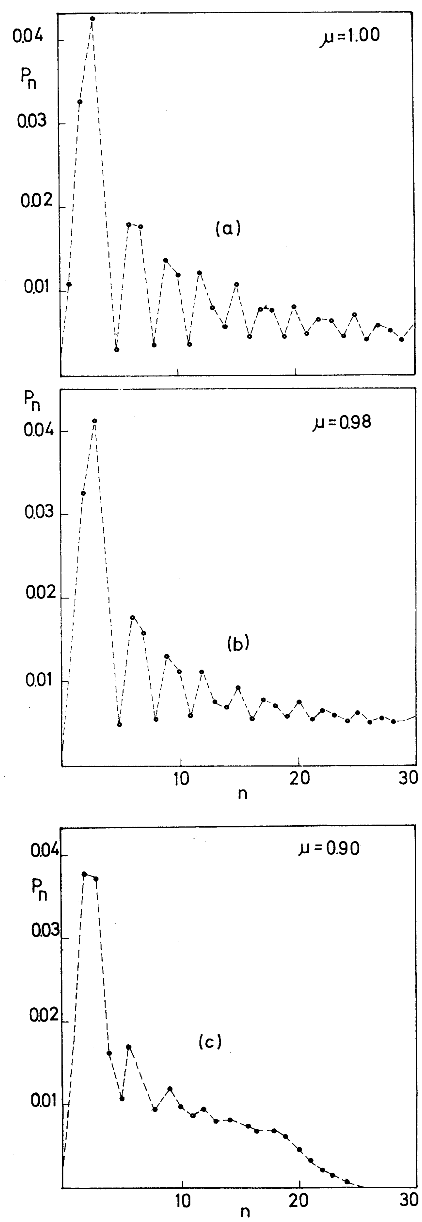

FIG. 1. Photon-counting distribution of a DSTS with the parameters $\bar{n}=2$, $|\alpha|^2=2$, and $r=2.5$. The quantum efficiency of the detector is $\mu=1.00$ (curve *a*), 0.98 (curve *b*), and 0.90 (curve *c*), while the phase difference is $2\arg(\alpha)-\varphi=\pi$.

the atomic relaxation processes are taken into account by using a suitable atom-reservoir interaction Hamiltonian [25]. The photon-counting distribution obtained is given by Eq. (5.3), too, with a detector efficiency depending also on the atom-reservoir coupling constant. We finally quote the comprehensive paper [11] of Srinivas and Davies. In the framework of Davies's theory of continuous measurements in quantum mechanics, they rederive the counting formula (5.3) together with expressions of more complicated and higher-order joint probabilities [26].

It is now convenient to introduce a fictitious time τ defined as

$$\mu \equiv e^{-\gamma\tau} , \tag{5.5}$$

so that Eqs. (5.2)–(5.4) provide the identity

$$p_n(t) = \rho_{nn}(\tau) . \tag{5.6}$$

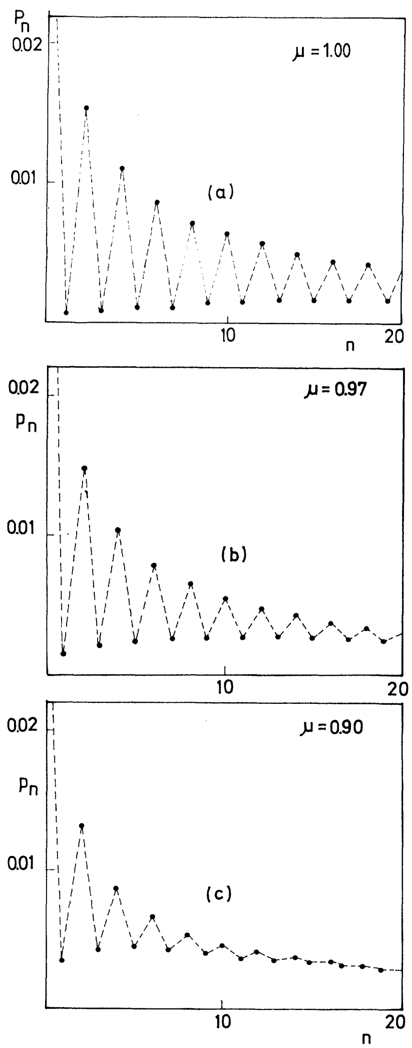

FIG. 2. As in Fig. 1, but for the parameters $\bar{n}=4$, $|\alpha|^2=10$, and $r=3$. The phase difference is now $2\arg(\alpha)-\varphi=0$, while the quantum efficiency of the detector is $\mu=1.00$ (curve *a*), 0.97 (curve *b*), and 0.90 (curve *c*).

When the incident field is in a DSTS, we carry out the summation (5.2) by employing the method described in Sec. IV. Then, the photon-counting distribution (5.3) is found from Eq. (3.12):

$$p_n(t) = \pi Q(0,\tau)[\tilde{A}(\tau)]^n \times \sum_{q=0}^{n} \frac{1}{q!} \begin{bmatrix} n \\ q \end{bmatrix} \left[\frac{|\tilde{B}(\tau)|}{2\tilde{A}(\tau)} \right]^q \times |H_q([2\tilde{B}(\tau)]^{-1/2}\tilde{C}(\tau))|^2 . \tag{5.7}$$

In Eq. (5.7) we must insert the functions (3.3) and (3.4) and use the scaling law (4.1).

Plots of the photon-counting distribution of a DSTS for some values of the detector quantum efficiency μ are drawn in Figs. 1 and 2. The oscillatory behavior of these distributions depends on both μ and the phase difference $2\arg(\alpha)-\varphi$. In Fig. 1 we have taken $2\arg(\alpha)-\varphi=\pi$. The quantum efficiency varies from $\mu=1$ in Fig. 1(a) to $\mu=0.9$ in Fig. 1(c), whereas parameters $\bar{n}$, $|\alpha|^2$, and r are kept fixed. Figure 2 presents the case $2\arg(\alpha)-\varphi=0$. Here the oscillations are of the pairwise type and remain significant even for $\mu=0.9$, Fig. 2(c), although the thermal noise is strong enough ($\bar{n}=4.0$). In fact, the occurrence of thermal noise does not affect the shape of the counting distribution, which resembles that found by Milburn and Walls in the case of a SCS [27]. Our graphs in Figs. 1 and 2 also agree with those reported by Agarwal and Adam for a DSTS with the same phase differences [28].

Notice finally that, according to the scaling property (A13), the factorial moments of the photon-counting distribution are proportional to the corresponding correlation functions of the incident field:

$$\sum_{n=l}^{\infty} \frac{n!}{(n-l)!} p_n(t) = \mu^l \langle (a^\dagger)^l a^l \rangle |_{t=0} . \tag{5.8}$$

In Eq. (5.8) the summation index n denotes the number of photons recorded in the time interval $[0,t]$.

VI. SUMMARY

Starting from the well-known master equation for a single-mode radiation field weakly coupled to a reservoir at temperature T_b, we have established that damping preserves the Gaussian form of the CF of a DSTS. As a consequence, we could employ the results of our preceding paper I to analyze some statistical and squeezing properties of a damped displaced squeezed thermal mode. In the special case of a field dissipating in contact with a zero-temperature reservoir, we have rederived the density matrix in the Fock basis, making use of a general formal solution of the master equation.

Photodetection is considered in most theoretical treatments as an attenuation of the incident field. This idea has enabled us to compute the photon-counting statistics of a DSTS, in close analogy with the photon-number distribution of the dissipating field mode. Of particular interest are the nonclassical features of the photon-number and photon-counting distributions. As a salient example, we have pointed out the persistence of the oscillations in the counting distribution for sufficiently high quantum efficiency of the detector.

APPENDIX A: THE MASTER EQUATION FOR FIELD DISSIPATION

In the case of a field mode in contact with a zero-temperature heat bath, the master equation (2.1) becomes

$$\frac{\partial \rho_I}{\partial t}=\frac{\gamma}{2}(2a_I\rho_I a_I^\dagger - a_I^\dagger a_I \rho_I - \rho_I a_I^\dagger a_I) . \quad \text{(A1)}$$

With the notation

$$\Phi_n(q;t)\equiv\left[\frac{(n+q)!}{n!}\right]^{1/2}(\rho_I)_{n,n+q}(t) , \quad \text{(A2)}$$

we write Eq. (A1) in the Fock basis as

$$\frac{\partial}{\partial t}\Phi_n(q;t)=\gamma\left[(n+1)\Phi_{n+1}(q;t)-\left[n+\frac{q}{2}\right]\Phi_n(q;t)\right] . \quad \text{(A3)}$$

To solve the system of differential equations (A3), we apply the Laplace-transform method. First, the Laplace transform of the function (A2),

$$\tilde{\Phi}_n(q;s)\equiv\int_0^\infty dt\, e^{-st}\Phi_n(q;t) \quad (\text{Re}s>0) , \quad \text{(A4)}$$

is readily seen to satisfy the recurrence relation

$$\left[s+\gamma\left[n+\frac{q}{2}\right]\right]\tilde{\Phi}_n(q;s)=\Phi_n(q;0)+\gamma(n+1)\tilde{\Phi}_{n+1}(q;s) . \quad \text{(A5)}$$

By iteration we obtain it as the sum of a series:

$$\tilde{\Phi}_n(q;s)=\frac{1}{\gamma}\sum_{p=0}^{\infty}\frac{(n+1)_p}{\left[\dfrac{s}{\gamma}+n+\dfrac{q}{2}\right]_{p+1}}\Phi_{n+p}(q;0) . \quad \text{(A6)}$$

The inverse Laplace transform of the function (A4),

$$\Phi_n(q;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}ds\, e^{ts}\tilde{\Phi}_n(q;s) , \quad \text{(A7)}$$

is an integral taken along a straight line $\text{Re}s=a$, from $a-iR$ to $a+iR$, where a is an arbitrary positive number and $R\to\infty$. We insert the series (A6) into the integral (A7) and interchange the order of operations. Then we evaluate the inverse Laplace transform of a current term of the series (A6) as a contour integral:

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}ds\, e^{ts}\frac{1}{\left[\dfrac{s}{\gamma}+n+\dfrac{q}{2}\right]_{p+1}}=\frac{1}{2\pi i}\lim_{R\to\infty}\int_{\mathcal{C}(R)}ds\, e^{ts}\frac{1}{\left[\dfrac{s}{\gamma}+n+\dfrac{q}{2}\right]_{p+1}} . \quad \text{(A8)}$$

In Eq. (A8), the path of integration $\mathcal{C}(R)$ lies in the half-plane $\text{Re}s\le a$ and encloses all the poles of the integrand. It includes two horizontal segments with $\text{Im}s=\pm R$ and $0\le\text{Re}s\le a$, whose end points $s=iR$ and $s=-iR$ are joined by a semicircle with the center at $s=0$. By means of the theorem of residues, and after performing a finite binomial sum, we get

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}ds\, e^{ts}\frac{1}{\left[\dfrac{s}{\gamma}+n+\dfrac{q}{2}\right]_{p+1}}=\frac{\gamma}{p!}e^{-(n+q/2)\gamma t}(1-e^{-\gamma t})^p . \quad \text{(A9)}$$

Consequently, the function (A7) is the sum of the series

$$\Phi_n(q;t)=e^{-(n+q/2)\gamma t}\sum_{p=0}^{\infty}\begin{bmatrix} n+p \\ p\end{bmatrix}(1-e^{-\gamma t})^p\Phi_{n+p}(q;0) . \quad \text{(A10)}$$

In view of the notation (A2), the result (A10) represents a formal solution of the master equation (A1):

$$\left[\frac{(n+q)!}{n!}\right]^{1/2}(\rho_I)_{n,n+q}(t)=(e^{-\gamma t})^{n+q/2}\sum_{N=n}^{\infty}\begin{bmatrix} N \\ n\end{bmatrix}(1-e^{-\gamma t})^{N-n}\times\left[\frac{(N+q)!}{N!}\right]^{1/2}\rho_{N,N+q}(0) . \quad \text{(A11)}$$

The relation (A11) provides an important time-scaling property of any correlation function. To prove it, we employ the expansion

$$\langle (a^\dagger)^l a^m\rangle|_t\equiv\text{Tr}\{\rho_I(t)[a_I^\dagger(t)]^l[a_I(t)]^m\}=e^{i(l-m)\omega t}\sum_{n=m}^{\infty}\frac{n!}{(n-m)!}\left[\frac{(n+l-m)!}{n!}\right]^{1/2}(\rho_I)_{n,n+l-m}(t) \quad (l\ge m) . \quad \text{(A12)}$$

After substituting Eq. (A11) into Eq. (A12), we interchange the summation order and perform a finite binomial sum. Using again Eq. (A12), taken at $t=0$, we obtain the formula

$$\langle (a^\dagger)^l a^m\rangle|_t=e^{i(l-m)\omega t}(e^{-\gamma t})^{(l+m)/2}\langle (a^\dagger)^l a^m\rangle|_{t=0} . \quad \text{(A13)}$$

As a consequence, dissipation has no influence on a normalized correlation function, defined by Eq. (7.5) of I, which undergoes just the free-field phase transformation,

$$g^{(lm,)}(0;t)=e^{i(l-m)\omega t}g^{(l,m)}(0;0) \ . \tag{A14}$$

APPENDIX B: A SERIES OF LAGUERRE POLYNOMIALS

We evaluate a series of the type

$$T_l^{(\alpha)}(t,u)\equiv \sum_{n=l}^{\infty}\begin{bmatrix} n \\ l \end{bmatrix} t^n L_n^{(\alpha)}(u)$$

$$(|t|<1,\ \alpha>-1,\ l=0,1,2,\ldots)\ , \tag{B1}$$

starting from the generating function of the Laguerre polynomials of order α [29],

$$T_0^{(\alpha)}(t,u)\equiv(1-t)^{-\alpha-1}\exp\left[-\frac{tu}{1-t}\right] \quad (|t|<1)\ . \tag{B2}$$

By applying Cauchy's integral formula in the special case $l=0$ of Eq. (B1), we get

$$L_n^{(\alpha)}(u)=\frac{1}{2\pi i}\int^{(0+)}dt\frac{1}{t^{n+1}}T_0^{(\alpha)}(s+t,u)\ . \tag{B3}$$

The contour of integration in Eq. (B3) is a closed loop encircling the origin $t=0$ in the counterclockwise sense. Next, taking note of the relation

$$T_l^{(\alpha)}(t,u)=\frac{1}{l!}t^l\left[\frac{\partial}{\partial t}\right]^l T_0^{(\alpha)}(t,u)\ , \tag{B4}$$

we employ once again Cauchy's integral formula to find, after an obvious change of the variable of integration,

$$T_l^{(\alpha)}(t,u)=\frac{t^l}{2\pi i}\int^{(0+)}ds\frac{1}{s^{l+1}}T_0^{(\alpha)}(s+t,u)\ . \tag{B5}$$

Making use of the factorization property

$$T_0^{(\alpha)}(s+t,u)=T_0^{(\alpha)}(t,u)T_0^{(\alpha)}\left[\frac{s}{1-t},\frac{u}{1-t}\right]\ , \tag{B6}$$

the function (B5) can be expressed as

$$T_l^{(\alpha)}(t,u)=\left[\frac{t}{1-t}\right]^l T_0^{(\alpha)}(t,u)\frac{1}{2\pi i}$$

$$\times\int^{(0+)}d\zeta\frac{1}{\zeta^{l+1}}T_0^{(\alpha)}\left[\zeta,\frac{u}{1-t}\right]\ . \tag{B7}$$

Equations (B7) and (B3) give the simple formula

$$T_l^{(\alpha)}(t,u)=\left[\frac{t}{1-t}\right]^l T_0^{(\alpha)}(t,u)L_l^{(\alpha)}\left[\frac{u}{1-t}\right]$$

$$(|t|<1\ ,\ \alpha>-1\ ,\ l=0,1,2,\ldots)\ . \tag{B8}$$

We finally remark that this method of calculating the series (B1) is analogous to that developed previously by one of us in order to evaluate a similar series of Legendre polynomials [30].

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