Squeezed states with thermal noise. I. Photon-number statistics

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(Received 18 May 1992; revised manuscript received 30 November 1992)

We investigate a free monochromatic electromagnetic field which is the superposition of a squeezed thermal radiation and a coherent one. The main tool in our analysis is the characteristic function that has a Gaussian form. We establish an analytic formula for an arbitrary correlation function, as well as its strong-squeezing limit. Besides the usual quasiprobability densities, the coherent-state, number-state, coordinate, and momentum representations of the density operator are derived. We point out the nonclassical oscillations of the photon-number distribution and find its generating function. Collaterally, displaced thermal states and squeezed thermal states are revisited as nontrivial particular cases. We examine finally the squeezing properties of the field using the distribution functions of the quadratures.

PACS number(s): 42.50.Dv

I. INTRODUCTION

Superposition of light in a squeezed thermal state (STS) with coherent light is an important problem from both theoretical and practical standpoints. Because thermal noise is inevitable and hard to quench, it is more realistic to consider a thermal-state instead of a vacuum-state input to a squeezing device. Had we prepared a free field in a STS, this could be driven by a classical current, providing a *displaced squeezed thermal state* (DSTS).

In the present paper we deal with a single-mode free radiation field of angular frequency ω, whose photon annihilation and creation operators are a and $a^\dagger$, respectively. The field is assumed to be in a DSTS, having the density operator

$$\rho_{\rm DST}=D(\alpha)S(\zeta)\rho_T S^\dagger(\zeta)D^\dagger(\alpha) \ . \tag{1.1}$$

In Eq. (1.1), $D(\alpha)$ is a Weyl displacement operator [1],

$$D(\alpha)=\exp(\alpha a^\dagger-\alpha^* a) \ , \tag{1.2}$$

with the complex parameter α, while $S(\zeta)$ is a Stoler squeeze operator [2],

$$S(\zeta)=\exp[\tfrac{1}{2}\zeta(a^\dagger)^2-\tfrac{1}{2}\zeta^* a^2] \ , \tag{1.3}$$

where

$$\zeta=re^{i\varphi} \quad (r\geq 0, \ -\pi<\varphi\leq\pi) \ . \tag{1.4}$$

Although its subscript T signifies thermal state (TS), ρ_T is the density operator of a more general chaotic state with the mean occupancy $\bar{n}$,

$$\rho_T=\frac{1}{\bar{n}+1}\sum_{n=0}^{\infty}\left[\frac{\bar{n}}{\bar{n}+1}\right]^n |n\rangle\langle n| \ . \tag{1.5}$$

Specifically, for thermal equilibrium, at the temperature T,

$$\bar{n}=\left[\exp\left[\frac{\hbar\omega}{k_B T}\right]-1\right]^{-1} \ . \tag{1.6}$$

The interchanging of the operators $D(\alpha)$ and $S(\zeta)$ in Eq. (1.1) leads to a *squeezed displaced thermal state* (SDTS), which is, in fact, a DSTS with a modified displacement parameter. Indeed, the well-known transformations of the annihilation operator a by the unitary operators (1.2),

$$D^\dagger(\alpha)aD(\alpha)=a+\alpha I \ , \tag{1.7}$$

and (1.3),

$$S^\dagger(\zeta)aS(\zeta)=(\cosh r)a+e^{i\varphi}(\sinh r)a^\dagger \ , \tag{1.8}$$

imply that

$$D^\dagger(\alpha)D(\lambda)D(\alpha)=\exp(\lambda\alpha^*-\lambda^*\alpha)D(\lambda) \tag{1.9}$$

and

$$S^\dagger(\zeta)D(\lambda)S(\zeta)=D(\lambda\cosh r-\lambda^* e^{i\varphi}\sinh r) \ , \tag{1.10}$$

respectively. The transformation (1.10) is equivalent to the identity [2]

$$S(\zeta)D(\alpha)=D(\alpha\cosh r+\alpha^* e^{i\varphi}\sinh r)S(\zeta) \ , \tag{1.11}$$

which proves the assertion made above.

Quite recently, nonclassical features of photon statistics in DSTS's have been studied [3–11]. Vourdas [3] considered the density operator of a chaotic field superposed on a squeezed coherent one, which can be written in the form (1.1), with specific values of the parameters. He expressed its matrix elements in the Fock basis in terms of Hermite polynomials of two variables. In the limit $r=0$, describing a superposition of coherent and chaotic fields, his result reduces to the classical formulas obtained by Lachs for the photon-number probability [12] and by Mollow and Glauber for the whole density

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matrix in the number-state representation [13]. Numerical calculations [5] led to the conclusion that even very small amounts of thermal noise destroy the oscillatory character of the photon-number distribution. This idea was not confirmed by the subsequent results of Agarwal and Adam [6,7]. These authors have examined the probabilities of finding and counting n photons for a class of squeezed states that possess a Gaussian Wigner function. They have pointed out that for certain ranges of the squeeze parameter r, the photon-number distribution displays oscillations. But, at the same time, we shall prove that the DSTS's belong to the above-mentioned class of quantum states. By analyzing the results of Ref. [7], a simpler formula for the photon-number distribution of a DSTS was found by Chaturvedi and Srinivasan [9]. Another noteworthy contribution to the study of these states is due to Janszky and Yushin [4]. Starting from a normally ordered characteristic function (CF) of Gaussian form, they have calculated the correlation function of any order, as well as its limit for weak and strong squeezing. The special case of STS's was investigated by Kim, de Oliveira, and Knight [8] and by one of us [11]. In Ref. [8], quasiprobability densities, second-order squeezing, and photon statistics have been discussed, while in Ref. [11], analytical formulas for photon-number distribution, higher-order squeezing, and correlation functions have been established. In a very recent paper [10], Ezawa *et al.*, after deriving the normally ordered CF for a DSTS, proved an interesting factorizability property of this function for the multimode case. The DSTS's are characterized by the invariance of the CF factorizability under an orthogonal transformation of the field operators. For a two-mode field, this transformation describes the action of a special kind of beam splitter.

The present work is devoted to an extensive theoretical study of a DSTS, starting from its CF. As a matter of fact, our discussion encompasses a broad class of quantum states, whose CF has precisely the form proposed by Janszky and Yushin [14]. In Sec. II we derive in a simpler way the CF of a DSTS. The degree of purity of this mixed state is also calculated and discussed in this section. A general formula for correlation functions is established in Sec. III. Then we specialize it for displaced thermal states (DTS's) and STS's. In Sec. IV we evaluate the density operator in the coherent-state representation, as well as the corresponding quasiprobability densities. Glauber's R function [1] is employed in Secs. V and VI to obtain the density matrices in the number-state, coordinate, and momentum representations. In Sec. V we also derive the generating function of the photon-number distribution. Using the probability densities of the quadratures, the squeezing properties in a DSTS are examined in Sec. VII. In addition, the strong-squeezing limit of the correlation functions is established. Section VIII outlines the results. In Appendix A we recall an infinite integral of a Gaussian-type function of several variables. An equivalent expression of the lth-order correlation function is derived in Appendix B.

II. CHARACTERISTIC FUNCTIONS

It is well known that the most useful particular cases of the s-ordered CF

$$\chi(\lambda,s)\equiv\exp\left[\frac{s}{2}|\lambda|^2\right]\mathrm{Tr}[\rho D(\lambda)]\quad (s\ \text{real})\ , \qquad (2.1)$$

defined by Cahill and Glauber [15], are the normally ordered, usual, and antinormally ordered CF's, introduced previously by Glauber [16,17], respectively, as

$$\chi(\lambda,1)\equiv\chi_N(\lambda)=\langle e^{\lambda a^\dagger}e^{-\lambda^* a}\rangle\ , \qquad (2.2)$$

$$\chi(\lambda,0)\equiv\chi(\lambda)\ , \qquad (2.3)$$

$$\chi(\lambda,-1)\equiv\chi_A(\lambda)=\langle e^{-\lambda^* a}e^{\lambda a^\dagger}\rangle\ . \qquad (2.4)$$

We derive the generalized CF (2.1) of a DSTS in two steps. First, the CF (2.3) for a chaotic mixture,

$$\chi_T(\lambda)=\mathrm{Tr}[\rho_T D(\lambda)]\ , \qquad (2.5)$$

is readily evaluated in the Fock basis. Indeed, after employing Eq. (1.5) and the diagonal matrix elements of the displacement operator (1.2) [18–20], the remaining summation in Eq. (2.5) gives, via the generating function of the Laguerre polynomials [21], a formula due to Glauber [22]:

$$\chi_T(\lambda)=\exp[-(\bar{n}+\tfrac{1}{2})|\lambda|^2]\ . \qquad (2.6)$$

Second, using Eq. (1.1) and performing a cyclic permutation of operators under the trace symbol, we can write the CF (2.1) for a DSTS as

$$\chi(\lambda,s)=\exp\left[\frac{s}{2}|\lambda|^2\right]\mathrm{Tr}[\rho_T S^\dagger(\zeta)D^\dagger(\alpha)D(\lambda)D(\alpha)S(\zeta)]\ . \qquad (2.7)$$

After successive use of Eqs. (1.9) and (1.10), we achieve the possibility of employing Eq. (2.6). We finally get

$$\chi(\lambda,s)=\exp\left[-\left[A+\frac{1-s}{2}\right]|\lambda|^2 -\tfrac{1}{2}[B^*\lambda^2+B(\lambda^*)^2]+C^*\lambda-C\lambda^*\right]\ , \qquad (2.8)$$

where

$$A=\bar{n}+(2\bar{n}+1)(\sinh r)^2\ , \qquad (2.9)$$

$$B=-(2\bar{n}+1)e^{i\varphi}\sinh r\cosh r\ , \qquad (2.10)$$

$$C=\alpha\quad (\text{DSTS})\ , \qquad (2.11a)$$

$$C=\alpha\cosh r+\alpha^* e^{i\varphi}\sinh r\quad (\text{SDTS}) \qquad (2.11b)$$

are dimensionless coefficients depending on the parameters $\bar{n}$, ζ, and α. The difference between expressions (2.11a) and (2.11b) originates in Stoler's identity (1.11). Equation (2.8) for $s=1$, with the notations (2.9), (2.10), and (2.11a), is a main result of Ezawa and co-workers [23].

In the special case $\bar{n}=0$, the quantum state (1.1) becomes a pure one. This is a displaced squeezed vacuum state (DSVS),

$$|\psi_{\rm DSV}\rangle = D(\alpha)S(\zeta)|0\rangle \ . \quad (2.12a)$$

Similarly, for $\bar{n}=0$, a SDTS reduces to a squeezed coherent state (SCS),

$$|\psi_{\rm SC}\rangle = S(\zeta)D(\alpha)|0\rangle \ . \quad (2.12b)$$

As nontrivial examples of simpler mixed states, we mention the DTS's ($r=0$) and the STS's ($\alpha=0$). Further, by equating two parameters to zero, we get as particular cases the coherent states (CS's) ($\bar{n}=0, r=0$), squeezed vacuum states (SVS's) ($\bar{n}=0, \alpha=0$), and thermal states (TS's) ($r=0, \alpha=0$).

We recall that for a definite state of the field the CF (2.3) determines uniquely the density operator. More generally, following Weyl [24], any operator F which has a finite Hilbert-Schmidt norm,

$$\|F\| \equiv [{\rm Tr}(F^\dagger F)]^{1/2} \ , \quad (2.13)$$

may be expressed as

$$F = \frac{1}{\pi}\int d^2\xi f(\xi) D(-\xi) \ . \quad (2.14)$$

In Eq. (2.14),

$$d^2\xi = d({\rm Re}\xi)d({\rm Im}\xi) \quad (2.15)$$

is the differential element of area in the complex ξ plane. The weight function $f(\xi)$ has the expression

$$f(\xi) = {\rm Tr}[FD(\xi)] \quad (2.16)$$

and is square-integrable in view of the relation

$${\rm Tr}(F^\dagger F) = \frac{1}{\pi}\int d^2\xi |f(\xi)|^2 \ . \quad (2.17)$$

The Weyl expansion of the Hilbert-Schmidt operators was used for the first time in quantum optics by Glassgold and Holliday [19] and then was analyzed very carefully by Cahill and Glauber [20]. In particular, the Weyl representation of the density operator has been applied long ago by Mollow and Glauber to their quantum treatment of the parametric amplification [25]. The density operator belongs to the trace class and the weight function (2.16) in its Weyl expansion,

$$\rho = \frac{1}{\pi}\int d^2\lambda\, \chi(\lambda) D(-\lambda) \ , \quad (2.18)$$

is precisely the CF (2.3) which is always square-integrable,

$${\rm Tr}(\rho^2) = \frac{1}{\pi}\int d^2\lambda |\chi(\lambda)|^2 \ . \quad (2.19)$$

According to Eq. (2.18), a DSTS is completely specified by the coefficients A, B, and C which occur in Eq. (2.8). In turn, due to Eqs. (2.9)–(2.11), they depend on the thermal mean occupancy $\bar{n}$, the complex squeeze parameter ζ, and the coherent amplitude α. Of particular interest is the degree of purity ${\rm Tr}(\rho^2)$ which can be evaluated making use of Eqs. (2.19), (2.8), and (A8):

$${\rm Tr}(\rho_{\rm DST}^2) = \tfrac{1}{2}[(A+\tfrac{1}{2})^2 - |B|^2]^{-1/2} \ . \quad (2.20)$$

Substitution of the coefficients (2.9) and (2.10) into Eq. (2.20) yields the explicit formula

$${\rm Tr}(\rho_{\rm DST}^2) = \frac{1}{2\bar{n}+1} \ , \quad (2.21)$$

while the specification (1.6) gives further

$${\rm Tr}(\rho_{\rm DST}^2) = \tanh\left[\frac{\hbar\omega}{2k_B T}\right] . \quad (2.22)$$

The result (2.21) is not surprising. Indeed, the square of the density operator (1.1) is

$$\rho_{\rm DST}^2 = D(\alpha)S(\zeta)\rho_T^2 S^\dagger(\zeta)D^\dagger(\alpha) \ , \quad (2.23)$$

so that

$${\rm Tr}(\rho_{\rm DST}^2) = {\rm Tr}(\rho_T^2) \ . \quad (2.24)$$

From Eq. (1.5), we get

$${\rm Tr}(\rho_T^2) = \frac{1}{2\bar{n}+1} \quad (2.25)$$

and thus the expression (2.21) is recovered. Equation (2.24) proves that the degree of purity of the input chaotic state (1.5) is left unchanged by the subsequent squeezing and displacement processes described by the density operator (1.1). Accordingly, the degree of purity of a DSTS depends only on the temperature of the initial reservoir, as shown by Eq. (2.22).

III. CORRELATION FUNCTIONS

The normally ordered CF, Eq. (2.2), has the Taylor expansion

$$\chi_N(\lambda) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{l!m!}\lambda^l(-\lambda^*)^m\langle (a^\dagger)^l a^m\rangle \ , \quad (3.1)$$

which allows one to evaluate the correlation function $\langle (a^\dagger)^l a^m\rangle$. Making use of the generating function of the Hermite polynomials [26], we first write the CF as a power series of λ and then expand its coefficients as a power series of $(-\lambda^*)$. After some simple algebra, we get, by comparison with Eq. (3.1),

$$\langle (a^\dagger)^l a^m\rangle = \sum_{k=0}^{\min\{l,m\}} k! \begin{bmatrix} l \\ k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} A^k(\tfrac{1}{2}B^*)^{(l-k)/2} \times(\tfrac{1}{2}B)^{(m-k)/2} H_{l-k}((2B^*)^{-1/2}C^*) \times H_{m-k}((2B)^{-1/2}C) \ , \quad (3.2)$$

where $\binom{n}{k}$ is a binomial coefficient. For definiteness, taking notice of Eq. (2.10), in Eq. (3.2) we choose

$$B^{1/2} = ie^{i(\varphi/2)}|B|^{1/2} \quad (3.3)$$

and

$$(B^*)^{1/2} = (B^{1/2})^* \ . \quad (3.4)$$

Accordingly, the *finite* sum in Eq. (3.2) involves products of Hermite polynomials of complex conjugate variables.

Note that the lth-order correlation function

$$\langle (a^\dagger)^l a^l \rangle = A^l \sum_{q=0}^{l} (l-q)! \binom{l}{q}^2 \left[\frac{|B|}{2A} \right]^q |H_q((2B)^{-1/2}C)|^2 \quad (3.5)$$

has also the equivalent form (B9). This expression was derived by Peřinova *et al.* for a DSVS using the generating function of the photon-number distribution [27].

With Eq. (3.5) and the average number of photons

$$\langle a^\dagger a \rangle = A + |C|^2 , \quad (3.6)$$

one can write explicitly the degree of coherence of lth order,

$$g^{(l)}(0) \equiv \frac{\langle (a^\dagger)^l a^l \rangle}{(\langle a^\dagger a \rangle)^l} . \quad (3.7)$$

For example, the second-order normalized correlation function is

$$g^{(2)}(0) = 2 + \frac{|B|^2 - |C|^4 - 2\,\mathrm{Re}(B^* C^2)}{(A + |C|^2)^2} . \quad (3.8)$$

Explicitly, Eqs. (3.6) and (3.8) read, for a DSTS,

$$\langle n \rangle = \bar{n} + (2\bar{n}+1)(\sinh r)^2 + |\alpha|^2 , \quad (3.9)$$

$$g^{(2)}(0) = 2 + \frac{(\bar{n}+\frac{1}{2})^2[\sinh(2r)]^2 - |\alpha|^4 + (2\bar{n}+1)\sinh(2r)\mathrm{Re}(\alpha^2 e^{-i\varphi})}{[\bar{n} + (2\bar{n}+1)(\sinh r)^2 + |\alpha|]^2} . \quad (3.10)$$

The function $g^{(2)}(0)$ is quite sensitive to the phase difference $2\arg(\alpha)-\varphi$, as shown in Figs. 1 and 2. In Fig. 1 we have plotted $g^{(2)}(0)$ as a function of the squeeze parameter for some values of $\bar{n}$ and $|\alpha|^2$ and for $2\arg(\alpha)-\varphi=\pi$. We note the existence of a minimum at small value of r, but the statistics is super-Poissonian. In Fig. 2, $g^{(2)}(0)$ is plotted versus r for several values of the parameters $\bar{n}$ and $|\alpha|^2$ and for a phase difference $2\arg(\alpha)-\varphi=\pi/2$.

According to Eq. (3.2), the expectation value of any power of the creation or annihilation operator is expressed in terms of a single Hermite polynomial. For instance,

$$\langle (a^\dagger)^l \rangle = (\tfrac{1}{2}B^*)^{l/2} H_l((2B^*)^{-1/2}C^*) . \quad (3.11)$$

The particular cases (3.5) and (3.11) of the general formula (3.2) have been derived previously by Janszky and Yushin [28].

We conclude this section with some mention about the limiting cases of a DTS and a STS.

(1) Displaced thermal states. A DTS corresponds to the superposition of a coherent and a chaotic radiation field. Such a *signal-plus-noise* field was investigated first in Refs. [12] and [17]. In the absence of the squeezing process ($r=0$), we have to take the limit $B=0$ of our result (3.2) to get an expression in terms of Laguerre polynomials, Eq. (B4),

$$\langle (a^\dagger)^l a^m \rangle = m!(\alpha^*)^{l-m}(\bar{n})^m L_m^{(l-m)}\left[-\frac{|\alpha|^2}{\bar{n}}\right]$$

$$= l!(\alpha)^{m-l}(\bar{n})^l L_l^{(m-l)}\left[-\frac{|\alpha|^2}{\bar{n}}\right] . \quad (3.12)$$

This formula was put forward by Cahill and Glauber in a more general context [29].

(2) Squeezed thermal states. Setting $C=0$ in Eq. (3.2), we find, after adequate algebra,

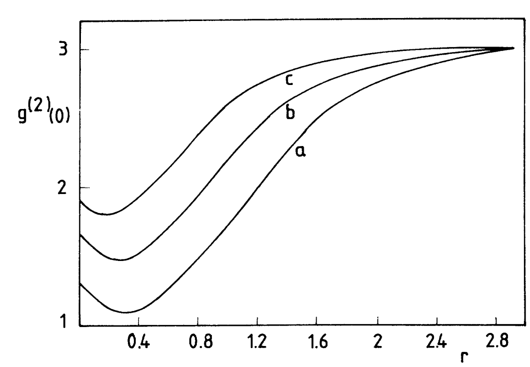

FIG. 1. Second-order degree of coherence $g^{(2)}(0)$ vs the squeeze parameter r for a DSTS with the parameters $|\alpha|^2=2$, $\bar{n}=0.4$ (curve *a*); $|\alpha|^2=2$, $\bar{n}=1.4$ (curve *b*); and $|\alpha|^2=2$, $\bar{n}=4$ (curve *c*). The phase difference is $2\arg(\alpha)-\varphi=\pi$.

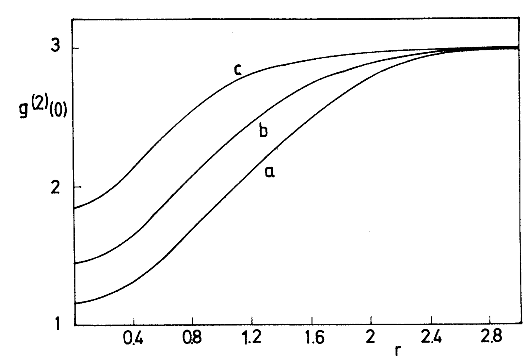

FIG. 2. As in Fig. 1, but for the phase difference $2\arg\alpha-\varphi=\pi/2$. The parameters are $|\alpha|^2=4$, $\bar{n}=0.4$ (curve *a*); $|\alpha|^2=4$, $\bar{n}=1.4$, (curve *b*); and $|\alpha|^2=4$, $\bar{n}=6$ (curve *c*).

$$\langle (a^\dagger)^l a^m\rangle = \begin{cases} m!\exp\left[-i\frac{(l-m)\varphi}{2}\right](A^2-|B|^2)^{(l+m)/4}P_{(l+m)/2}^{(l-m)/2}\left[\frac{A}{(A^2-|B|^2)^{1/2}}\right] & (l+m \text{ even}) & (3.13a)\\ 0 \quad (l+m \text{ odd}) . & & (3.13b)\end{cases}$$

In Eq. (3.13a), $P_L^M(z)$ is the Hobson's associated Legendre function of the first kind, of degree L and order M [30]. The sign of the root $(A^2-|B|^2)^{1/2}$ in Eq. (3.13a) could be arbitrary. However, for definiteness, we make the same assumption as in a previous work [31]: The square root of a positive quantity is positive, while

$$(A^2-|B|^2)^{1/2}=i(|B|^2-A^2)^{1/2} \quad \text{if } B|>A . \tag{3.14}$$

For $m=l$, Eq. (3.13a) becomes

$$\langle (a^\dagger)^l a^l\rangle = l!(A^2-|B|^2)^{l/2}P_l\left[\frac{A}{(A^2-|B|^2)^{1/2}}\right] , \tag{3.15}$$

where $P_l(z)$ is a Legendre polynomial. We mention that the result (3.15) has been obtained previously by a quite different method [32].

Substitution of the coefficients (2.9) and (2.10) into Eq. (3.13a) yields the explicit form

$$\langle (a^\dagger)^l a^m\rangle = \begin{cases} m!e^{-i[(l-m)/2]\varphi}(2\bar{n}+1)^{(l+m)/4}[(\sinh r_s)^2-(\sinh r)^2]^{(l+m)/4} \\ \times P_{(l+m)/2}^{(l-m)/2}\left[\frac{\bar{n}+(2\bar{n}+1)(\sinh r)^2}{(2\bar{n}+1)^{1/2}[(\sinh r_s)^2-(\sinh r)^2]^{1/2}}\right] \quad (l+m \text{ even}) \\ 0 \quad (l+m \text{ odd}) ,\end{cases} \tag{3.16}$$

where we have used the parameter introduced in Ref. [11],

$$r_s=\tfrac{1}{2}\ln(2\bar{n}+1) . \tag{3.17}$$

For $r<r_s$, the argument of the Legendre function is positive and greater than unity, while for $r>r_s$, it becomes imaginary. As a particular case, we get either from Eq. (3.11) or from Eq. (3.16) the expectation value [33]

$$\langle (a^\dagger)^l\rangle = \begin{cases} (l-1)!![e^{-i\varphi}(\bar{n}+\tfrac{1}{2})\sinh(2r)]^{l/2} & (l \text{ even}) \\ 0 \quad (l \text{ odd}) . \end{cases} \tag{3.18}$$

IV. COHERENT-STATE REPRESENTATION OF THE DENSITY OPERATOR

The density operator ρ is represented uniquely in the coherent-state basis by Glauber's R function [1],

$$R(\beta^*,\beta')\equiv\exp[\tfrac{1}{2}(|\beta|^2+|\beta'|^2)]\langle\beta|\rho|\beta'\rangle . \tag{4.1}$$

This is a complex-valued entire function of two complex variables β^* and β'. The Weyl expansion (2.18) of the density operator allows one to express the R representation in terms of the normally ordered CF [34]:

$$R(\beta^*,\beta')=\exp(\beta^*\beta')\frac{1}{\pi}\times\int d^2\lambda\,\chi_N(\lambda)\exp(-|\lambda|^2-\beta^*\lambda+\beta'\lambda^*) . \tag{4.2}$$

Taking into account the explicit expression (2.8), we carry out the integral (4.2) by use of Eqs. (A6) and (A8) to get the result

$$R(\beta^*,\beta')=R(0,0)\exp\{\tilde{A}\beta^*\beta'-\tfrac{1}{2}[\tilde{B}(\beta^*)^2+\tilde{B}^*(\beta')^2]+\tilde{C}\beta^*+\tilde{C}^*\beta'\} , \tag{4.3}$$

where

$$R(0,0)=[(1+A)^2-|B|^2]^{-1/2}\times\exp\left\{-\frac{(1+A)|C|^2+\tfrac{1}{2}[B(C^*)^2+B^*C^2]}{(1+A)^2-|B|^2}\right\} , \tag{4.4}$$

$$\tilde{A}=\frac{A(1+A)-|B|^2}{(1+A)^2-|B|^2} , \tag{4.5}$$

$$\tilde{B}=\frac{B}{(1+A)^2-|B|^2} , \tag{4.6}$$

$$\tilde{C}=\frac{(1+A)C+BC^*}{(1+A)^2-|B|^2} . \tag{4.7}$$

Note that the validity condition (A7) reads

$$(1+A)^2-|B|^2>0 \tag{4.8}$$

and is always fulfilled, as shown by Eqs. (2.20) and (2.21). We write also the explicit expressions of the coefficients (4.5)–(4.7) as follows:

$$\tilde{A}=\frac{\bar{n}(\bar{n}+1)}{\bar{n}^2+(\bar{n}+\tfrac{1}{2})[1+\cosh(2r)]} , \tag{4.9}$$

$$\widetilde{B}=-\frac{e^{i\varphi}(\bar{n}+\frac{1}{2})\sinh(2r)}{\bar{n}^{2}+(\bar{n}+\frac{1}{2})[1+\cosh(2r)]}\ , \tag{4.10}$$

$$\widetilde{C}=\frac{C[\frac{1}{2}+(\bar{n}+\frac{1}{2})\cosh(2r)]-C^{*}e^{i\varphi}(\bar{n}+\frac{1}{2})\sinh(2r)}{\bar{n}^{2}+(\bar{n}+\frac{1}{2})[1+\cosh(2r)]}\ . \tag{4.11}$$

In Eq. (4.11) one has to insert one of equations (2.11), in accordance with the state studied.

It is instructive to evaluate the quasiprobability densities as Fourier transforms [35]:

$$W(\beta,s)=\frac{1}{\pi}\int d^{2}\lambda\,\exp(\beta\lambda^{*}-\beta^{*}\lambda)\chi(\lambda,s)\quad(s\ \text{real})\ . \tag{4.12}$$

Substitution of Eq. (2.8) followed by application of Eq. (A8) gives

$$W(\beta,s)=\left[\left[\frac{1-s}{2}+A\right]^{2}-|B|^{2}\right]^{-1/2}\exp\left\{-\frac{\left[\frac{1-s}{2}+A\right]|\beta-C|^{2}+\frac{1}{2}[B^{*}(\beta-C)^{2}+B(\beta^{*}-C^{*})^{2}]}{\left[\frac{1-s}{2}+A\right]^{2}-|B|^{2}}\right\}\ , \tag{4.13}$$

while the requirement (A7) reads

$$\frac{1-s}{2}+A>|B| \tag{4.14}$$

or, explicitly,

$$s<(2\bar{n}+1)e^{-2r}\ . \tag{4.15}$$

The condition (4.15), which is independent of the coherent amplitude α, is satisfied for every value $s\le 0$. Hence, two of the three usual quasiprobability densities [36], the Q function

$$Q(\beta)=\frac{1}{\pi}W(\beta,-1) \tag{4.16}$$

and the Wigner function multiplied by π^{-1},

$$\frac{1}{\pi}W(\beta)=\frac{1}{\pi}W(\beta,0)\ , \tag{4.17}$$

exist for every value of the parameters $\bar{n}$ and r. On the contrary, according to the same general condition (4.15), the third one, namely, Glauber's P distribution,

$$P(\beta)=\frac{1}{\pi}W(\beta,1)\ , \tag{4.18}$$

exists only for values of the squeeze parameter not exceeding the threshold (3.17),

$$r<r_{s}\ . \tag{4.19}$$

This threshold is independent of α and decreases with the degree of purity (2.21) of the STS. Note that whenever the restriction (4.14) holds, the quasiprobability function (4.13) is positive definite.

We recall that the distribution $Q(\beta)$ is proportional to the average value of the density operator in a coherent state [37],

$$Q(\beta)=\frac{1}{\pi}\langle\beta|\rho|\beta\rangle\ , \tag{4.20}$$

and thereby is connected to the R function, Eq. (4.1),

$$Q(\beta)=\frac{1}{\pi}e^{-|\beta|^{2}}R(\beta^{*},\beta)\ . \tag{4.21}$$

Its explicit form for a DSTS,

$$Q(\beta)=Q(0)\exp\{-(1-\widetilde{A})|\beta|^{2}-\tfrac{1}{2}[\widetilde{B}(\beta^{*})^{2}+\widetilde{B}^{*}\beta^{2}]+\widetilde{C}\beta^{*}+\widetilde{C}^{*}\beta\}\ , \tag{4.22}$$

is quite similar to the CF $\chi_{N}(\lambda)$. The expectation value of the density operator (1.1) in the vacuum state,

$$\pi Q(0)=R(0,0)\ , \tag{4.23}$$

has the expression (4.4).

V. NUMBER-STATE REPRESENTATION OF THE DENSITY OPERATOR

The power-series expansion of the R function [1],

$$R(\beta^{*},\beta')=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\rho_{lm}\frac{1}{(l!m!)^{1/2}}(\beta^{*})^{l}(\beta')^{m}\ , \tag{5.1}$$

allows one to evaluate the density matrix elements ρ_{lm} in the Fock basis as derivatives of the R function. The similitude of expressions (2.8) and (4.3) leads to a formula of the same kind as Eq. (3.2):

$$\rho_{lm}=\frac{\pi Q(0)}{(l!m!)^{1/2}}\sum_{k=0}^{\min\{l,m\}}k!\begin{bmatrix}l\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}\widetilde{A}^{k}(\tfrac{1}{2}\widetilde{B})^{(l-k)/2}(\tfrac{1}{2}\widetilde{B}^{*})^{(m-k)/2}H_{l-k}((2\widetilde{B})^{-1/2}\widetilde{C})H_{m-k}((2\widetilde{B}^{*})^{-1/2}\widetilde{C}^{*})\ , \tag{5.2}$$

with $Q(0)$ given by Eqs. (4.23) and (4.4). As regards the square roots of the coefficient (4.10) and its complex conjugate, we make the same choice as in Eqs. (3.3) and (3.4), i.e.,

$$\widetilde{B}^{1/2}=ie^{i(\varphi/2)}|\widetilde{B}|^{1/2} \tag{5.3}$$

and

$$(\widetilde{B}^{*})^{1/2}=(\widetilde{B}^{1/2})^{*} . \tag{5.4}$$

In particular, the probability of finding l photons in a DSTS of the field is

$$\rho_{ll}=\pi Q(0)\widetilde{A}^{l}\sum_{q=0}^{l}\frac{1}{q!}\begin{bmatrix} l \\ q \end{bmatrix}\left[\frac{|\widetilde{B}|}{2\widetilde{A}}\right]^{q}|H_q((2\widetilde{B})^{-1/2}\widetilde{C})|^2 . \tag{5.5}$$

We mention an equivalent form of the l-photon probability, written in close analogy with Eq. (B9),

$$\begin{aligned}\rho_{ll}=\pi Q(0)(-1)^{l}2^{-2l}(\widetilde{A}+|\widetilde{B}|)^{l} &\times\sum_{k=0}^{l}\frac{1}{k!(l-k)!}\left[\frac{\widetilde{A}-|\widetilde{B}|}{\widetilde{A}+|\widetilde{B}|}\right]^{k} \\ &\times H_{2k}\left[i\frac{\mathrm{Im}(\widetilde{C}e^{-i(\varphi/2)})}{(\widetilde{A}-|\widetilde{B}|)^{1/2}}\right] \\ &\times H_{2l-2k}\left[i\frac{\mathrm{Re}(\widetilde{C}e^{-i(\varphi/2)})}{(\widetilde{A}+|\widetilde{B}|)^{1/2}}\right] ,\end{aligned} \tag{5.6}$$

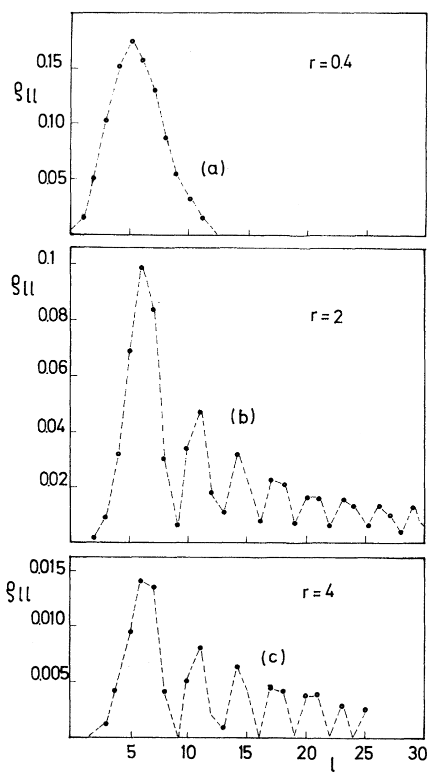

FIG. 3. Photon-number distribution of a DSTS with the parameters $|\alpha|^2=5$, $\bar{n}=0.4$, and $2\arg(\alpha)-\varphi=\pi$. The squeeze parameter is $r=0.4$ (curve a), $r=2$ (curve b), and $r=4$ (curve c).

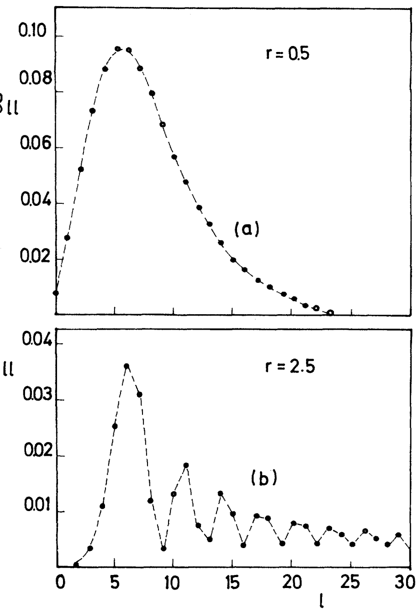

FIG. 4. As in Fig. 3, but for the parameters $|\alpha|^2=5$, $\bar{n}=2$, and $2\arg(\alpha)-\varphi=\pi$. The squeeze parameter is $r=0.5$ (curve a) and $r=2.5$ (curve b).

and its limit for $|B|=A$,

$$\begin{aligned}\rho_{ll}=\pi Q(0)(-\tfrac{1}{2}\widetilde{A})^{l} &\times\sum_{k=0}^{l}\frac{1}{k!(l-k)!}\left[-\frac{2}{\widetilde{A}}\right]^{k}[\mathrm{Im}(\widetilde{C}e^{-i(\varphi/2)})]^{2k} \\ &\times H_{2l-2k}(i(2\widetilde{A})^{-1/2}\mathrm{Re}(\widetilde{C}e^{-i(\varphi/2)})) .\end{aligned} \tag{5.7}$$

Figures 3–5 present the photon-number distribution ρ_{ll} plotted for several values of the thermal and coherent mean occupancies $\bar{n}$ and $|\alpha|^2$, respectively. It is quite remarkable that for large values of the squeeze parameter r the function ρ_{ll} is oscillating. Even for strong thermal noise, oscillations occur, as shown in Fig. 5, where the photon-number distribution is plotted for $\bar{n}=|\alpha|^2=10$

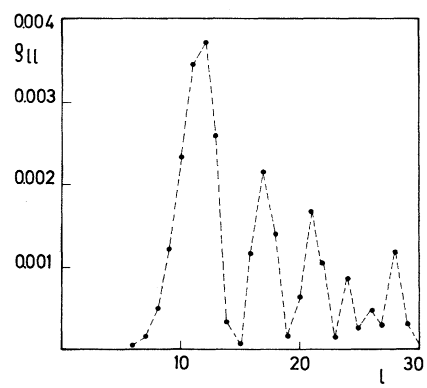

FIG. 5. As in Fig. 3, but for the parameters $|\alpha|^2=10$, $\bar{n}=10$, and $r=4$. The phase difference is again $2\arg(\alpha)-\varphi=\pi$.

and a large squeeze parameter $r=4$. Our figures are in full agreement with similar ones reported in Ref. [7]. This is not the case with the numerical calculations of ρ_{ll} performed by Vourdas and Weiner [5]. These authors came to the conclusion that small amounts of thermal noise destroy the oscillatory character of the photon-number distribution. In our opinion they did not find oscillations of ρ_{ll} because they chose too small values of the squeeze parameter. Also, our Figs. 3(a) and 4(a) display a nonoscillating behavior of ρ_{ll} at small values of r. As a matter of fact, oscillations in photon-number distribution set in at values of r exceeding the threshold r_s [Eq. (3.17)]. On the other hand, Glauber's P representation does not exist for $r>r_s$. Thus, the oscillations of ρ_{ll} can be connected with the nonclassical character of the field state.

To get further insight, we evaluate the generating function of the photon-number distribution,

$$G(s)=\sum_{l=0}^{\infty}\rho_{ll}s^l \quad (|s|\le 1) . \tag{5.8}$$

Substitution of Eq. (5.5) into Eq. (5.8), followed by the interchange of summations, gives

$$G(s)=\pi Q(0)\sum_{q=0}^{\infty}\frac{1}{(q!)^2}\left[\frac{|\tilde{B}|}{2\tilde{A}}\right]^q|H_q[(2\tilde{B})^{-1/2}\tilde{C}]|^2 \times\sum_{l=q}^{\infty}\frac{l!}{(l-q)!}(s\tilde{A})^l . \tag{5.9}$$

After performing the last summation in Eq. (5.9), we use Mehler's formula [38] to carry out the remaining sum. The result is

$$G(s)=\pi Q(0)[(1-s\tilde{A})^2-s^2|\tilde{B}|^2]^{-1/2}\times\exp\left\{\frac{s|\tilde{C}|^2-s^2[\tilde{A}|\tilde{C}|^2+\mathrm{Re}(\tilde{B}^*\tilde{C}^2)]}{(1-s\tilde{A})^2-s^2|\tilde{B}|^2}\right\} \quad (|s|\le 1) . \tag{5.10}$$

If we write explicitly in Eq. (5.10) the coefficients (4.5)–(4.7) and the factor (4.23), we finally get the alternative expression

$$G(s)=\{[1-(s-1)A]^2-(s-1)^2|B|^2\}^{-1/2}\exp\left\{\frac{(s-1)|C|^2-(s-1)^2[A|C|^2+\mathrm{Re}(B^*C^2)]}{[1-(s-1)A]^2-(s-1)^2|B|^2}\right\} \quad (s\le 1) . \tag{5.11}$$

Equation (5.11) may be obtained formally from Eq. (5.10) by operating simultaneously the following changes:

$$s\to s-1,\quad \tilde{A}\to A,\quad \tilde{B}\to B,\quad \tilde{C}\to C,\quad \pi Q(0)\to 1 . \tag{5.12}$$

Since the correlation functions (3.5) are the factorial moments of the photon-number distribution, the following Taylor expansion holds:

$$G(s)=\sum_{l=0}^{\infty}\frac{1}{l!}(s-1)^l\langle(a^\dagger)^l a^l\rangle . \tag{5.13}$$

Equations (5.8) and (5.13) on the one hand, and the similitude of expressions (5.5) and (3.5), on the other hand, account for the existence of the transformation (5.12) which preserves the form of the generating function.

Now we discuss briefly three particular cases.

(1) Squeezed coherent states. Taking $\bar{n}=0$ in Eqs. (4.9)–(4.11) and then substituting Eq. (2.11b) into Eq. (4.11), we get

$$\tilde{A}=0,\quad \tilde{B}=-e^{i\varphi}\tanh r,\quad \tilde{C}=\frac{\alpha}{\cosh r} . \tag{5.14}$$

Accordingly, for a SCS, the sum (5.2) reduces to its first term which reads

$$\rho_{lm}=\frac{1}{\cosh r}\exp[-|\alpha|^2-\mathrm{Re}(\alpha^2e^{-i\varphi})\tanh r]\frac{(ie^{i(\varphi/2)})^{l-m}}{[l!m!]^{1/2}}(\tfrac{1}{2}\tanh r)^{(l+m)/2}H_l\left[-\frac{i\alpha e^{-i(\varphi/2)}}{[\sinh(2r)]^{1/2}}\right]H_m\left[\frac{i\alpha^* e^{i(\varphi/2)}}{[\sinh(2r)]^{1/2}}\right] . \tag{5.15}$$

The expression of the probability for l photons coincides with Yuen's result [39].

(2) Displaced thermal states. For a DTS, Eqs. (4.9)–(4.11) become

$$\tilde{A}=\frac{\bar{n}}{\bar{n}+1},\quad \tilde{B}=0,\quad \tilde{C}=\frac{\alpha}{\bar{n}+1} . \tag{5.16}$$

Hence, we get a formula analogous to Eq. (3.12):

$$\rho_{lm}=\left[\frac{l!}{m!}\right]^{1/2}\exp\left[-\frac{|\alpha|^2}{\bar{n}+1}\right](\alpha^*)^{m-l}\frac{\bar{n}^{\,l}}{(\bar{n}+1)^{m+1}}L_l^{(m-l)}\left[-\frac{|\alpha|^2}{\bar{n}(\bar{n}+1)}\right]$$
$$=\left[\frac{m!}{l!}\right]^{1/2}\exp\left[-\frac{|\alpha|^2}{\bar{n}+1}\right]\alpha^{l-m}\frac{\bar{n}^{\,m}}{(\bar{n}+1)^{l+1}}L_l^{(l-m)}\left[-\frac{|\alpha|^2}{\bar{n}(\bar{n}+1)}\right] . \tag{5.17}$$

Equation (5.17), which includes the result of Lachs for $l=m$ [12], has been established by Mollow and Glauber in their classic work on parametric amplification [13].

(3) Squeezed thermal states. Equations (2.11) and (4.11) show that for a STS, $\tilde{C}=0$. The discussion is similar to that concerning the correlation functions of a STS in Sec. III. Accordingly, we get, in close analogy with Eqs. (3.13), the following density matrix:

$$\rho_{lm}=\begin{cases}[(1+A)^2-|B|^2]^{-1/2}\left[\dfrac{m!}{l!}\right]^{1/2}e^{i[(l-m)/2]\varphi}(\tilde{A}^2-|\tilde{B}|^2)^{(l+m)/4}P_{(l+m)/2}^{(l-m)/2}\left[\dfrac{\tilde{A}}{(\tilde{A}^2-|\tilde{B}|^2)^{1/2}}\right], & (l+m \text{ even}) \quad (5.18a)\\ 0 \quad (l+m \text{ odd}) . & (5.18b)\end{cases}$$

In Eq. (5.18a), $P_L^M(z)$ is an associated Legendre function of the first kind [30]. We make the same assumption as in Sec. III about the square roots: They are positive for positive arguments and

$$(\tilde{A}^2-|\tilde{B}|^2)^{1/2}=i(|\tilde{B}|^2-\tilde{A}^2)^{1/2} \text{ if } |\tilde{B}|>\tilde{A} . \tag{5.19}$$

This convention coincides with that of Ref. [11]. Making use of Eqs. (4.5), (4.6), (2.9) and (2.10) our result (5.18) can be written explicitly as

$$\rho_{lm}=\begin{cases}\left[\dfrac{m!}{l!}\right]^{1/2}e^{i[(l-m)/2]\varphi}\{(\bar{n}+\tfrac{1}{2})[\cosh(2r_s)+\cosh(2r)]\}^{-1/2}\\ \left[\dfrac{[\sinh(2r_s)]^2-[\sinh(2r)]^2}{\cosh(2r_s)+\cosh(2r)}\right]^{(l+m)/4}P_{(l+m)/2}^{(l-m)/2}\left[\left[1-\left[\dfrac{\sinh(2r)}{\sinh(2r_s)}\right]^2\right]^{-1/2}\right] & \text{for } (l+m) \text{ even}\\ 0 \quad \text{for } (l+m) \text{ odd} .\end{cases} \tag{5.20}$$

The l-photon probability ρ_{ll}, which is proportional to a Legendre polynomial of degree l, has been derived by a different method and extensively discussed previously [11]. As already noted in Ref. [11], for the diagonal case, the value (3.17) of the squeeze parameter is a special one regarding the density-matrix elements (5.20). Indeed, the argument of the Legendre function in Eq. (5.20), which is positive and greater than unity for $r<r_s$, becomes pure imaginary for $r>r_s$. It is noticeable that the situation facing us here is exactly the same as for the correlation function (3.16).

VI. COORDINATE AND MOMENTUM REPRESENTATIONS OF THE DENSITY OPERATOR

We use the quadrature operators

$$\hat{X}_1=\tfrac{1}{2}(a+a^\dagger), \quad \hat{X}_2=\frac{1}{2i}(a-a^\dagger) . \tag{6.1}$$

For a harmonic oscillator of mass M and classical frequency ω, the quadratures $\hat{X}_1$ and $\hat{X}_2$ are proportional to the coordinate and momentum operator, respectively,

$$\hat{X}_1=\left[\frac{M\omega}{2\hbar}\right]^{1/2}\hat{q}, \quad \hat{X}_2=\frac{1}{(2M\hbar\omega)^{1/2}}\hat{p} . \tag{6.2}$$

In spite of their purely mechanical meaning, we shall refer in what follows to the observables q and p rather than to the quadratures X_1 and X_2, which are suited for the radiation field.

We evaluate the coordinate and momentum density matrices starting from the R function (4.3) and taking into account the relation (4.1):

$$\langle q|\rho|q'\rangle=\frac{1}{\pi^2}\int\int d^2\beta\, d^2\beta'\langle q|\beta\rangle\langle\beta|\rho|\beta'\rangle\langle\beta'|q'\rangle , \tag{6.3a}$$

$$\langle p|\rho|p'\rangle=\frac{1}{\pi^2}\int\int d^2\beta\, d^2\beta'\langle p|\beta\rangle\langle\beta|\rho|\beta'\rangle\langle\beta'|p'\rangle . \tag{6.3b}$$

In Eqs. (6.3) we also have to insert the wave functions describing a coherent state of the harmonic oscillator in the coordinate and momentum representations [40]:

$$\langle q|\beta\rangle=\left[\frac{M\omega}{\pi\hbar}\right]^{1/4}\exp\left\{-\left[\left[\frac{M\omega}{2\hbar}\right]^{1/2}q-\beta\right]^2+\tfrac{1}{2}\beta(\beta-\beta^*)-i\left[\varphi_0+\frac{\omega t}{2}\right]\right\} \tag{6.4a}$$

$$\langle p|\beta\rangle=\left[\frac{1}{\pi M\hbar\omega}\right]^{1/4}\exp\left\{-\left[\left[\frac{1}{2M\hbar\omega}\right]^{1/2}p+i\beta\right]^2-\tfrac{1}{2}\beta(\beta+\beta^*)-i\left[\varphi_0+\frac{\omega t}{2}\right]\right\} . \tag{6.4b}$$

Thus, we have to carry out quadruple integrals of the type (A1). The convergence conditions (A4) are satisfied for a DSTS. Application of Eq. (A5) and insertion of the expression (4.4) provide the results

$$\langle q|\rho|q'\rangle=\left[\frac{M\omega}{\pi\hbar}\right]^{1/2}[1+2(A-\mathrm{Re}B)]^{-1/2}\exp\left[-\frac{2(\mathrm{Re}C)^2}{1+2(A-\mathrm{Re}B)}\right]$$
$$\times\exp\left[-\frac{1}{1+2(A-\mathrm{Re}B)}\left[\frac{M\omega}{4\hbar}\{(q+q')^2+4[(A+\tfrac{1}{2})^2-|B|^2](q-q')^2+4i\,\mathrm{Im}B(q+q')(q-q')\}\right.\right.$$
$$\left.\left.-\left[\frac{M\omega}{2\hbar}\right]^{1/2}2\{\mathrm{Re}C(q+q')+2i[(A+\tfrac{1}{2})\mathrm{Im}C+\mathrm{Im}(BC^*)](q-q')\}\right]\right] \tag{6.5a}$$

and

$$\langle p|\rho|p'\rangle=(\pi M\hbar\omega)^{-1/2}[1+2(A+\mathrm{Re}B)]^{-1/2}\exp\left[-\frac{2(\mathrm{Im}C)^2}{1+2(A+\mathrm{Re}B)}\right]$$
$$\times\exp\left[-\frac{1}{1+2(A+\mathrm{Re}B)}\left[\frac{1}{4M\hbar\omega}\{(p+p')^2+4[(A+\tfrac{1}{2})^2-|B|^2](p-p')^2-4i\,\mathrm{Im}B(p+p')(p-p')\}\right.\right.$$
$$\left.\left.+\frac{2}{(2M\hbar\omega)^{1/2}}\{-\mathrm{Im}C(p+p')+2i[(A+\tfrac{1}{2})\mathrm{Re}C+\mathrm{Re}(BC^*)](p-p')\}\right]\right]. \tag{6.5b}$$

In particular, for a DSTS, the probability densities of the coordinate and momentum are Gaussian distributions:

$$\langle q|\rho|q\rangle=\left[\frac{M\omega}{2\pi\hbar}\right]^{1/2}(\tfrac{1}{2}+A-\mathrm{Re}B)^{-1/2}\exp\left\{-\frac{\left[\left[\frac{M\omega}{2\hbar}\right]^{1/2}q-\mathrm{Re}C\right]^2}{\frac{1}{2}+A-\mathrm{Re}B}\right\} \tag{6.6a}$$

and

$$\langle p|\rho|p\rangle=(2\pi M\hbar\omega)^{-1/2}(\tfrac{1}{2}+A+\mathrm{Re}B)^{-1/2}\exp\left\{-\frac{\left[\frac{1}{(2M\hbar\omega)^{1/2}}p-\mathrm{Im}C\right]^2}{\frac{1}{2}+A+\mathrm{Re}B}\right\}. \tag{6.6b}$$

In the simpler case of a DTS, their form has been found by Lachs [41].

Equations (6.6) give also the distribution functions of the quadratures (6.2) for a DSTS of the radiation field:

$$P(X_1)=[\pi(\tfrac{1}{2}+A-\mathrm{Re}B)]^{-1/2}\exp\left[-\frac{(X_1-\mathrm{Re}C)^2}{\frac{1}{2}+A-\mathrm{Re}B}\right] \tag{6.7a}$$

and

$$P(X_2)=[\pi(\tfrac{1}{2}+A+\mathrm{Re}B)]^{-1/2}\exp\left[-\frac{(X_2-\mathrm{Im}C)^2}{\frac{1}{2}+A+\mathrm{Re}B}\right]. \tag{6.7b}$$

We finally note that the expectation values of these quadratures,

$$\langle X_1\rangle=\mathrm{Re}C,\quad \langle X_2\rangle=\mathrm{Im}C\ , \tag{6.8}$$

do not depend on the average photon number $\bar{n}$ of the thermal field, whereas their variances,

$$(\Delta X_1)^2=\tfrac{1}{2}(\tfrac{1}{2}+A-\mathrm{Re}B),\quad (\Delta X_2)^2=\tfrac{1}{2}(\tfrac{1}{2}+A+\mathrm{Re}B) \tag{6.9}$$

are independent of the coherent amplitude α. The probability density of each quadrature (6.2) is entirely determined by the corresponding mean values (6.8) and (6.9),

$$P(X)=[2\pi(\Delta X)^2]^{-1/2}\exp\left[-\frac{(\delta X)^2}{2(\Delta X)^2}\right]. \tag{6.10}$$

In Eq. (6.10) X stands for X_1 or X_2 and δX denotes the deviation of a quadrature from its average value,

$$\delta X\equiv X-\langle X\rangle\ . \tag{6.11}$$

VII. SQUEEZING

The moments of a random variable X having a Gaussian distribution function (6.10) depend only on the variance $(\Delta X)^2$ as

$$\langle(\delta X)^N\rangle=\begin{cases}(N-1)!![(\Delta X)^2)]^{N/2} & (N\text{ even})\\ 0\quad (N\text{ odd})\ .\end{cases} \tag{7.1}$$

As stated by Hong and Mandel [42], the field in a definite state is squeezed to any even order N in the quadrature X_1 if the Nth-order moment $\langle(\delta X_1)^N\rangle$ is smaller than its value for a coherent state:

$$\langle (\delta X_2)^N \rangle < (N-1)!!2^{-N} \quad (N \text{ even}) . \tag{7.2}$$

In the case of a DSTS, according to Eq. (7.1) the existence of Nth-order squeezing reduces to a condition independent of N,

$$(\Delta X_1)^2 < \tfrac{1}{4} . \tag{7.3}$$

We substitute the coefficients (2.9) and (2.10) in the first equation (6.9) to obtain

$$(\Delta X_1)^2 = \tfrac{1}{4}(2\bar{n}+1)[\cosh(2r)+\cos\varphi \sinh(2r)] . \tag{7.4}$$

When the variance (7.4), as a function of the phase φ, achieves its minimum at $\varphi=\pi$, the squeezing sets in at the threshold (3.17) of the squeeze parameter. Thus, we get effective squeezing to any even order N, as soon as the P function no longer exists, i.e., for $r > r_s$.

We evaluate now the normalized mixed-order correlation function

$$g^{(l,m)}(0) = \frac{\langle (a^\dagger)^l a^m \rangle}{(\langle a^\dagger a \rangle)^{(l+m)/2}} \tag{7.5}$$

in the strong-squeezing limit ($r \to \infty$), when $A \sim |B| \sim \frac{1}{4}(2\bar{n}+1)e^{2r}$, as shown by Eqs. (2.9) and (2.10).

In the case of a DSTS,

$$(2B)^{-1/2}C \sim \frac{\alpha e^{-i(\varphi/2)}e^{-r}}{\mathrm{Re}(\alpha e^{-i(\varphi/2)})} \to 0 , \tag{7.6a}$$

so that we have to take the limit $C=0$ in Eq. (3.2). This value corresponds to a STS, whose correlation functions are given by Eqs. (3.13). The limit $A=|B|$ of Eqs. (3.13) [30] leads to the result

$$g^{(l,m)}(0) = \begin{cases} (l+m-1)!! e^{-i[(l-m)/2]\varphi} & (l+m \text{ even}) \\ 0 \quad (l+m \text{ odd}) , \end{cases} \tag{7.7a}$$

which is independent of the displacement parameter α.

In the case of a SDTS,

$$(2B)^{-1/2}C \sim \frac{-i\,\mathrm{Re}(\alpha e^{-i(\varphi/2)})}{(\bar{n}+\frac{1}{2})^{1/2}} \equiv -i\xi . \tag{7.6b}$$

Due to the equality $|B|=A$, we are in a position to apply a summation formula for products of Hermite polynomials [43] yielding the strong-squeezing limit for a SDTS,

$$g^{(l,m)}(0) = e^{-i[(l-m)/2]\varphi} \frac{H_{l+m}(i\xi)}{[H_2(i\xi)]^{(l+m)/2}} . \tag{7.7b}$$

Taking note of the value in origin of a Hermite polynomial [44], we remark that Eq. (7.7b) coincides with Eq. (7.7a) only for $\xi=0$, which corresponds simply to a STS.

Finally, we write the strong-squeezing limit of the lth-order degree of coherence (3.7) in both cases:

$$g^{(l)}(0) = (2l-1)!! \quad (\text{DSTS}) \tag{7.8a}$$

$$g^{(l)}(0) = \frac{H_{2l}(i\xi)}{[H_2(i\xi)]^l} \quad (\text{SDTS}) . \tag{7.8b}$$

VIII. SUMMARY

We have analyzed a broad class of quantum states of a single-mode radiation field (DSTS's and SDTS's). It includes, as particular cases, pure and mixed states studied a long time ago (CS's, TS's, DTS's, and SCS's) or more recently (STS's). The starting point of our study is Glauber's CF $\chi(\lambda)$, which is a Gaussian distribution of two real variables, determined by three coefficients, Eqs. (2.9)–(2.11).

We have derived three continuous representations of the density operator, namely, the coherent-state, coordinate, and momentum representations, all of them being exponentials with algebraic quadratic exponents. The correlation functions, Eq. (3.2), and the elements of the number-state density matrix, Eq. (5.2), have similar expressions. We have also evaluated the generating function of the photon-number distribution, Eqs. (5.10) and (5.11). The onset of squeezing takes place, to any even order, at a critical value r_s, Eq. (3.17), of the squeeze parameter, irrespective of the coherent amplitude. At the same threshold, Glauber's P function ceases existing. When the squeezing is strong enough, the photon-number distribution displays nonclassical oscillations, regardless of the amount of noise.

APPENDIX A: AN INFINITE INTEGRAL

In this appendix we use Einstein's summation convention. Let us consider the infinite integral

$$I(A,B) \equiv \int d^n x \exp\{b_j x_j - \tfrac{1}{2}a_{kl}x_k x_l\} , \tag{A1}$$

where A is a $n \times n$ symmetric nonsingular matrix with complex elements,

$$a_{kl} \equiv c_{kl} + i d_{kl} \quad (c_{kl}, d_{kl} \text{ real}) , \tag{A2}$$

and B is a row matrix with n complex elements b_j. The n real variables of integration x_j form a column matrix X. The necessary and sufficient condition for the convergence of the integral (A1) is the absolute integrability of the exponential. This is realized if the quadratic form whose matrix is the real part C of the matrix A is positively definite:

$$c_{kl}x_k x_l > 0 \quad \text{for } X \neq 0 . \tag{A3}$$

The inequality (A3) implies that all the principal minors of the matrix C should be positive:

$$\det C^{(l)} > 0, \quad (l=1,2,\ldots,n) , \tag{A4}$$

with $C^{(1)}=c_{11}$ and $C^{(n)}=C$. The integral (A1) generalizes slightly a similar one discussed by Landau and Lifshitz [45]. Following closely their line of reasoning, we get the final result:

$$I(A,B) = (2\pi)^{n/2}(\det A)^{-1/2}\exp[\tfrac{1}{2}(A^{-1})_{kl}b_k b_l] . \tag{A5}$$

Now we apply Eq. (A5) to evaluate the integral

$$I \equiv \int d^2\lambda \exp[-K|\lambda|^2 - \tfrac{1}{2}L(\lambda^*)^2 - \tfrac{1}{2}L'\lambda^2 - M\lambda^* - M'\lambda] . \tag{A6}$$

The convergence conditions (A4) require that the complex numbers K, L, and L' should fulfill the inequality

$$\mathrm{Re}K > \tfrac{1}{2}|L + L'^*| , \tag{A7}$$

while M and M' may be arbitrary. The result (A5) reads, in this particular case,

$$I=\pi(K^2-LL')^{-1/2}\exp\left\{\frac{KMM'-\frac{1}{2}[L(M')^2+L'M^2]}{K^2-LL'}\right\}. \tag{A8}$$

APPENDIX B: ALTERNATIVE FORM OF THE lTH-ORDER CORRELATION FUNCTION

Starting with the generating function of the Hermite polynomials [26], an integral representation of these polynomials can be obtained:

$$H_n(\xi)=\frac{2^n}{\sqrt{\pi}}\int_{-\infty}^{\infty}dv\, e^{-v^2}(\xi\pm iv)^n . \tag{B1}$$

We use Eq. (B1) to carry out the finite summation in our formula (3.5):

$$\langle (a^\dagger)^l a^l\rangle = l!A^l\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}du\,dv e^{-u^2-v^2}L_l^{(0)}\left[-\frac{1}{A}f^*(u)f(v)\right], \tag{B2}$$

where we have denoted

$$f(v)\equiv C-i(2B)^{1/2}v \tag{B3}$$

and $L_l^{(0)}$ is a Laguerre polynomial [46],

$$L_n^{(\alpha)}(z)=\sum_{\nu=0}^{n}\frac{1}{(n-\nu)!}\frac{\Gamma(n+\alpha+1)}{\Gamma(\nu+\alpha+1)}\frac{(-z)^\nu}{\nu!} . \tag{B4}$$

Taking note of Eq. (B3), we get the formula

$$-\frac{1}{A}f^*(u)f(v)=\left[\left[\frac{|B|}{2A}\right]^{1/2}(u-v)-iA^{-1/2}\mathrm{Im}(Ce^{-i(\varphi/2)})\right]^2+\left\{i\left[\left[\frac{|B|}{2A}\right]^{1/2}(u+v)+A^{-1/2}\mathrm{Re}(Ce^{-i(\varphi/2)})\right]\right\}^2, \tag{B5}$$

which suggests an orthogonal transformation of the variables of integration:

$$w=\frac{1}{\sqrt{2}}(u-v),\quad z=\frac{1}{\sqrt{2}}(u+v) . \tag{B6}$$

This change of variables gives the equality

$$\langle (a^\dagger)^l a^l\rangle = l!A^l\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dw\,dz\, e^{-(w^2+z^2)} \times L_l^{(0)}\left[\left[\left[\frac{|B|}{A}\right]^{1/2}w-iA^{-1/2}\mathrm{Im}(Ce^{-i(\varphi/2)})\right]^2 + \left[i\left[\frac{|B|}{A}\right]^{1/2}z+iA^{-1/2}\mathrm{Re}(Ce^{-i(\varphi/2)})\right]^2\right] . \tag{B7}$$

The finite decomposition of a Laguerre polynomial [47] allows us to write the correlation function (B7) as a sum of products of two integrals:

$$\langle (a^\dagger)^l a^l\rangle=(-1)^l\frac{A^l}{2^{2l}}\sum_{k=0}^{l}\begin{bmatrix} l\\ k\end{bmatrix}\frac{1}{\pi}\int_{-\infty}^{\infty}dwe^{-w^2}H_{2k}\left[\left[\frac{|B|}{A}\right]^{1/2}w-iA^{-1/2}\mathrm{Im}(Ce^{-i(\varphi/2)})\right] \times\int_{-\infty}^{\infty}dz\, e^{-z^2}H_{2l-2k}\left[i\left[\frac{|B|}{A}\right]^{1/2}z+iA^{-1/2}\mathrm{Re}(Ce^{-i(\varphi/2)})\right] . \tag{B8}$$

The integrals in Eq. (B8) can be performed [48] and the final expression of the correlation function is obtained:

$$\langle (a^\dagger)^l a^l\rangle=(-1)^l2^{-2l}(A+|B|)^l\sum_{k=0}^{l}\begin{bmatrix} l\\ k\end{bmatrix}\left[\frac{A-|B|}{A+|B|}\right]^k H_{2k}\left[\frac{i\,\mathrm{Im}(Ce^{-i(\varphi/2)})}{(A-|B|)^{1/2}}\right]H_{2l-2k}\left[\frac{i\,\mathrm{Re}(Ce^{-i(\varphi/2)})}{(A+|B|)^{1/2}}\right], \tag{B9}$$

while the limit $|B|=A$ leads to the formula

$$\langle (a^\dagger)^l a^l\rangle=\left[-\frac{A}{2}\right]^l\sum_{k=0}^{l}\begin{bmatrix} l\\ k\end{bmatrix}\left[-\frac{2}{A}\right]^k\left[\mathrm{Im}(Ce^{-i(\varphi/2)})\right]^{2k}H_{2l-2k}(i(2A)^{-1/2}\mathrm{Re}(Ce^{-i(\varphi/2)})) . \tag{B10}$$

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