Three-dimensional relativistic model of a bound particle in an intense laser field

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We analyze a three-dimensional model of a Klein-Gordon particle in a short-range separable potential and interacting with an intense plane-wave electromagnetic field. In the specific case of the circular polarization of the radiation, we find an *exact* solution of the Klein-Gordon equation of the system and derive analytic expressions for obtaining the total and partial rates of particle ejection by N-photon absorption, the energy spectrum of the ejected particle, as well as the amplitudes for stimulated bremsstrahlung and its inverse.

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I. INTRODUCTION

Recent developments in high-intensity lasers [1] make it likely that in the near future the intensities will approach and exceed the limit in which the vibrational (quiver) motion of a free electron in the field would be comparable to its rest energy mc^2. In this situation it is necessary to treat the laser-atom interaction problem both relativistically *and* nonperturbatively. Although exact solution of the Dirac and Klein-Gordon equations for a *free* electron in a plane-wave electromagnetic field is known for a long time [2–5], there appears to be no known exact solution of these equations for a relativistic particle interacting *simultaneously* with a binding potential and a plane-wave electromagnetic field. It would be desirable, therefore, to be able to construct and to find exact solutions of nontrivial relativistic model problems. Such solutions would not only be of intrinsic mathematical interest but could give useful qualitative insights regarding the physical processes such as ionization, detachment, or breakup processes, as well as stimulated bremsstrahlung, inverse bremsstrahlung, and related radiative scattering processes. They can also help in developing and testing approximation methods which would be necessary to tackle the real systems. The difficulty of finding exact solutions of appropriate model problems in the relativistic case is not surprising, especially if one recalls the scarcity of such soluble models even in the nonrelativistic Schrödinger case. There are two well-investigated soluble models in the Schrödinger case for a charged particle interacting with a plane-wave field (in the dipole approximation): a δ potential

$$\hat{V}(\mathbf{r}) = V_0\,\delta^{(3)}(\mathbf{r})\cdot r\,\frac{\partial}{\partial r} \tag{1}$$

and a separable binding potential

$$\hat{V} = V_0|\phi\rangle\langle\phi|. \tag{2}$$

One may expect that a relativistic generalization of these models can be solved exactly too. Since the Dirac electron introduces the additional mathematical complication due to the direction of spin, in this work we shall restrict ourselves to the investigation of the Klein-Gordon (KG) equation only (see, however, Ref. [6]). A first observation in this context is that the δ-potential models are not applicable to the KG equation since the square of the potential appears in this case. But we shall show below that the separable potential model can, in fact, be generalized to obtain an exact solution of the KG equation in the case of the circularly polarized light. The reason for the choice of the circular polarization is the same as in nonrelativistic case [7], namely, the existence of the planar symmetry of the radiation compared to the axial symmetry in the case of the linear polarization. This leads to an azimuthal invariance in the "photon space" in the presence of an s-type potential (such as δ potential or a separable S potential). This will be shown explicitly below. The theory of separable potentials has been developed extensively and employed in the context of nuclear reaction theory in the past [8]. The potential $\hat{V}$ above, being a rank-one potential, supports only one bound state (and all the continuum states). This is analogous to the case of the δ potential, which also supports one bound state. Such a potential may, therefore, be used to model quantum systems which have effectively only one bound state, e.g., the hydrogen negative ion H^- or the deuteron nucleus. Introduction of higher-rank separable potentials can accommodate any number of bound states and has been used for systematic variational treatment of the problem in the nonrelativistic case [9].

The three-dimensional (3D) relativistic model problem to be investigated in this paper is defined by the KG equation ($\hbar = c = 1$):

$$\{(i\partial_t - \hat{V})^2 - [\hat{\mathbf{p}} - e\mathbf{A}(\mathbf{x},t)]^2 - m^2\}\Psi(t) = 0\,. \tag{3}$$

We choose a short-range separable pseudopotential with one bound state

$$\hat{V} = V_0|\tilde{\phi}\rangle\langle\tilde{\phi}| , \tag{4}$$

where

$$\tilde{\phi}(\mathbf{x}) = N_0 \frac{1}{\mathbf{x}} e^{-\lambda \mathbf{x}} , \quad N_0 = \sqrt{\frac{\lambda}{2\pi}}, \tag{5}$$

and the vector potential $\mathbf{A}(\mathbf{x}, t)$ is chosen to be circularly polarized

$$\mathbf{A}(x) = A_0[\mathbf{e}_1 \cos(k \cdot x + \delta) - \mathbf{e}_2 \sin(k \cdot x + \delta)] . \tag{6}$$

x and k are four-coordinates and -momenta, respectively. With the normalization (5) we have

$$\langle\tilde{\phi}|\tilde{\phi}\rangle = 1 . \tag{7}$$

The solution of (3) obtained here will be then used to derive analytic expressions for the rate of multiphoton detachment, radiative scattering, multiphoton bremsstrahlung, and inverse bremsstrahlung. We shall also derive the corresponding expression of the above-threshold-detachment spectrum (i.e., the probability distribution of the energy of the ejected electron in the continuum by absorption of any number of photons).

II. KLEIN-GORDON PARTICLE IN A SEPARABLE POTENTIAL

Before proceeding further, in this section we consider the solution of the model problem in the absence of the field and fix the parameters of the model potential (5) in order to be able to reproduce the (unique) bound-state energy of the system of interest. The KG equation to be satisfied by the wave function in the absence of the field ($\mathbf{A} = 0$) is from (3)

$$\left[(i\partial_t - V_0|\tilde{\phi}\rangle\langle\tilde{\phi}|)^2 - \hat{\mathbf{p}}^{\,2} - m^2\right] |\Psi(t)\rangle = 0 . \tag{8}$$

For a stationary solution $|\Psi_E\rangle$, where $|\Psi(t)\rangle = e^{-iEt}|\Psi_E\rangle$, we get

$$(E^2 - \hat{\mathbf{p}}^{\,2} - m^2)|\Psi_E\rangle = V_0(2E - V_0)|\tilde{\phi}\rangle\langle\tilde{\phi}|\Psi_E\rangle . \tag{9}$$

This leads to the integral equation for Ψ_E in the coordinate representation (in which $\hat{V}$ is an integral operator)

$$\Psi_E(\mathbf{x}) = \int d^3x' \int d^3z \, \triangle(E; \mathbf{x} - \mathbf{x}') V_0(2E - V_0) \times \tilde{\phi}(\mathbf{x}')\tilde{\phi}(\mathbf{z})\Psi_E(\mathbf{z}) , \tag{10}$$

where the free-particle Green's function

$$\triangle(E; \mathbf{x} - \mathbf{x}') = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \times \exp\left[-\sqrt{m^2 - E^2}\, |\mathbf{x} - \mathbf{x}'|\right] . \tag{11}$$

The eigenvalue equation for the energy, obtained by projecting both sides of Eq. (10) onto $\tilde{\phi}$ and canceling the common factor, is

$$1 = V_0(2E - V_0) \int d^3x \int d^3x' \tilde{\phi}(\mathbf{x}) \triangle(E; \mathbf{x} - \mathbf{x}') \tilde{\phi}(\mathbf{x}') . \tag{12}$$

Our potential defined by (4) and (5) contains two free parameters: λ and V_0. To be specific, we will fix λ to be equal to $\sqrt{m^2 - E_0^2}$, where E_0 is the solution of (12). This choice of λ simplifies (12) into

$$1 = -\frac{V_0(2E_0 - V_0)}{4(m^2 - E_0^2)} , \tag{13}$$

so that

$$E_0 = \sqrt{m^2 - \frac{3}{16}V_0^2} + \frac{1}{4}V_0 . \tag{14}$$

For V_0 negative and $\frac{3}{16}V_0^2 < m^2$ we have $E_0 < m$ and the solution corresponds to a bound state.

The corresponding wave function of the (unique) bound state is

$$\begin{aligned}\Psi(\mathbf{x}, t) &= e^{-iE_0 t}\Psi_{E_0}(\mathbf{x}) \\ &= \frac{(m^2 - E_0^2)^{3/4}}{\sqrt{2\pi E_0}} \\ &\quad \times \exp\left[-\sqrt{m^2 - E_0^2}\, |\mathbf{x}|\right] e^{-iE_0 t} .\end{aligned} \tag{15}$$

III. KLEIN-GORDON PARTICLE IN A LASER FIELD

Solutions of a free relativistic particle in plane-wave electromagnetic field are well known in both Dirac and Klein-Gordon cases (the so-called Volkov solutions) [2–5]. For our present purpose we shall instead require and derive an alternative representation (Floquet representation) of the KG–Volkov Green's function which will allow us to make further progress in obtaining the full solution of the model problem defined by (3).

The Green's function [in the absence of $\hat{V}$ in (3)] satisfies the equation

$$[(i\partial^\mu - eA^\mu)(i\partial_\mu - eA_\mu) - m^2]G^{(0)}(x, x') = \delta^{(4)}(x - x') , \tag{16}$$

with

$$A^\mu = [0, \mathbf{A}(\mathbf{x})] , \quad x^\mu = (t, \mathbf{x}) , \tag{17}$$

where $\mathbf{A}(\mathbf{x})$ is given by Eq. (6). The Floquet representation of the Green's function is obtained by first removing the periodic space-time dependence due to the external electromagnetic potential in (16) by putting

$$G^{(0)}(x, x') = \sum_{n=-\infty}^{+\infty} e^{in(k\cdot x + \delta)} G^{(0)}_{n0}(x, x') . \tag{18}$$

The function $G^{(0)}_{n0}$ depends now on x and x' only through the combination $x - x'$. We could expand $G^{(0)}(x, x')$ in $k \cdot x'$ equally well and obtain another function $\tilde{G}^{(0)}_{0n}$ different from the first one by the factor $e^{ink\cdot(x - x')}$. If we now put the expansion (18) into Eq. (16) and go to the momentum representation we get

$$\{p^2 - e^2A_0^2 - m^2 - 2nk\cdot p + eA_0[(S_n^+ + S_n^-)p_1 - i(S_n^+ - S_n^-)p_2]\}G_{nn'}^{(0)}(p) = \delta_{nn'} , \tag{19}$$

where the more general function $G_{nn'}^{(0)}$ is used instead of $G_{n0}^{(0)}$. In this formula p_1 and p_2 are three-vector components of the momentum and $S_n^{\pm}$ are operators increasing and lowering the index n

$$S_n^{\pm}\phi_n = \phi_{n\pm1} . \tag{20}$$

To solve the equation for the Green's function we first consider the simplified one

$$\{-2nk\cdot p + eA_0[(S_n^+ + S_n^-)p_1 - i(S_n^+ - S_n^-)p_2]\}\chi_n = 0 . \tag{21}$$

Going to the representation in which

$$S_n^+ \to \frac{1}{\xi} , \qquad S_n^- \to \xi , \qquad n \to \xi\frac{d}{d\xi} , \tag{22}$$

where ξ is certain complex variable, we obtain

$$\frac{d\chi}{d\xi} = \frac{eA_0}{2k\cdot p}\left[\left(1+\xi^{-2}\right)p_1 + i\left(1-\xi^{-2}\right)p_2\right]\chi(\xi) , \tag{23}$$

and hence

$$\chi(\xi) = \chi_0 \exp\left[\frac{eA_0}{2k\cdot p}[\left(\xi - \xi^{-1}\right)p_1 + i\left(\xi + \xi^{-1}\right)p_2]\right] . \tag{24}$$

Taking $\xi = \exp(i\tau)$ we get

$$\chi(\tau) = \chi_0 \exp\left[i\frac{eA_0}{k\cdot p}|\mathbf{p}|\sin\theta_p \sin(\tau + \phi_p)\right] , \tag{25}$$

and χ_n, which we are looking for, is now the nth coefficient in the expansion of $\chi(\tau)$ in the powers of $\exp(i\tau)$

$$\chi_n \sim J_n(\alpha_0^p|\mathbf{p}|\sin\theta_p)e^{in\phi_p} , \tag{26}$$

where

$$\alpha_0^p = \frac{eA_0}{k\cdot p} , \tag{27}$$

and J_n is the nth Bessel function. The angles ϕ_p and θ_p are measured with respect to the propagation vector of radiation $\mathbf{k}$ (z axis).

Having found χ_n we can write down the Green's function satisfying the Feynman boundary condition

$$G_{nn'}^{(0)}(p) = \sum_{N=-\infty}^{+\infty} J_{n-N}(\alpha_0^p|\mathbf{p}|\sin\theta_p) \times \frac{e^{i(n-n')\phi_p}}{(p-Nk)^2 - e^2A_0^2 - m^2 + i\varepsilon} \times J_{n'-N}(\alpha_0^p|\mathbf{p}|\sin\theta_p) . \tag{28}$$

The ε prescription refers now to the positive or negative values of *quasienergy* p_0, which is conserved in the present case, instead of the *free* energy. The corresponding time-dependent Green's function $G^{(0)}(x,y)$ can be constructed from $G_{n0}^{(0)}$:

$$\begin{aligned} G^{(0)}(x,y) &= \sum_{n=-\infty}^{+\infty} e^{in(k\cdot x+\delta)}G_{n0}^{(0)}(x,y) \\ &= \sum_{n=-\infty}^{+\infty} e^{in(k\cdot x+\delta)}\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\sum_{N=-\infty}^{+\infty}\frac{J_{n-N}(\alpha_0^p|\mathbf{p}|\sin\theta_p)e^{in\phi_p}J_{-N}(\alpha_0^p|\mathbf{p}|\sin\theta_p)}{(p_0-Nk_0)^2 - (\mathbf{p}-N\mathbf{k})^2 - e^2A_0^2 - m^2 + i\varepsilon} \end{aligned} \tag{29}$$

or

$$\begin{aligned} G^{(0)}(x,y) &= \int\frac{d^4p}{(2\pi)^4}\Psi_p^{(0)}(x)\sum_{N=-\infty}^{+\infty}e^{iN(\phi_p+\delta)}e^{i(p+Nk)\cdot y}\frac{J_{-N}(\alpha_0^p|\mathbf{p}|\sin\theta_p)}{p_0^2 - \mathbf{p}^2 - e^2A_0^2 - m^2 + i\varepsilon} \\ &= \int\frac{d^4p}{(2\pi)^4}\Psi_p^{(0)}(x)\frac{1}{p_0^2 - \mathbf{p}^2 - e^2A_0^2 - m^2 + i\varepsilon}\Psi_p^{(0)*}(y) , \end{aligned} \tag{30}$$

where

$$\Psi_p^{(0)} = \exp\left[-ip\cdot x + i\alpha_0^p|\mathbf{p}|\sin\theta_p\sin(k\cdot x + \phi_p + \delta)\right] . \tag{31}$$

We now see that $G^{(0)}(x,y)$ has similar properties to those of the Feynman Green's function. Since the "square" of the wave function is given by the residue of the Green's function at its poles, Eq. (31) corresponds to the Volkov wave function of the scalar particle with the *quasi*-four-momentum p, where $p_0^2 = \mathbf{p}^2 + m^2 + e^2A_0^2$.

The formula (28) is the relativistic version of that obtained in previous works [7, 10] and reduces to the latter in the nonrelativistic and dipole approximations. For completeness we will give here also the relativistic result for the linear polarization

$$G^{(0)}_{nn'}(p) = \sum_{N=-\infty}^{+\infty} \frac{J_{n-N}(a_p, b_p) J_{n'-N}(a_p, b_p)}{[(p-(N-\lambda_p)k]^2 - m^2 + i\varepsilon} , \quad (32)$$

where the two-arguments functions $J_n(a_p, b_p)$ are the generalized Bessel functions defined by

$$J_n(a_p, b_p) = \sum_{m=-\infty}^{+\infty} J_{n+2m}(a_p) J_m(b_p) , \quad (33)$$

with

$$a_p = \frac{eA_0 \boldsymbol{\epsilon} \cdot \mathbf{p}}{2k \cdot p}, \quad b_p = \frac{e^2 A_0^2}{4k \cdot p}, \quad \lambda_p = -\frac{e^2 A_0^2}{8k \cdot p} , \quad (34)$$

and $\boldsymbol{\epsilon}$ is the linear polarization vector. The important feature of the function (28) is its simple ϕ_p dependence, only through the factor $e^{i(n-n')\phi_p}$. This is not the case in (32), where ϕ_p is hidden in a_p, the argument of the Bessel function. This point will turn out to be of crucial importance in the following sections, since the integration of $G^{(0)}_{nn'}$, with spherically symmetric objects $\tilde{\phi}$ as in Eq. (5), produces a diagonal object in n and n'. Otherwise we would have to do with infinite matrices. In Sec. IV we deal with this point explicitly.

IV. KLEIN-GORDON PARTICLE IN BOTH THE EXTERNAL FIELDS

The main goal of this section is to find the full Green's function for a scalar particle in both the external fields: the separable potential and the laser field. In the Floquet representation the full Green's function associated with Eq. (3) satisfies

$$\left\{(E-nk_0)^2 - \left[-i\boldsymbol{\partial} - n\mathbf{k} - \frac{eA_0}{2}[\mathbf{e}_1(S_n^+ + S_n^-) - i\mathbf{e}_2(S_n^+ - S_n^-)]\right]^2 - m^2\right\} G_{nn'}(E;\mathbf{x},\mathbf{x}')$$
$$-V_0[2(E-nk_0)-V_0]\, e^{in\mathbf{k}\cdot\mathbf{x}}\,\tilde{\phi}(\mathbf{x}) \int d^3z\, \tilde{\phi}(\mathbf{z})\, e^{-in\mathbf{k}\cdot\mathbf{z}}\, G_{nn'}(E;\mathbf{z},\mathbf{x}') = \delta_{nn'}\delta^{(3)}(\mathbf{x}-\mathbf{x}') . \quad (35)$$

In this representation, the equation is fully separable and can be solved exactly. Using (28) we invert the operator in large curly brackets on the left-hand side and get

$$G_{nn'}(E;\mathbf{x},\mathbf{x}') = G^{(0)}_{nn'}(E;\mathbf{x},\mathbf{x}') + \sum_{m=-\infty}^{+\infty} V_0[2(E-mk_0)-V_0] \int d^3y\, G^{(0)}_{nm}(E;\mathbf{x},\mathbf{y}) e^{im\mathbf{k}\cdot\mathbf{y}}\tilde{\phi}(\mathbf{y})$$
$$\times \int d^3z\, \tilde{\phi}(\mathbf{z}) e^{-im\mathbf{k}\cdot\mathbf{z}} G_{mn'}(E;\mathbf{z},\mathbf{x}') . \quad (36)$$

Projecting both sides of (36) onto $\tilde{\phi}(\mathbf{x})e^{-in\mathbf{k}\cdot\mathbf{x}}$ one obtains the (infinite) matrix equation

$$\sum_{m=-\infty}^{+\infty} \left[\delta_{nm} - V_0[2(E-mk_0)-V_0] \int d^3x \int d^3y\, \tilde{\phi}(\mathbf{x}) e^{-in\mathbf{k}\cdot\mathbf{x}} G^{(0)}_{nm}(E;\mathbf{x},\mathbf{y}) e^{im\mathbf{k}\cdot\mathbf{y}} \tilde{\phi}(\mathbf{y})\right]$$
$$\times \int d^3z\, \tilde{\phi}(\mathbf{z}) e^{-im\mathbf{k}\cdot\mathbf{z}} G_{mn'}(E;\mathbf{z},\mathbf{x}') = \int d^3x\, \tilde{\phi}(\mathbf{x}) e^{-in\mathbf{k}\cdot\mathbf{x}} G^{(0)}_{nn'}(E;\mathbf{x},\mathbf{x}') . \quad (37)$$

The crucial observation, already spoken of above, is that for the Klein-Gordon particle considered here, circularly polarized light and spherically symmetric $\tilde{\phi}$'s, the matrix in large square brackets in (37) becomes diagonal and the infinite set of equations reduces to a single one. This can be easily seen if we use the formula for $G^{(0)}_{nm}$, Eq. (28), and calculate

$$\int d^3x \int d^3y\, \tilde{\phi}(\mathbf{x}) e^{-in\mathbf{k}\cdot\mathbf{x}} G^{(0)}_{nm} e^{im\mathbf{k}\cdot\mathbf{y}} \tilde{\phi}(\mathbf{y})$$
$$= \sum_{N=-\infty}^{+\infty} \int_0^\infty \frac{dp\, p^2}{4\pi^2} \int_0^\pi d\theta_p \sin\theta_p \frac{\check{\phi}^2(\mathbf{p}-n\mathbf{k}) J^2_{n-N}(\alpha_0^p |\mathbf{p}| \sin\theta_p)}{(E-Nk_0)^2 - (\mathbf{p}-N\mathbf{k})^2 - e^2A_0^2 - m^2 + i\varepsilon} \delta_{nm} , \quad (38)$$

where $\breve{\phi}(\mathbf{q})$ denotes the Fourier transform of the profile $\tilde{\phi}(\mathbf{x})$.

Introducing now the object $W_n(E)$

$$W_n(E) = 1- V_0[2(E-nk_0)-V_0] \times \sum_{N=-\infty}^{+\infty} \int_0^\infty \frac{dp\,p^2}{4\pi^2} \int_0^\pi d\theta_p \sin\theta_p \frac{\breve{\phi}^2(\mathbf{p}-n\mathbf{k})J^2_{n-N}(\alpha_0^{E,\mathbf{p}}|\mathbf{p}|\sin\theta_p)}{(E-Nk_0)^2-(\mathbf{p}-N\mathbf{k})^2-m^2-e^2A_0^2+i\varepsilon}\,, \tag{39}$$

we can write the final formula for the full Green's function in the explicit form:

$$G_{nn'}(E;\mathbf{x},\mathbf{x}') = G^{(0)}_{nn'}(E;\mathbf{x},\mathbf{x}') + \sum_{m=-\infty}^{+\infty} V_0[2(E-mk_0)-V_0] \int d^3y\, G^{(0)}_{nm}(E;\mathbf{x},\mathbf{y})e^{im\mathbf{k}\cdot\mathbf{y}}\tilde{\phi}(\mathbf{y})[W_m(E)]^{-1} \times \int d^3z\, \tilde{\phi}(\mathbf{z})e^{-im\mathbf{k}\cdot\mathbf{z}}G^{(0)}_{mn'}(E;\mathbf{z},\mathbf{x}')\,. \tag{40}$$

Equation (40) is one of the key results of this paper. It is now used to derive analytic expressions for the amplitudes of various processes of physical interest.

V. THE PROCESS OF DETACHMENT

A. Total rate of detachment

The total ejection probability per unit time (i.e., the rate of detachment in the field) can be found by investigating the eigenvalue equation for the *quasienergy*. Since the state of the bound particle in the external laser field becomes unstable, we expect this "energy" to be complex: $E = E_R - i\Gamma/2$. According to the theory of unstable states, Γ may be identified as the total decay rate and E_R as the dressed energy of the initial state. The energy eigenvalue equation is obtained from the Floquet equation for the wave function [c.f. Eq. (35)]:

$$\left\{(E-nk_0)^2 - \left[-i\boldsymbol{\partial} - n\mathbf{k} - \frac{eA_0}{2}[\mathbf{e}_x(S_n^+ + S_n^-) - i\mathbf{e}_y(S_n^+ - S_n^-)]\right]^2 - m^2\right\}\Psi_n(\mathbf{x}) = V_0[2(E-nk_0)-V_0]e^{in\mathbf{k}\cdot\mathbf{x}}\tilde{\phi}(\mathbf{x}) \int d^3z\, \tilde{\phi}(\mathbf{z})e^{-in\mathbf{k}\cdot\mathbf{z}}\Psi_n(\mathbf{z})\,. \tag{41}$$

This leads to the integral equation

$$\Psi_n(\mathbf{x}) = \int d^3x' \sum_{m=-\infty}^{+\infty} G^{(0)}_{nm}(E;\mathbf{x},\mathbf{x}')V_0[2(E-mk_0)-V_0]e^{im\mathbf{k}\cdot\mathbf{x}'}\tilde{\phi}(\mathbf{x}') \int d^3z\, \tilde{\phi}(\mathbf{z})e^{-im\mathbf{k}\cdot\mathbf{z}}\Psi_m(\mathbf{z})\,, \tag{42}$$

where, as required, the Green's function $G^{(0)}_{nm}$ satisfies the Feynman boundary condition. After projecting both sides of this relation onto $\tilde{\phi}(\mathbf{x})e^{-in\mathbf{k}\cdot\mathbf{x}}$ and canceling the common factor we obtain the eigenvalue equation for the complex energy,

$$V_0(2E-V_0) \sum_{N=-\infty}^{+\infty} \int_0^\infty \frac{dp}{4\pi^2}p^2 \int_0^\pi d\theta_p \frac{\sin\theta_p\;\breve{\phi}^2(\mathbf{p})J^2_{-N}(\alpha_0^{E,\mathbf{p}}|\mathbf{p}|\sin\theta_p)}{(E-Nk_0)^2-(\mathbf{p}-N\mathbf{k})^2-e^2A_0^2-m^2+i\varepsilon} = 1\,. \tag{43}$$

For our special choice of $\tilde{\phi}$ in (5) this reads

$$4V_0(2E-V_0)N_0^2 \sum_{N=-\infty}^{+\infty} \int_0^\infty \frac{dp\,p^2}{(\mathbf{p}^2+\lambda^2)^2} \int_0^\pi d\theta_p \sin\theta_p \frac{J^2_{-N}(\alpha_0^{E,\mathbf{p}}|\mathbf{p}|\sin\theta_p)}{(E-Nk_0)^2-(\mathbf{p}-N\mathbf{k})^2-e^2A_0^2-m^2+i\varepsilon} = 1\,. \tag{44}$$

The relevant eigenvalue E_λ is the one which goes over adiabatically into the unperturbed bound state when $A_0 \to 0$. The rate of detachment is then given by $\Gamma = -2\,\mathrm{Im}(E_\lambda)$.

This eigenvalue equation (44) is reminiscent of the result obtained for the δ potential [11] and the rank-one separable potential in the Schrödinger case [9,10] and in fact reproduces the latter in the nonrelativistic limit and dipole approximation.

It is important to note that the eigenvalues E_λ and the corresponding eigenvectors $C_N(E_\lambda)$ satisfy the relativistic analog of the "twin-transformation" invariance of the Floquet system known in the nonrelativistic case [12] (p. 250). The relativistic twin-transformation is

$$\begin{aligned} p_\lambda \quad &\rightarrow \quad p_\lambda + Mk, \\ C_N(p_\lambda) &\rightarrow C_{N+M}(p_\lambda + Mk), \end{aligned} \tag{45}$$

where $p_\lambda = (E_\lambda, \mathbf{p})$, $k = (\omega, \mathbf{k})$.

Thus, for every irreducible eigenvalue E_λ there corresponds infinitely many eigenvalues $E_\lambda + M\omega$ ($\mathbf{p} \rightarrow \mathbf{p} + M\mathbf{k}$) and eigenvectors $C_{N+M}(E_\lambda + M\omega, \mathbf{p} + M\mathbf{k})$, which constitute an equivalent set of solutions of the Floquet-KG equation. The invariance of the eigenvalue equation (44) under (45) can be easily checked by substitution in (44), shifting the summation index from N to $N - M$ and using the fact that $\alpha_0^{E,\mathbf{p}}|\mathbf{p}|\sin\theta_p$ remains unchanged by addition of a constant multiple of the four-vector k to p.

B. Partial widths and branching ratios

The investigation of the asymptotic properties of the Green function in (42) allows us to find also the partial widths Γ_n connected with detachment processes in separate channels, i.e., with certain fixed number of photons absorbed. Let us consider the asymptotic behavior of $G^{(0)}_{nn'}(E;\mathbf{r},\mathbf{r}')$

$$G^{(0)}_{nn'}(E;\mathbf{r},\mathbf{r}') = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} \sum_{N=-\infty}^{+\infty} \frac{J_{n-N}(\alpha_0^p|\mathbf{p}|\sin\theta_p) e^{i(n-n')\phi_p} J_{n'-N}(\alpha_0^p|\mathbf{p}|\sin\theta_p)}{(E-Nk_0)^2 - (\mathbf{p}-N\mathbf{k})^2 - e^2A_0^2 - m^2 + i\varepsilon} . \tag{46}$$

A shift of the variable $\mathbf{p} \rightarrow \mathbf{p} + N\mathbf{k}$ and the integration over $d\Omega_p$ for large r gives

$$\begin{aligned} G^{(0)}_{nn'}(E;\mathbf{r},\mathbf{r}') &\sim \frac{i}{4\pi^2 r} \sum_{N=-\infty}^{+\infty} e^{iN\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \int_0^\infty dp\, p \Bigg[e^{ipr} e^{-ip\mathbf{r}'\cdot\hat{\mathbf{r}}} e^{i(n-n')\phi_r} \frac{J_{n-N}(\alpha^- p\sin\theta_r) J_{n'-N}(\alpha^- p\sin\theta_r)}{(p\hat{\mathbf{r}})^2 - (E-Nk_0)^2 + e^2A_0^2 + m^2 - i\varepsilon} \\ &\qquad - e^{-ipr} e^{ip\mathbf{r}'\cdot\hat{\mathbf{r}}} (-1)^{n-n'} e^{i(n-n')\phi_r} \\ &\qquad \times \frac{J_{n-N}(\alpha^+ p\sin\theta_r) J_{n'-N}(\alpha^+ p\sin\theta_r)}{(-p\hat{\mathbf{r}})^2 - (E-Nk_0)^2 + e^2A_0^2 + m^2 - i\varepsilon} \Bigg] \\ &= \frac{i}{4\pi^2 r} \sum_{N=-\infty}^{+\infty} e^{iN\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \int_{-\infty}^{+\infty} dp\, p \left[e^{ipr} e^{-ip\mathbf{r}'\cdot\hat{\mathbf{r}}} e^{i(n-n')\phi_r} \frac{J_{n-N}(\alpha^- p\sin\theta_r) J_{n'-N}(\alpha^- p\sin\theta_r)}{p^2 - (E-Nk_0)^2 + e^2A_0^2 + m^2 - i\varepsilon} \right] , \end{aligned} \tag{47}$$

where we have introduced the symbols

$$\begin{aligned} \alpha^- &= \frac{eA_0}{k_0(E-Nk_0) - p\mathbf{k}\cdot\hat{\mathbf{r}}}, \\ \alpha^+ &= \frac{eA_0}{k_0(E-Nk_0) + p\mathbf{k}\cdot\hat{\mathbf{r}}} . \end{aligned} \tag{48}$$

In the remaining p integral we can close the contour in the upper half-plane and get

$$G^{(0)}_{nn'}(E;\mathbf{r},\mathbf{r}') \sim \sum_{N=-\infty}^{+\infty} e^{iN\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \frac{e^{ip_N r}}{r} J_{n-N}(\alpha_0^{p_N} \sin\theta_r p_N) e^{i(n-n')\phi_r} J_{n'-N}(\alpha_0^{p_N} p_N \sin\theta_r) e^{-ip_N\hat{\mathbf{r}}\cdot\mathbf{r}'} . \tag{49}$$

This formula holds even if E is complex: $E \rightarrow \tilde{E} = E_R - i\frac{\Gamma}{2}$. With this situation we have to do while solving (42). In this case we have to analytically continue (49) from $E + i\varepsilon$ to $E_R - i\frac{\Gamma}{2}$.

As it can be seen from (42) the asymptotic form of the wave function is dictated by the expression

$$\Psi_n(\mathbf{x}) = CV_0[2\tilde{E} - V_0] \int d^3x' G^{(0)}_{n0}(\tilde{E};\mathbf{x},\mathbf{x}')\tilde{\phi}(\mathbf{x}') , \tag{50}$$

where C is certain (coordinates independent) constant. Using the asymptotic form of the Green function (49) we get

$$\begin{aligned} \Psi_n(\mathbf{x}) \sim -\frac{CV_0[2\tilde{E} - V_0]}{4\pi} &\times \sum_{N=-\infty}^{+\infty} e^{iN\mathbf{k}\cdot\mathbf{x}} \frac{e^{ip_N x}}{x} \check{\phi}(p_N\hat{\mathbf{x}} + N\mathbf{k}) \\ &\times J_{n-N}(\alpha_0^{p_N} p_N \sin\theta_x) \\ &\times e^{in\phi_x} J_{-N}(\alpha_0^{p_N} p_N \sin\theta_x) , \end{aligned} \tag{51}$$

where $p_N = \sqrt{(E-Nk_0)^2 - e^2A_0^2 - m^2}$ [13].

Using the fact that the asymptotic behavior of the scalar Volkov state (31) (only outgoing waves) is

$$\frac{e^{ipx}}{2x} e^{in\phi_x} J_n(\alpha_0^p |\mathbf{p}| \sin\theta_x) , \tag{52}$$

we can identify the ionization amplitude in the Nth channel as

$$f_N(\phi_x, \theta_x) = D e^{iN(\mathbf{k}\cdot\mathbf{x}+\phi_x)} \breve{\phi}(p_N \hat{\mathbf{x}} + N\mathbf{k}) \times J_N(\alpha_0^{p_N} p_N \sin\theta_x) , \tag{53}$$

where D is a new constant. For the probability of detachment by absorption of N photons we get

$$\Gamma_N = \int d\Omega_x f_N^* f_N p_N = |D|^2 \int d\Omega_x p_N |\breve{\phi}(p_N \hat{x} + N\mathbf{k}) J_{-N}(\alpha_0^{p_N} p_N \sin\theta_x)|^2 , \tag{54}$$

and, for our special choice of binding potential (5),

$$\Gamma_N = |D'|^2 \int d\Omega_x p_N \left| \frac{J_{-N}(\alpha_0^{p_N} p_N \sin\theta_x)}{(p_N \hat{\mathbf{x}} + N\mathbf{k})^2 + \lambda^2} \right|^2 . \tag{55}$$

Since Γ_N's are detachment probabilities in separate photon absorption channels, the unknown constants may be found from the normalization condition: $\sum_N \Gamma_N = \Gamma$, where Γ is the total rate considered in Sec. V A.

The branching ratios B_N's in individual photon channels can be found independent of the normalization constant: $B_N = \frac{\Gamma_N}{\Gamma} = \frac{\gamma_N}{\gamma}$, where

$$\gamma_N = \int d\Omega_x p_N \left| \frac{J_{-N}(\alpha_0^{p_N} p_N \sin\theta_x)}{(p_N \hat{\mathbf{x}} + N\mathbf{k})^2 + \lambda^2} \right|^2 \tag{56}$$

and $\gamma = \sum_N \gamma_N$.

C. The probability distribution for outgoing electrons

In Secs. V A and V B we have dealt with the detachment process in the language of the eigenvalue equation and in terms of a constant (in time) rate of detachment. The rate concept is essentially an approximate one and is appropriate so long as $\Gamma \ll \omega, E_R$. For high field intensities for which this condition may be violated one requires us to describe the process in terms of the probability distribution of the energy of the ejected particle in the continuum (or the so-called above-threshold detachment spectrum). This leads us to consider the full time evolution of the wave function. We assume that at certain initial moment ($t < 0$) the electron is in the unperturbed bound state. At $t = 0$ the laser field is turned on and the state evolves now in a way governed by the full Hamiltonian. The full time-dependent wave function can be written as

$$\Psi(\mathbf{x}, t) = -\int d^3x' G(\mathbf{x}, t; \mathbf{x}', t') (\overleftrightarrow{\partial}_{t'} + 2i\hat{V}) \Phi_i(\mathbf{x}', t') , \tag{57}$$

where the total Green's function in the Floquet representation has the form

$$G(x, x') = \sum_{n=-\infty}^{+\infty} e^{in(k\cdot x+\delta)} \times \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} e^{-ip_0(t-t')} G_{n0}(p_0; \mathbf{x}, \mathbf{x}') , \tag{58}$$

with G_{n0} given by (40). Since the initial state Φ_i is a positive *quasienergy* state we do not distinguish between *retarded* and *Feynman* functions.

Substituting (58) into (57), carrying out the operation $\overleftrightarrow{\partial}_{t'}$ and putting the initial time $t' = 0$, we get

$$\Psi(\mathbf{x}, t) = \sum_{n=-\infty}^{+\infty} e^{in(k\cdot x+\delta)} \Psi_n(\mathbf{x}, t) , \tag{59}$$

with

$$\Psi_n(\mathbf{x}, t) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dp_0 e^{-ip_0 t} \int d^3x' G_{n0}(p_0; \mathbf{x}, \mathbf{x}') \left[(p_0 + E_0) \Phi_i(\mathbf{x}') - \frac{V_0}{E_0^{1/2}} \tilde{\phi}(\mathbf{x}') \right] . \tag{60}$$

Using $G_{n0}(p_0; \mathbf{x}, \mathbf{x}')$ from Eq. (40), changing the integration variables $p \to p + Nk$ as well as the summation index $n \to n + N$, performing the integrations over $\mathbf{x}'$, and putting the result in (59) one gets a useful form of the full wave function

$$\Psi(\mathbf{x}, t) = \sum_{N=-\infty}^{+\infty} e^{iN\delta} \int \frac{d^3p}{(2\pi)^3} \Psi_{\epsilon,\mathbf{p}}^{(0)}(\mathbf{x}, t) \frac{-1}{2\pi i} \int_{-\infty}^{\infty} dp_0 \frac{e^{-i[p_0 - \epsilon(\mathbf{p})]t}}{p^2 - m^2 - e^2 A_0^2 + i\varepsilon} A^{(N)}(p_0, \mathbf{p}) , \tag{61}$$

where we have used Floquet expansion of the Volkov wave function

$$\Psi_{\epsilon,\mathbf{p}}^{(0)}(\mathbf{x}, t) = \sum_{n=-\infty}^{+\infty} e^{in(k\cdot x+\delta)} J_n(\alpha_0^p |\mathbf{p}| \sin\theta_p) e^{in\phi_p} e^{-i\epsilon(\mathbf{p})t + i\mathbf{p}\cdot\mathbf{x}} , \tag{62}$$

at the energy

$$\epsilon = \epsilon(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2 + e^2 A_0^2}\,, \tag{63}$$

and defined

$$A^{(N)}(p) = J_{-N}(\alpha_0^p |\mathbf{p}| \sin\theta_p) e^{iN\phi_p} \left[(E_0 + p_0 + Nk_0)\breve{\Phi}_i(\mathbf{p} + N\mathbf{k}) - \frac{V_0}{E_0^{1/2}} \breve{\phi}(\mathbf{p} + N\mathbf{k}) \right]$$
$$+ V_0 \cdot [2(P_0 + Nk_0) - V_0] J_{-N}(\alpha_0^p |\mathbf{p}| \sin\theta_p) e^{iN\phi_p} \breve{\phi}(\mathbf{p} + N\mathbf{k}) [W_0(p_0 + Nk_0)]^{-1} C^{(N)}(p_0)\,, \tag{64}$$

with

$$C^{(N)}(p_0) = \sum_{M=-\infty}^{+\infty} \int \frac{d^3q}{2\pi^3} \frac{\breve{\phi}(\mathbf{q} + N\mathbf{k}) J_{-M}(\alpha_0^{p_0,\mathbf{q}} |\mathbf{q}| \sin\theta_q) [(E + p_0 + Nk_0)\breve{\Phi}_i(\mathbf{q} + N\mathbf{k}) - \frac{V_0}{E_0^{1/2}} \breve{\phi}(\mathbf{q} + N\mathbf{k})]}{\{[p_0 - (M - N)k_0]^2 - [\mathbf{q} - (M - N)\mathbf{k}]^2 - m^2 - e^2 A_0^2 + i\varepsilon\}}\,, \tag{65}$$

and $p_0 = \epsilon(\mathbf{p})$, and $W_0(p_0 + Nk_0)$ is given by Eq. (39) with $n = 0$. We may now use the limits

$$\lim_{t\to+\infty} \frac{e^{-ixt}}{x \pm i\varepsilon} = \begin{cases} -2\pi i \delta(x) \\ 0 \end{cases} \tag{66}$$

to carry out the integration over p_0 in (61) for $t \to +\infty$ and thus find the long-time behavior of the total wave function

$$\Psi(\mathbf{x}, t) = \sum_{N=-\infty}^{+\infty} e^{iN\delta} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\epsilon(\mathbf{p})}} \Psi_{\epsilon,\mathbf{p}}^{(0)}(\mathbf{x}, t) \times \frac{A^{(N)}(\epsilon(\mathbf{p}), \mathbf{p})}{\sqrt{2\epsilon(\mathbf{p})}} \quad (\text{large } t)\,, \tag{67}$$

where $\frac{1}{\sqrt{2\epsilon(\mathbf{p})}} \Psi_{\epsilon,\mathbf{p}}^{(0)}(\mathbf{x}, t)$ is the Volkov wave function normalized to one particle in unit volume which is appropriate for a KG particle. Identifying the coefficient of the normalized Volkov wave function in (67), taking the modulus square, and averaging over the arbitrary initial phase (δ) of the field [9], we obtain the differential probability of transition to the continuum between ϵ and $\epsilon + d\epsilon$:

$$dW = \sum_{n=-\infty}^{+\infty} \left| \frac{A^{(N)}(\epsilon, \mathbf{p})}{\sqrt{2\epsilon}} \right|^2 \rho(\epsilon) d\epsilon d\Omega_p\,, \tag{68}$$

where $\rho(\epsilon) = \frac{\epsilon|\mathbf{p}|}{(2\pi)^3}$ is the density of the final states at the energy $\epsilon = \epsilon(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2 + e^2 A_0^2}$. Thus we arrive at the final formula for the probability distribution of the ejected particle (or the above threshold detachment spectrum) by absorption of any number of photons:

$$\frac{dW}{d\epsilon} = \sum_{N=-\infty}^{+\infty} \int \frac{d\Omega_p}{(2\pi)^3} \frac{|\mathbf{p}|}{2} |A^{(N)}(\epsilon, \mathbf{p})|^2\,, \tag{69}$$

where $A^{(N)}(\epsilon, \mathbf{p})$ is given by (64) and (65).

For the particular choice of the bound-state wave function (15) and the corresponding potential function (5), used in this paper, the respective Fourier transforms are

$$\breve{\Phi}_i(\mathbf{p}) = \frac{8\pi\lambda^2 N_0}{\sqrt{E_0}} \frac{1}{[\mathbf{p}^2 + \lambda^2]^2} \tag{70}$$

and

$$\breve{\phi}(\mathbf{p}) = \frac{4\pi N_0}{\mathbf{p}^2 + \lambda^2}\,, \tag{71}$$

with $N_0 = \sqrt{\frac{\lambda}{2\pi}}$ and $\lambda = \sqrt{m^2 - E_0^2}$. The final "momentum" $|\mathbf{p}|$ can be also given in terms of the "kinetic energy" of the ejected particle in the presence of the field: $|\mathbf{p}| = \left[\epsilon_{\text{kin}}^2 + 2\epsilon_{\text{kin}}\sqrt{m^2 + e^2 A_0^2}\right]^{1/2}$, where $\epsilon_{\text{kin}} = \epsilon(\mathbf{p}) - \sqrt{m^2 + e^2 A_0^2}$.

VI. RADIATIVE SCATTERING

The amplitudes for the scattering of an electron on the separable potential with simultaneous emission (stimulated bremsstrahlung) or absorption (inverse bremsstrahlung) of certain number of photons can also be found for the present model exactly. In the nonrelativistic case this process was considered in [10, 12]. The starting point is the equation satisfied by the wave function in the Floquet representation

$$\Psi_n(\mathbf{x}, t) = \Psi_n^{(0)}(\mathbf{x}, t) + \int d^3x' e^{-ip_0 t} \sum_{m=-\infty}^{+\infty} G_{nm}^{(0)}(p_0, \mathbf{x}, \mathbf{x}') V_0 [2(p_0 - mk_0) - V_0]$$
$$\times e^{im\mathbf{k}\cdot\mathbf{x}'} \tilde{\phi}(\mathbf{x}') \int d^3z\, \tilde{\phi}(\mathbf{z}) e^{-im\mathbf{k}\cdot\mathbf{z}} \Psi_m(\mathbf{z})\,, \tag{72}$$

where $G_{nm}^{(0)}$ is still the Feynman function. For $\Psi_n^{(0)}(\mathbf{x}, t)$—the wave function of the incoming particle in the field—we take the scalar Volkov solution $\Psi_p^n(\mathbf{x}, t)$

$$\Psi_p^n(\mathbf{x},t) = e^{-i(p_0 t - \mathbf{p}\cdot\mathbf{x})} J_n(\alpha_0^p |\mathbf{p}| \sin\theta_p) e^{in\phi_p} \ . \tag{73}$$

To find the appropriate scattering amplitude we will write the equation for $\Psi(\mathbf{x},t)$, for very large times, in the form

$$\Psi(\mathbf{x},t) = \Psi_p^{(0)}(\mathbf{x},t) + \int \frac{d^3q}{(2\pi)^3 2q_0} \Psi_q^{(0)}(\mathbf{x},t) f(t,p,q) \ . \tag{74}$$

From (72) we get

$$\begin{aligned}\Psi(\mathbf{x},t) = \sum_{n=-\infty}^{+\infty} e^{in(k\cdot x+\delta)} \Psi_n(\mathbf{x},t) &= \Psi_p^{(0)}(\mathbf{x},t) + \sum_{n=-\infty}^{+\infty} e^{in(k\cdot x+\delta)} \int d^3x' e^{-ip_0 t} \sum_{N=-\infty}^{+\infty} \int \frac{d^3q}{(2\pi)^3} \\ &\times e^{i(\mathbf{q}+N\mathbf{k})\cdot(\mathbf{x}-\mathbf{x}')} \sum_{m=-\infty}^{+\infty} \frac{J_{n-N}(\alpha_0^{p_0-Nk_0,\mathbf{q}}|\mathbf{q}|\sin\theta_q) e^{i(n-m)\phi_q} J_{m-N}(\alpha_0^{p_0-Nk_0,\mathbf{q}}|\mathbf{q}|\sin\theta_q)}{(p_0-Nk_0)^2 - \mathbf{q}^2 - e^2A_0^2 - m^2 + i\varepsilon} \\ &\times V_0[2(p_0 - mk_0) - V_0] e^{im\mathbf{k}\cdot\mathbf{x}'} \tilde{\phi}(\mathbf{x}') \int d^3z\, \tilde{\phi}(\mathbf{z}) e^{-im\mathbf{k}\cdot\mathbf{z}} \Psi_m(\mathbf{z}) \ ,\end{aligned} \tag{75}$$

where we have shifted the integration variable $\mathbf{q}$ in the definition of the Green's function: $\mathbf{q} \to \mathbf{q} + N\mathbf{k}$. After having performed the summation over n one obtains

$$\begin{aligned}\Psi(\mathbf{x},t) = \Psi_p^{(0)}(\mathbf{x},t) + \int \frac{d^3q}{(2\pi)^3} e^{-iq_0 t} \\ \times \sum_{N=-\infty}^{+\infty} e^{iN\delta} \exp\left[i\alpha_0^{p_0-Nk_0,\mathbf{q}}|\mathbf{q}|\sin\theta_q \sin(k\cdot x + \phi_p + \delta)\right] \\ \times \sum_{m=-\infty}^{+\infty} \int d^3x' e^{-i(\mathbf{q}+N\mathbf{k})\cdot\mathbf{x}'} e^{i(N-m)\phi_q} \frac{e^{-i(p_0-Nk_0-q_0)t}}{(p_0-Nk_0-q_0+i\varepsilon)(p_0-Nk_0+q_0-i\varepsilon)} \\ \times J_{m-N}(\alpha_0^{p_0-Nk_0,\mathbf{q}}|\mathbf{q}|\sin\theta_q) V_0[2(p_0-mk_0)-V_0] \\ \times e^{im\mathbf{k}\cdot\mathbf{x}'} \tilde{\phi}(\mathbf{x}') \int d^3z\, \tilde{\phi}(\mathbf{z}) e^{-im\mathbf{k}\cdot\mathbf{z}} \Psi_m(\mathbf{z}) \ .\end{aligned} \tag{76}$$

In this formula $q_0 = \sqrt{\mathbf{q}^2 + e^2A_0^2 + m^2}$. Using now (66) one arrives at the following relation

$$\begin{aligned}\Psi(\mathbf{x},t) = \Psi_p^{(0)}(\mathbf{x},t) + \int \frac{d^3q}{(2\pi)^3 2q_0} \Psi_q^{(0)}(\mathbf{x},t) \\ \times \sum_{N=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} e^{iN\delta}(-2\pi i)\delta(p_0 - q_0 - Nk_0) \\ \times e^{i(N-m)\phi_q} J_{m-N}(\alpha_0^q|\mathbf{q}|\sin\theta_q) V_0[2(p_0-mk_0)-V_0] \breve{\phi}(\mathbf{q}+(N-m)\mathbf{k}) \\ \times \int d^3z\, \tilde{\phi}(\mathbf{z}) e^{-im\mathbf{k}\cdot\mathbf{z}} \Psi_m(\mathbf{z}) \ .\end{aligned} \tag{77}$$

Now we can find the large time limit of $f(t,p,q)$

$$\lim_{t\to+\infty} f(t,p,q) = f(p,q) = \sum_{N=-\infty}^{+\infty} e^{iN\delta} 2\pi\delta(p_0 - q_0 - Nk_0) f^{(N)}(p,q) \ . \tag{78}$$

The partial amplitudes for the scattering with simultaneous absorption or emission of $|N|$ photons are therefore

$$\begin{aligned}f^{(N)}(p,q) = -i \sum_{m=-\infty}^{+\infty} e^{i(N-m)\phi_q} J_{m-N}(\alpha_0^q|\mathbf{q}|\sin\theta_q) V_0[2(p_0-mk_0)-V_0] \\ \times \breve{\phi}(\mathbf{q}+(N-m)\mathbf{k}) \int d^3z\, \tilde{\phi}(\mathbf{z}) e^{-im\mathbf{k}\cdot\mathbf{z}} \Psi_m(\mathbf{z}) \ .\end{aligned} \tag{79}$$

We have still to get rid of the factor $\int d^3z\, \tilde{\phi}(\mathbf{z}) e^{-im\mathbf{k}\cdot\mathbf{z}} \Psi_m(\mathbf{z})$, which contains the unknown function Ψ_m. One can do so by projecting Eq. (72) onto $\tilde{\phi}(\mathbf{x}) e^{-im\mathbf{k}\cdot\mathbf{x}}$. Thus,

$$\int d^3z\,\tilde{\phi}(\mathbf{z})e^{-im\mathbf{k}\cdot\mathbf{z}}\Psi_m(\mathbf{z}) = [W_m(p_0)]^{-1}\int d^3z\,\tilde{\phi}(\mathbf{z})e^{-im\mathbf{k}\cdot\mathbf{z}}\Psi_m^p(\mathbf{z}) = [W_m(p_0)]^{-1}\breve{\phi}(\mathbf{p}-m\mathbf{k})J_m(\alpha_0^p|\mathbf{p}|\sin\theta_p)e^{im\phi_p}\;, \tag{80}$$

which yields the final formula for the radiative scattering amplitude for emission ($N > 0$, stimulated bremsstrahlung) and absorption ($N < 0$, inverse bremsstrahlung):

$$f^{(N)}(p,q) = -i\sum_{m=-\infty}^{+\infty} e^{i(N-m)\phi_q}V_0[2(p_0-mk_0)-V_0]J_{m-N}(\alpha_0^q|\mathbf{q}|\sin\theta_q) \times\breve{\phi}(\mathbf{q}+(N-m)\mathbf{k})[W_m(p_0)]^{-1}J_m(\alpha_0^p|\mathbf{p}|\sin\theta_p)e^{im\phi_p}\breve{\phi}(\mathbf{p}-m\mathbf{k})\;, \tag{81}$$

where for the special choice of the potential (5) $\breve{\phi}$ is given by (71). The above equation should be compared with that obtained in the nonrelativistic case [10, 12], which it reproduces in the limit of the low energy and dipole approximation of the field, as it should. The corresponding cross-sections are obtained in terms of (81) from

$$\frac{d\sigma^{(N)}}{d\Omega} = \frac{1}{16\pi^2}\left|f^{(N)}(p,q)\right|^2. \tag{82}$$

VII. SUMMARY

In this paper we have analyzed a 3D relativistic model of interaction of a bound Klein-Gordon particle in a short-range separable pseudopotential subjected simultaneously to a plane-wave electromagnetic field of arbitrary frequency, wave number, and field strength. The corresponding KG equation is solved exactly and analytical expressions are derived for obtaining the total rate Γ [see Eq. (43)], the partial rates of detachment by absorption of N photons Γ_N [Eq. (54)], the branching ratios $B_N = \frac{\Gamma_N}{\Gamma}$, the radiative scattering cross sections for stimulated bremsstrahlung and inverse bremsstrahlung [Eq. (81)], and the above-threshold detachment spectrum [Eq. (64)]. The work is based on the Floquet representation of relativistic Green's functions and wave functions. The obtained formulas show the expected correspondence with the nonrelativistic ones.

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[13] We neglect here the small imaginary contribution of p_N coming from the fact that the energy of an unstable state (and hence the momentum of outgoing electrons too) is not precisely defined. We cannot observe outgoing electrons earlier than the ionization process started or, in other words, the laser field was turned on. The approach developed in this section, however, does not specify the initial moment of the process. Normally we would also have the factor $e^{-(\Gamma/2)t}$ and large x would correspond to large t so the exploding exponents would be damped.