Quantum phenomena in nonstationary media

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(Received 25 March 1992; revised manuscript received 28 December 1992)

The problem of electromagnetic-field quantization in time-dependent nonuniform linear nondispersive media is investigated. The explicit formulas for the number of photons generated from the initial vacuum state due to the change in time of dielectric permeability of the medium are obtained in the case when the spatial and temporal dependences are factorized. The concrete time dependences include adiabatic and sudden changes of permeability, the parametric resonance at twice the eigenfrequency of the mode, Epstein's symmetric and transition profiles, "temporal Fabry-Pérot resonator," and some others. The upper and lower bounds for the squeezing and correlation coefficients of the field in the final state are given in terms of the reflection coefficient from an equivilent potential barrier or the number of created quanta. The problem of impulse propagation in a spatially uniform but time-dependent dielectric medium is discussed.

PACS number(s): 42.50.Dv

I. INTRODUCTION

The aim of this paper is to consider the problem of photon creation and generation of squeezed and correlated states of the electromagnetic field in media in which dielectric properties vary in time (due to some external action).

The problem of electromagnetic-field quantization is usually considered in textbooks under the assumption that the field occupies some empty box. The case when the box is filled with a uniform dielectric medium was considered in [1,2]. The quantization of the field in a medium consisting of two uniform dielectrics with different permeabilities was studied in [3–5]. The case of an arbitrary inhomogeneous dielectric medium was investigated in [6,7] and especially in [8,9]. In all the above-mentioned papers the properties of the medium were believed to be time independent.

The most general case of nonuniform and time-dependent linear media was investigated in [10]. However, the authors of that paper considered only approximate solutions of the Heisenberg equations for field operators valid for some polarization of the medium. Here we want to consider the case when a nonuniform time-dependent medium is described with some space-time *factorized* dielectric permeabilities ϵ and magnetic permeabilities μ. Then explicit results can be found for arbitrary magnitudes of parameters ϵ and μ.

In Sec. II we present a scheme of electromagnetic-field quantization in a general case of nonuniform and nonstationary (although linear and nondispersive) medium. In Sec. III we apply this scheme to the case of the medium in which dielectric permeability can be represented as a product of two arbitrary functions: one dependent only on space coordinates and another dependent only on time. Different specific time dependences of the permeability are considered in detail in this section too. The main results of the paper are summarized in Sec. IV. In the Appendix we consider a classical problem of impulse propagation in a uniform but time-dependent dielectric medium.

II. QUANTIZATION OF ELECTROMAGNETIC FIELD IN NONSTATIONARY MEDIA

The basis of the subsequent consideration is the system of Maxwell's equations in linear, passive, nondispersive, time-dependent dielectric and magnetic media without sources (the field quantization in nonlinear stationary dielectric media was investigated in [6,8,11], and the most general approach suitable for nonstationary nonlinear media was proposed recently in [12]),

$$\begin{gathered}\mathrm{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\ ,\quad \mathrm{rot}\mathbf{H}=\frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}\ ,\\ \mathrm{div}\mathbf{D}=0\ ,\quad \mathrm{div}\mathbf{B}=0\ ,\\ \mathbf{D}=\epsilon(r,t)\mathbf{E}\ ,\quad \mathbf{B}=\mu(r,t)\mathbf{H}\ .\end{gathered} \tag{2.1}$$

Introducing the vector potential according to the relations

$$\mathbf{B}=\mathrm{rot}\,\mathbf{A}\;,\quad \mathbf{E}=-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{2.2}$$

and imposing gauge conditions

$$\mathrm{div}\left[\epsilon\frac{\partial \mathbf{A}}{\partial t}\right]=0\;,\quad \phi=0\;, \tag{2.3}$$

we can replace the system of first-order equations (2.1) with the single second-order equation

$$\mathrm{rot}\left[\frac{1}{\mu}\mathrm{rot}\,\mathbf{A}\right]+\frac{1}{c^2}\frac{\partial}{\partial t}\left[\epsilon\frac{\partial \mathbf{A}}{\partial t}\right]=0\;. \tag{2.4}$$

The subsequent quantization procedure is based on the following important property of Eq. (2.4): it admits a time-independent scalar product of any two different solutions in the form

$$((\mathbf{A}_1,\mathbf{A}_2))=-\frac{i}{2}\int \epsilon(\mathbf{r},t)\left[\mathbf{A}_1\frac{\partial \mathbf{A}_2^*}{\partial t}-\mathbf{A}_2^*\frac{\partial \mathbf{A}_1}{\partial t}\right]d^2\mathbf{r}\;. \tag{2.5}$$

It is essential that the dielectric permeability be a real function, i.e., the medium is assumed lossless. Besides, the vector potential has to go to zero at the surfaces confining the integration domain. Moving boundaries (considered in [13] in the special case of a free space) are included into the general scheme automatically.

Suppose that before some instant of time (let it be $t=0$) both the medium and the boundaries were time independent. Then solutions of (2.4) could be factorized:

$$\mathbf{A}(\mathbf{r},t)=\mathbf{g}(\mathbf{r})\exp(-i\omega t)\;, \tag{2.6}$$

$$\mathrm{rot}\left[\frac{1}{\mu}\mathrm{rot}\mathbf{g}\right]-\frac{\omega^2\epsilon(\mathbf{r})\mathbf{g}}{c^2}=0\;. \tag{2.7}$$

The scalar product (2.5) was proportional to the usual scalar product

$$((\mathbf{A}_1,\mathbf{A}_2))=-\tfrac{1}{2}(\omega_1+\omega_2)\exp[i(\omega_2-\omega_1)t](\mathbf{g}_2,\mathbf{g}_1)\;, \tag{2.8}$$

$$(\mathbf{g}_2,\mathbf{g}_1)=\int \epsilon(\mathbf{r})\mathbf{g}_2^*\mathbf{g}_1 d^3\mathbf{r}\;. \tag{2.9}$$

But it is known that solutions of Eq. (2.7) form the complete orthogonal set of vector functions with respect to scalar product (2.9). Therefore any real vector field can be decomposed over this set of functions:

$$\mathbf{A}(\mathbf{r},t)=\sum_n [a_n\mathbf{g}_n(\mathbf{r})\exp(-i\omega_n t)+a_n^*\mathbf{g}^*(\mathbf{r})\exp(i\omega_n t)]\;. \tag{2.10}$$

Comparing (2.8) and (2.9) we conclude that the set of basis functions $\{\mathbf{A}_n\}$ satisfying generalized wave equation (2.4) can be normalized as follows ($n=1,2,\ldots$):

$$((\mathbf{A}_n,\mathbf{A}_m))=\delta_{nm}\;,\quad ((\mathbf{A}_n,\mathbf{A}_m^*))=0\;. \tag{2.11}$$

After the instant when the properties of the medium became time dependent, the basis functions change their explicit expressions, but the scalar products (2.11) will not change. Then for $t>0$ we can write instead of (2.10) the following decomposition:

$$\mathbf{A}(\mathbf{r},t)=\sum_n [a_n\mathbf{A}_n(\mathbf{r},t)+a_n^*\mathbf{A}_n^*(\mathbf{r},t)]\;. \tag{2.12}$$

Then we proclaim that the (time-independent) coefficients of this expansion operators satisfy bosonic commutation relations and thus obtain the *quantized* field from a classical one.

If in some period of time the medium will become time independent again; then the physical states will be described with monochromatic mode functions of the type (2.6), which will not coincide in general with the basis functions of expansion (2.12). Therefore we have two different decompositions of the field operator: expansion (2.12) over the states corresponding to the physical photons in remote past, and an expansion like (2.10) over the physical states arising in future. Designating the "physical" states with the superscript 0, we can expand each set of basis functions into a series with respect to another one:

$$\mathbf{A}_n=\sum_m [\alpha_{nm}\mathbf{A}_m^{(0)}+\beta_{nm}\mathbf{A}_m^{(0)*}]\;. \tag{2.13}$$

The corresponding expansion of "new" creation and annihilation operators over the set of "old" ones is as follows:

$$\hat{a}_m^{(0)}=\sum_n [\hat{a}_n\alpha_{nm}+\hat{a}_n^{\dagger}\beta_{nm}^*]\;. \tag{2.14}$$

All values entering this relation do not depend on time.

Remember that the initial state of the quantized field was determined with respect to the set of "old" operators (without the superscript 0). Then using expansion (2.14) we can calculate all quantum statistical characteristics of the field in the final state. Taking into account conditions (2.11) and the evident properties of the scalar product (2.5),

$$((\mathbf{A}_1,\mathbf{A}_2))=((\mathbf{A}_2,\mathbf{A}_1))^*=-((\mathbf{A}_2^*,\mathbf{A}_1^*))\;, \tag{2.15}$$

one can express the coefficients of expansions (2.13) or (2.14) as

$$\alpha_{nm}=((\mathbf{A}_n,\mathbf{A}_m^{(0)}))\;,\quad \beta_{nm}=((\mathbf{A}_n^*,\mathbf{A}_m^{(0)}))^*\;. \tag{2.16}$$

If an external current $\mathbf{J}(\mathbf{r},t)$ is present, then the basic second-order equation (2.4) becomes nonuniform,

$$\mathrm{rot}\left[\frac{1}{\mu}\mathrm{rot}\,\mathbf{A}\right]+\frac{1}{c^2}\frac{\partial}{\partial t}\left[\epsilon\frac{\partial \mathbf{A}}{\partial t}\right]=\frac{4\pi}{c}\mathbf{J}\;. \tag{2.17}$$

Let us designate by $\mathbf{A}\{\mathbf{J}\}$ its unique solution, which is proportional to $\mathbf{J}$ (i.e., which goes to zero when $\mathbf{J}=0$). Then only minor changes are necessary: it is sufficient to add $\mathbf{A}\{\mathbf{J}\}$ to the right-hand side of (2.12) and extract $\mathbf{A}\{\mathbf{J}\}$ from functions $\mathbf{A}_1$ and $\mathbf{A}_2$ in the right-hand side of formula (2.5) defining the scalar product.

The quantization scheme based on introducing "old" and "new" Heisenberg operators connected with some linear canonical transformation is rather usual for the quantum field theory. For example, this is the main tool in the theory of particle creation in an external field or in a nonstationary universe (see, e.g., [13–15] and references therein). However, it was not applied to the electrodynamics in nonuniform and nonstationary dielectrics until recent years [10,12].

III. SPECIFIC EXAMPLES FOR FACTORIZED MEDIA

To calculate the coefficients of the canonical transformation (2.14) (through which all physical quantities related to the system under study can be expressed), one needs the explicit form of the field mode functions satisfying Eq. (2.4) and determining scalar product integrals (2.5) and (2.16). Unfortunately, the complete explicit set of mode functions can be found only for rather simple special cases. One of them corresponds to the electromagnetic field inside an empty resonator with ideal walls moving according to the given law of motion [13,16–19]. Here we want to consider another case admitting exact solutions, namely, the case of media with factorized electric and magnetic permeabilities:

$$\epsilon(\mathbf{r},t)=\bar{\epsilon}(\mathbf{r})\chi(t)\ ,\quad \mu(\mathbf{r},t)=\bar{\mu}(\mathbf{r})\nu(t) \tag{3.1}$$

(the boundaries do not move). Then mode functions can be also sought in a factorized form:

$$\mathbf{A}(\mathbf{r},t)=\mathbf{g}(\mathbf{r})\xi(t)\ ,\quad \mathbf{D}(\mathbf{r},t)=\bar{\epsilon}(\mathbf{r})\mathbf{g}(\mathbf{r})\eta(t)\ . \tag{3.2}$$

Let us demand the function $g(r)$ to satisfy the equation

$$\mathrm{rot}(\bar{\mu}^{-1}\mathrm{rot}\mathbf{g})=k^2\bar{\epsilon}(\mathbf{r})\mathbf{g}\ ,\quad k=\mathrm{const}\ . \tag{3.3}$$

Then Eqs. (2.2) and (2.4) result in the following ordinary differential equations for time-dependent factors of the vector potential and electric displacement:

$$\frac{d\eta}{dt}=k^2c\xi/\nu(t)\ ,\quad \frac{d\xi}{dt}=-c\eta/\chi(t)\ . \tag{3.4}$$

Equations (3.4) resemble equations of motion of an oscillator with time-dependent mass and frequencies. The role of the generalized coordinate is played by the electric displacement time-dependent factor, while the vector potential time-dependent factor plays the role of generalized momentum. Equations (3.4) can be replaced by the following second-order differential equation:

$$\begin{gathered}\frac{d^2\eta}{dt^2}+\gamma(t)\frac{d\eta}{dt}+\Omega^2(t)\eta=0\ ,\\ \gamma=\frac{1}{\nu}\frac{d\nu}{dt}\ ,\quad \Omega^2=\frac{k^2c^2}{\nu(t)\chi(t)}\ .\end{gathered} \tag{3.5}$$

We shall consider the field inside a resonator. Then solutions of Eq. (3.3) can be chosen to be real vector functions satisfying the orthogonality conditions

$$\int \epsilon(\mathbf{r})\mathbf{g}_k(\mathbf{r})\mathbf{g}_l(\mathbf{r})d^3\mathbf{r}=k^2\delta_{kl}\ . \tag{3.6}$$

Complex solutions of Eq. (3.5) can be normalized as follows:

$$\nu(t)\left[\eta^*\frac{d\eta}{dt}-\eta\frac{d\eta^*}{dt}\right]=-2i\ . \tag{3.7}$$

This means that we choose the solution of Eq. (3.5) in the stationary case in the form of

$$\eta_0(t)=(\nu_0\Omega_0)^{-1/2}\exp(-i\Omega_0 t)\ . \tag{3.8}$$

Because of (3.6), coefficients (2.16) are not equal to zero only for coinciding indices (intermode interactions are absent), so we may omit the indices. Taking into account Eqs. (2.5), (3.4), and (3.8), one can represent these coefficients as

$$\alpha=\frac{1}{2}\left[\frac{\nu_0}{\Omega_0}\right]^{1/2}\left[\Omega_0\eta+i\frac{d\eta}{dt}\right]\exp(i\Omega_0 t)\ , \tag{3.9}$$

$$\beta=\frac{1}{2}\left[\frac{\nu_0}{\Omega_0}\right]^{1/2}\left[\Omega_0\eta-i\frac{d\eta}{dt}\right]\exp(-i\Omega_0 t)\ . \tag{3.10}$$

Let us introduce the quadrature components and their variances as follows:

$$\begin{gathered}\hat{X}_1(t)=2^{-1/2}[\hat{a}_0\exp(-i\Omega_0 t)+\hat{a}_0^\dagger\exp(i\Omega_0 t)]\ ,\\ \hat{X}_2(t)=i2^{-1/2}[\hat{a}_0^\dagger\exp(i\Omega_0 t)-\hat{a}_0\exp(-i\Omega_0 t)]\ ,\end{gathered} \tag{3.11}$$

$$\sigma_{ij}=\tfrac{1}{2}\langle \hat{x}_i\hat{x}_j+\hat{x}_j\hat{x}_i\rangle-\langle\hat{x}_i\rangle\langle\hat{x}_j\rangle\ . \tag{3.12}$$

Suppose for simplicity that initially the field was in the coherent quantum state. Taking into account Eq. (2.14), one can easily obtain the expressions

$$\begin{aligned}\sigma_{11}(t)&=\tfrac{1}{2}|\alpha\exp(-i\Omega_0 t)+\beta\exp(i\Omega_0 t)|^2\\ &=\tfrac{1}{2}\nu_0\Omega_0|\eta|^2\ ,\end{aligned} \tag{3.13}$$

$$\begin{aligned}\sigma_{22}(t)&=\tfrac{1}{2}|\alpha\exp(-i\Omega_0 t)-\beta\exp(i\Omega_0 t)|^2\\ &=\tfrac{1}{2}\nu_0\Omega_0^{-1}\left|\frac{d\eta}{dt}\right|^2\ ,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\sigma_{12}(t)&=\mathrm{Im}[\alpha\beta^*\exp(-2i\Omega_0 t)]\\ &=\tfrac{1}{2}\nu_0\mathrm{Re}\left[\eta^*\frac{d\eta}{dt}\right]\ .\end{aligned} \tag{3.15}$$

We see that a time-dependent medium transforms an initially coherent state to a "correlated quantum state" characterized by a nonzero covariance (3.15) and unequal variances (3.13) and (3.14).

This state minimizes the generalized uncertainty relation by Schrödinger and Robertson [20]:

$$\sigma_{11}\sigma_{22}-\sigma_{12}^2\geq\tfrac{1}{4} \tag{3.16}$$

[the equality takes place in the case under study due to Eq. (3.7)]. For a detailed review of various forms of uncertainty relations, see [21]. Properties of correlated quantum states were investigated in [22–25]. These states can be considered as a generalization of "squeezed states," properties of which were investigated (although they were known under different names, especially in the earliest papers) by many authors; see, e.g., original papers [26–32] and reviews [33–36].

It is worth noting that for a quite arbitrary dependence $\Omega(t)$ the combination $I=\sigma_{11}\sigma_{22}-\sigma_{12}^2$ does not depend on time because of (3.7). This combination is in fact the simplest example of so-called universal quantum invariants, i.e., certain functions of variances which are conserved in time independently of the concrete parameters of quantum canonical transformations. For a general multidimensional canonical transformation (2.14) such invariants were studied in [37]. For vacuum or coherent initial states $I=\frac{1}{4}$.

The invariant I has a simple geometrical interpretation [37]. Suppose we have some Gaussian distribution in the phase space (x_1,x_2). Then the curves of equal (quasi-) probability have the form of ellipses. If one calculates the area confined within such a curve, it appears proportional to $I^{1/2}$. Therefore conservation of invariant I is equivalent to the conservation of the phase volume in the process of evolution due to the famous Liouville theorem of classical mechanics.

In the case of a harmonic oscillator with time-independent frequency all equal-probability ellipses rotate in the phase space without changing their shapes. This means that any Glauber's coherent state which is represented by a circle in the phase space cannot be transformed into a squeezed or correlated state (represented by an ellipse) in the processes with time-independent parameters of the system. Consequently, generation of correlated or squeezed states from an initial vacuum or coherent state requires time dependence of the system's parameters.

Thus let us consider as the first example the case of a parametric excitation when the properties of the medium harmonically oscillate at twice the frequency with respect to some (resonance) field mode. This can be achieved, for example, by means of a change in the density of the medium due to the action of a powerful external monochromatic classical pumping wave going in the transverse direction. Since the magnetic effects are extremely weak, we can write

$$\Omega^2(t)=\Omega_0^2[1+\kappa\cos(2\Omega_0 t)]\ ,\quad \gamma=0\ . \tag{3.17}$$

We look for the solution of Eq. (3.5) in the form

$$\eta(t)=(\nu_0\Omega_0)^{-1/2}[u(t)\exp(-i\Omega_0 t)+v(t)\exp(i\Omega_0 t)] \tag{3.18}$$

with slowly varying time-dependent amplitudes. Substituting (3.17) and (3.18) into (3.5), neglecting the second-order derivatives of slowly varying amplitudes, and performing averaging over fast oscillations with frequency Ω_0 [this approximation is valid provided the depth of modulation in (3.17) is small, i.e., $|\kappa|\ll 1$], we arrive at the equations

$$\frac{du}{dt}=-i\Omega_0\kappa v/4\ ,\quad \frac{dv}{dt}=i\Omega_0\kappa u/4\ , \tag{3.19}$$

whose solutions are

$$u(t)=\cosh(\Omega_0\kappa t/4)\ ,\quad v(t)=i\sinh(\Omega_0 t/4)\ . \tag{3.20}$$

The variances (3.13) and (3.14) oscillate with twice the resonance frequency, but their ratio (the so-called squeezing coefficient) is confined at every instant between the values

$$\exp(-\Omega_0\kappa t)\le\sigma_{11}/\sigma_{22}\le\exp(\Omega_0\kappa t)\ . \tag{3.21}$$

If the initial quantum state of the field was a vacuum, then the number of photons generated in the mode under study is equal to

$$N=\langle\hat{a}^{(0)\dagger}\hat{a}^{(0)}\rangle=|v|^2=[\sinh(\Omega_0\kappa t/4)]^2\ , \tag{3.22}$$

and for large values of parameter $\Omega_0\kappa t$ the number of photons increases exponentially with time. In real cases of resonators with finite Q factor, formula (3.22) is valid provided time t is less than the relaxation time $\tau=Q/\Omega_0$. Then the maximal number of photons created from the vacuum state is

$$N_{\max}=\tfrac{1}{4}\exp(\tfrac{1}{2}\kappa Q)\ ,\quad \kappa Q\gg 1\ . \tag{3.23}$$

Certain inequalities for the squeezing coefficients can be found for arbitrary time dependence of the frequency in Eq. (3.5) (for nonmagnetic medium) if one takes into account that these equations turn into the Helmholtz equation describing the one-dimensional wave propagation through a nonhomogeneous medium after the replacement $t\to x$, $\Omega(t)\to k(x)$. Suppose that function $\Omega(t)$ assumes constant values Ω_i in the remote past and Ω_0 in the future and the initial value of $\eta(t)$ is $\Omega_i^{-1/2}\exp(-i\Omega t)$ when $t\to-\infty$. Then for $t\to\infty$ we have

$$\eta(t)=\Omega_0^{-1/2}[\alpha\exp(-i\Omega_0 t)+\beta\exp(i\Omega_0 t)] \tag{3.24}$$

with time-independent coefficients satisfying the relation

$$|\alpha|^2-|\beta|^2=1\ , \tag{3.25}$$

resulting from (3.7). The ratio β/α can be treated as the amplitude reflection coefficient from the effective "potential barrier" represented by function $\Omega^2(t)$. For the energy reflecting coefficient we get from (3.24)

$$R=\left|\frac{\beta}{\alpha}\right|^2=\frac{\Omega_0^2|\eta|^2+\left|\frac{d\eta}{dt}\right|^2-2\Omega_0}{\Omega_0^2|\eta|^2+\left|\frac{d\eta}{dt}\right|^2+2\Omega_0}\ , \tag{3.26}$$

where relation (3.7) (with $\nu=1$) was taken into account. Let us introduce the notation

$$s=(\sigma_{11}/\sigma_{22})^{1/2}\ ,\quad r=\sigma_{12}/(\sigma_{11}\sigma_{22})^{1/2}\ .$$

From (3.7) and (3.13)–(3.15) we get

$$|\eta|^2=\frac{s}{\Omega_0}(1-r^2)^{-1/2}\ ,\quad \left|\frac{d\eta}{dt}\right|^2=\frac{\Omega_0}{s}(1-r^2)^{-1/2}\ . \tag{3.27}$$

Then (3.26) and (3.27) result in the relation

$$s=\frac{(1+R)(1-r^2)^{1/2}\pm[4R-r^2(1+R)^2]^{1/2}}{1-R}\ . \tag{3.28}$$

As a consequence we obtain the limitations on the possible values of the "correlation coefficient" r and the "squeezing coefficient" s for the given value of the energy reflection coefficient from the effective barrier [25]:

$$|r| \le \frac{2R^{1/2}}{1+R} , \tag{3.29}$$

$$\frac{1-R^{1/2}}{1+R^{1/2}} - \le s \le \frac{1+R^{1/2}}{1-R^{1/2}} . \tag{3.30}$$

Thus we see that neither squeezing nor correlation can be obtained for the reflectionless "barrier." However, one should remember that the time-dependent barrier is in a sense inverse with respect to the space-dependent one: function $\Omega^2(t)$ is proportional to $1/\epsilon(t)$, whereas the analogous function $k^2(x)$ in the usual Helmholtz equation is proportional to $\epsilon(x)$.

If the initial quantum state of the field was a vacuum, then the number of photons generated in the mode under study is uniquely related to the energy reflection coefficient due to Eqs. (3.22) and (3.23) ($\nu=1$),

$$N = \langle \hat{a}^{(0)\dagger}\hat{a}^{(0)} \rangle = |\beta|^2 = \frac{R}{1-R} . \tag{3.31}$$

The maximal squeezing coefficient can be expressed in terms of the number of quanta due to (3.30) and (3.31) as follows:

$$s_{\max} = [N^{1/2} + (1+N)^{1/2}]^2 , \tag{3.32}$$

so that

$$1-2N^{1/2} \le s \le 1+2N^{1/2} \quad \text{for } N \ll 1 \tag{3.33}$$

and

$$(4N)^{-1} \le s \le 4N \quad \text{for } N \gg 1 . \tag{3.34}$$

If dielectric permeability varies with time more or less monotonously and sufficiently slow, so that

$$\frac{d\Omega}{dt} \ll \Omega^2 , \tag{3.35}$$

then Eq. (3.5) can be solved in adiabatic approximation corresponding to the approximation of geometrical optics for the Helmholtz equation. The solutions of the zeroth order have the form

$$\eta(t) = \Omega^{-1/2}(t) \exp\left[\pm i \int_0^t \Omega(\tau)\, d\tau \right] . \tag{3.36}$$

The first-order corrections to these solutions yield the reflection coefficient (see, e.g., [38])

$$R = \left| \frac{1}{4} \int_{-\infty}^{\infty} \frac{d\tau}{\Omega^{1/2}(\tau)} \frac{d}{d\tau} \left[\frac{d\Omega/d\tau}{\Omega^{3/2}} \right] \times \exp\left[-2i \int_0^\tau \Omega(x)\, dx \right] \right|^2 . \tag{3.37}$$

This formula holds provided $R \ll 1$. Then the average number of created photons is also given by (3.37), as well as the probability to register a photon.

As an example let us consider the case when the time-dependent factor in dielectric permeability (3.1) decreases in time according to exponential law in the interval $0 \le t \le t_0$:

$$\chi(t) = \begin{cases} 1 , & t<0 \\ \exp(-\kappa t) , & 0 \le t \le t_0 \\ \exp(-\kappa t_0) , & t > t_0 . \end{cases} \tag{3.38}$$

In this case the main contribution to the integral (3.37) is given by two delta functions $\delta(t)$ and $\delta(t-t_0)$ arising due to the discontinuity of $d\Omega/dt$ at points $t=0$ and $t=t_0$. The number of created photons is

$$N(t_0) = \frac{\kappa^2}{64}[1+\exp(-\kappa t_0) - 2\exp(-\kappa t_0/2)\cos(2\varphi)](kc)^{-2} , \tag{3.39}$$

$$\varphi = \frac{2}{\kappa}[\exp(\kappa t_0/2)-1](kc) .$$

This formula is valid provided $\kappa \ll kc$.

Yablonovich [39] proposed to use a medium with refractive index decreasing in time (the so-called "plasma window") to simulate the *Unruh* effect, i.e., creation of quanta in an accelerated frame of reference. Using some heuristic reasoning he claimed that the spectrum of photons created in such a plasma window would resemble Planck's spectrum with effective temperature proportional to $|(1/\chi)/(d\chi/dt)|$, i.e., parameter κ in the example discussed. Equations (3.39) show that the real spectrum of photons created in the exponential model of a "plasma window" when $(1/\chi)(d\chi/dt)=\text{const}$ has nothing in common with Planck's spectrum even in the asymptotical limit of infinitely long time, $t_0 \to \infty$. An exponentially small reflection coefficient, as is known from the theory of adiabatic invariants, is possible only for those functions $\chi(t)$ that have continuous derivatives of all orders [40]. The analytical solution of (3.5) for smooth functions $\Omega(t)$ of such kind are known, e.g., for the symmetric Epstein profile [38,41]:

$$\Omega^2(t) = (kc)^2\{1 - M[\cosh(\gamma t/2)]^{-2}\} . \tag{3.40}$$

If $\gamma^2/(4kc)^2 < M < 1$, then the energy reflection coefficient is given by the expression [38,41]

$$R = \frac{\cosh^2(\pi d_1)}{\cosh[\pi(d_1+s)]\cosh[\pi(d_1-s)]} , \tag{3.41}$$

where

$$s = 2kc/\gamma , \quad d_1 = [M(2kc/\gamma)^2 - \tfrac{1}{4}]^{1/2} . \tag{3.42}$$

The number of created quanta due to (3.31) equals

$$N = \left[\frac{\cosh(\pi d_1)}{\sinh(\pi s)} \right]^2 , \tag{3.43}$$

and in the adiabatic limit, since $M<1$, we get indeed "Wien's spectrum" with effective temperature" proportional to γ,

$$N(k) = \exp[-(1-M^{1/2})4\pi kc/\gamma] , \quad 4\pi kc/\gamma \gg 1 . \tag{3.44}$$

It should be mentioned, however, that the case $M>0$ corresponds not to a plasma window, but to a "dielectric window," since in this case $\epsilon(t) > \epsilon_{in}$. In the plasma case of $M < 0$, the following formula is valid instead of (3.41)

$$R = \frac{\cos^2(\pi d_2)}{\cos^2(\pi d_2)\cosh^2(\pi s) + \sin^2(\pi d_2)\sinh^2(\pi s)} ,\quad d_2 = [\tfrac{1}{4} - M(2kc/\gamma)^2]^{1/2} . \tag{3.45}$$

This leads to the strongly oscillating number of quanta

$$N = \left[\frac{\cos(\pi d_2)}{\sinh(\pi s)} \right]^2 . \tag{3.46}$$

In the adiabatic limit

$$N = 4\cos^2(2\pi kc|M|^{1/2}/\gamma)\exp(-4\pi kc/\gamma) ,\quad 2\pi kc|M|^{1/2}\gamma \gg 1 . \tag{3.47}$$

We see that in certain modes photons are not generated at all due to a peculiar "interference in time." In the case of a rapid change of dielectric permeability, when $\gamma^2/(4kc)^2 \gg |M|$, oscillations disappear and (3.46) yields

$$N = (2kcM/\gamma)^2 \ll 1 , \tag{3.48}$$

which resembles the "Rayleigh-Jeanes spectrum," but with effective temperature *inversely proportional* to the square of parameter γ characterizing the rate of change of dielectric permeability.

Another exactly solvable case corresponds to the transitional Epstein's profile:

$$\Omega^2(t) = (kc)^2 \left\{ 1 - p\frac{\exp(\gamma t)}{1+\exp(\gamma t)} \right\} , \tag{3.49}$$

which can be considered as a smoothed variant of the exponential dependence (3.38). The reflection coefficient is given by [38,41]

$$R = \left\{ \frac{\sinh\{\frac{1}{2}\pi s[1-(1-p)^{1/2}]\}}{\sinh\{\frac{1}{2}\pi s[1+(1-p)^{1/2}]\}} \right\}^2 ,\quad s = 2kc/\gamma . \tag{3.50}$$

Consequently,

$$N = \frac{(\sinh\{\frac{1}{2}\pi s[1-(1-p)^{1/2}]\})^2}{\sinh(\pi s)\sinh[\pi s(1-p)^{1/2}]} . \tag{3.51}$$

In the adiabatic limit we have again Wien's spectrum

$$N = \exp[-4\pi kc(1-p)^{1/2}/\gamma] ,\quad 4\pi kc/\gamma \gg 1 . \tag{3.52}$$

The opposite limit $\gamma \to \infty$ transforms (3.49) into the sharp step-function barrier. Such a frequency-jump case was considered in connection with the problem of squeezed-state generation in [42–44]. If the ratio of the final dielectric permeability to the initial one is ϵ, then the reflection coefficient is given by the usual Fresnel formula, and the number of created quanta equals

$$N = (\epsilon^{1/2} - 1)^2/(4\epsilon^{1/2}) , \tag{3.53}$$

without any dependence on the wave number. But an interesting effect arises if in some time τ the dielectric permeability restores (also instantly) its initial value. This situation is described by the usual formulas for the ideal Fabry-Pérot resonator, and after simple algebra we get

$$N = \frac{(\epsilon-1)^2}{4\epsilon}\sin^2\left[\frac{kc\tau}{\epsilon^{1/2}}\right] \tag{3.54}$$

(one should remember that the effective refractive index in the time-dependent case is not $\epsilon^{1/2}$ but $\epsilon^{-1/2}$). We see again that due to a kind of temporal interference photons in certain modes are not created at all, as well as in the case of a smooth barrier (3.46).

In the case when dielectric permeability varies according to (3.38) the exact solution of (3.5) can be obtained as a linear combination of Bessel's functions with argument $z = 2kc\,\exp(\kappa t/2)/\kappa$:

$$\eta(z) = a_1 J_0(z) + a_2 Y_0(z) ,\quad \eta(z_0) = 1 ,\quad \frac{d\eta}{dz}(z_0) = -i ,\quad z_0 = 2kc/\kappa . \tag{3.55}$$

Substituting (3.55) into (3.10) and then into (3.31) we can get the coefficient β and, consequently, the occupation number

$$N = \left[\Omega_0|\eta|^2 + \left|\frac{d\eta}{dt}\right|^2/\Omega_0 \right] - \frac{1}{2} . \tag{3.56}$$

In the limit $\kappa \ll kc$ the decomposition of (3.55) for $z \to \infty$ produces the same result as (3.39). In the opposite limit ($\kappa \gg kc$) of rapid permeability decrease to zero [it would correspond to $\epsilon = 0$ in (3.53)] the adiabatic condition (3.35) is no more valid. The expression for the photon number can be obtained on decomposing (3.55) over $z_0 \to 0$. Taking into account the initial condition $\eta \approx J_0(z)$ one can see that for $t \to \infty$, which is equivalent to $z \to \infty$, the number of quanta tends to the constant (although large) asymptotic value

$$N \to \frac{\kappa}{4\pi kc} . \tag{3.57}$$

IV. CONCLUSION

Here we list the main results of the paper. We have shown that in the case of space-time factorized media the problem of generating squeezed and correlated states of an electromagnetic field is reduced in fact to the problem of harmonic oscillator with a time-dependent frequency, which in turn is intimately related to the problem of one-dimensional wave propagation in a medium with variable refraction index. Using this analogy we have expressed the number of created quanta in terms of the energy reflection coefficient from a certain effective potential barrier. In addition, we have derived inequalities estimating the maximal possible degrees of squeezing and correlation in terms of the same parameter.

Using both approximate (adiabatic) and exact solutions we have considered several important specific time dependences of dielectric permeability. These examples show that the number of created quanta (as well as correlation and squeezing coefficients) depends, in general, on rather fine details of the dielectric permeability time dependence. In particular, this quantity can strongly oscillate and go to zero for certain values of the wave number due to a kind of temporal interference.

In real experiments time dependence of the dielectric permeability arises as a result of action on the nonlinear medium by an external pumping field. Thus the situation considered in this paper, when the dielectric permeability is some function of time prescribed beforehand, is in fact a model of real experimental situations. However, this model presents a correct qualitative description of the processes of squeezing and photon creation. It is seen, e.g., in the example of the parametric resonance at the twice resonator eigenfrequency. In this case our results relating to the rate of photon generation are in qualitative agreement with the results of [45], where the pumping field was taken into account explicitly (the case of classical pumping was investigated in detail, e.g., in [46]).

ACKNOWLEDGMENT

A.B. Klimov acknowledges the support from CONACYT.

APPENDIX

Here we want to discuss some interesting features relating to the electromagnetic wave propagation in spatially uniform but time-dependent media. The electric displacement for traveling waves can be expressed as

$$\mathbf{D}_k(x,t)=e^{ikx}\eta_k(t) , \tag{A1}$$

where the time-dependent factor satisfies the equation

$$\frac{d^2\eta_k}{dt^2}+k^2c^2\eta_k/\epsilon(t)=0 . \tag{A2}$$

Suppose $\epsilon(t)=1$ for $t<0$. Then the traveling-wave solution is

$$\mathbf{D}_k^{(i)}(x,t)=\exp[ik(x-ct)] , \quad t<0 . \tag{A3}$$

If the dielectric permeability changes in the interval $0<t<T$ but assumes some constant value ϵ_0 for $t>T$, then function (A3) will be transformed for $t>T$, into the superposition of two waves traveling in opposite directions,

$$\mathbf{D}_k^{(f)}(x,t)=\alpha\exp[ik(x-ct\epsilon_0^{-1/2})] +\beta\exp[ik(x+ct\epsilon_0^{-1/2})] . \tag{A4}$$

We see that the ratio $|\beta/\alpha|^2$, which was treated in Sec. III as the reflection coefficient from some conventional "potential barrier," appears in the case under study as a genuine reflection coefficient, since the wave going in the opposite direction really exists. This is seen distinctly in the simplest example of an instant change of the dielectric permeability (which can be used by an instant change of the medium density, temperature, or other parameters due to some external action; see, e.g., [47]). Then an arbitrary initial wave packet $\mathbf{D}^{(i)}(x-ct)$, $t<0$, will be transformed due to the nonstationary wave equation

$$\frac{\partial^2\mathbf{D}}{\partial x^2}=\frac{\epsilon(t)}{c^2}\frac{\partial^2\mathbf{D}}{\partial t^2} \tag{A5}$$

and the continuity conditions into the function ($t>0$)

$$\mathbf{D}(x,t)=\tfrac{1}{2}[(1+\epsilon_0^{1/2})\mathbf{D}^{(i)}(x-ct\epsilon_0^{1/2}) +(1-\varepsilon_0^{1/2})\mathbf{D}^{(i)}(x+ct\epsilon_0^{1/2})] . \tag{A6}$$

Since we consider nondispersive media, this solution is physically acceptable provided $\varepsilon_0>1$. It is interesting to notice that transmitted wave packet [the first term on the right-hand side of (A6)] is amplified in comparison with the initial one. Moreover, for $\varepsilon_0>9$ the reflected wave packet is amplified too. The forms of both transmitted and reflected impulses are the same as the form of the initial packet, since the reflection coefficient does not depend on the wave number.

Now suppose that in time T function $\epsilon(t)$ restores its initial unit value. Then coefficients α and β in (A4) become dependent on the wave number:

$$\alpha=\tau_+\exp(ik\delta_-)-\tau_-\exp(ik\delta_+) ,$$
$$\beta=\rho[\exp(-ik\delta_-)-\exp(-ik\delta_+)] , \tag{A7}$$

where

$$\tau_\pm=\frac{(\epsilon_0^{1/2}\pm1)^2}{4\epsilon_0^{1/2}} , \quad \rho=\frac{\epsilon_0-1}{4\epsilon_0^{1/2}} , \quad \delta_\pm=cT(1\pm\epsilon_0^{-1/2}) . \tag{A8}$$

The reflected wave disappears provided the condition

$$k(\delta_+-\delta_-)=2\pi m , \quad m=1,2,\ldots \tag{A9}$$

is fulfilled. However, this is true only for a monochromatic initial wave (with an infinite extent in space). For packets bounded in space the situation can be elucidated in the frame of an exactly solvable example of a Gaussian initial packet

$$D^{(i)}(x,t)=\exp[-(x-ct)^2/\sigma^2+ik_0(x-ct)] . \tag{A10}$$

Calculating the Fourier transform of (A10) (which is a Gaussian exponential again) and replacing each Fourier component by expression (A4) with coefficients given in (A7) and (A8) one can easily obtain the following explicit expressions for the transmitted D_t and reflected D_r waves:

$$D_t(x,t)=\tau_+\exp[-(x-ct+\delta_-)^2/\sigma^2+ik_0(x-ct+\delta_-)] -\tau_-\exp[-(x-ct+\delta_+)^2/\sigma^2 +ik_0(x-ct+\delta_+)] , \tag{A11}$$

$$D_r(x,t)=\rho\{\exp[-(x+ct-\delta_+)^2/\sigma^2+ik_0(x+ct-\delta_+)] -\exp[-(x+ct-\delta_-)^2/\sigma^2 +ik_0(x+ct-\delta_-)]\} . \quad (A12)$$

We see that the initial impulse is split into four packets with the same shape: two transmitted and two reflected packets. This phenomenon manifests itself in the most distinct form for narrow packets satisfying the condition

$$\sigma \ll \delta_+ - \delta_- = 2cT\epsilon_0^{-1/2} . \quad (A13)$$

In this case no disappearance of the reflected wave is observed.

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