Perturbations of optical solitons

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A perturbation theory is developed to investigate the effects of various perturbations on soliton propagation down an optical fiber. The theory is formulated within the "natural" framework of inverse scattering theory. The perturbative effects of third-order dispersion, the soliton self-frequency shift, bandwidth-limited gain, periodic modulation, and stochastic fluctuations are analyzed in detail. In each case, the analytic results presented here are in excellent agreement with simulations carried out here and elsewhere.

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I. INTRODUCTION

The purpose of this article is to develop a perturbation theory to investigate aspects of soliton propagation down a nonideal optical fiber. In so doing, the advantages of using a "natural" mathematical framework based on inverse scattering theory will be emphasized [1,2]. The equation under study is the perturbed nonlinear Schrödinger equation, whose unperturbed form is known to be integrable using the techniques of inverse scattering theory [1–3]. In particular, it has the exact single-soliton solution, Eq. (2) below. Additional perturbations modify this in two distinct ways.

(i) The soliton parameters, which were constants of the motion in the unperturbed case, now vary with distance down the fiber. If the perturbation is small, this change is adiabatic [4,5].

(ii) The perturbation is responsible also for the generation of a background radiation field, which is superimposed on the soliton pulse. Depending on the nature of the perturbation, this can exhibit quite complicated resonance features [6–13].

The propagation of optical pulses down a nonideal anomalously dispersive single-mode optical fiber is described by the perturbed nonlinear Schrödinger equation [2]

$$i\frac{\partial q}{\partial x}-\frac{\partial^2 q}{\partial t^2}-2q|q|^2=iF\ . \tag{1}$$

q is the complex field envelope, $x>0$ is the propagation distance down the fiber, and t is retarded time; all variables appear in normalized form. Perturbing influences are represented by the complex term iF, with appropriate choices for F. With F set to zero, Eq. (1) is integrable using the techniques of inverse scattering theory [1,2]. In particular, it has the exact single-soliton solution

$$q(x,t)\equiv q_s=2\eta_1 e^{-2i\xi_1 t+4i(\xi_1^2-\eta_1^2)x}\,\mathrm{sech}[2\eta_1(t-4\xi_1 x)]\ , \tag{2}$$

hereafter denoted q_s. The parameters η_1 and ξ_1 characterize the soliton; $2\eta_1$ is its height and inverse width, $4\xi_1$ is its velocity relative to a convenient reference frame.

We will be interested in the case when $F\neq 0$, but is assumed to represent a small perturbing influence on the propagating soliton. Many different forms for F have been considered in the literature, including the following.

(i)

$$F=\epsilon\frac{\partial^3 q}{\partial t^3}\ , \tag{3a}$$

ϵ real, representing the perturbing effects of third-order dispersion in the fiber [6,7,10,14].

(ii)

$$F=i\epsilon q\frac{\partial}{\partial t}|q|^2\ , \tag{3b}$$

ϵ real, causing a self-frequency shift in the propagating soliton [15,16]. A more general nonlocalized form for F has been considered elsewhere, but is not discussed here [17–19].

(iii)

$$F=\Gamma q+\gamma\frac{\partial^2 q}{\partial t^2}\ , \tag{3c}$$

Γ,γ real, representing soliton propagation through a doped fiber offering bandwidth-limited gain. For simple gain (loss), γ is zero and Γ is positive (negative) [14,20–23].

(iv)

$$F=A(x)q\ , \tag{3d}$$

where

$$A(x)=\Gamma-G\sum_{n=1}^{\infty}\delta(x-nx_a) \tag{3e}$$

is periodic in x with period x_a. $A(x)$ has zero mean when $G=\exp(\Gamma x_a)$. This form for F represents soliton amplification (loss) when $\Gamma>0$ (<0), with discrete loss (gain) at locations $x=nx_a$ [8,9,11,24–26,13].

(v) Stochastic perturbations [27–33]: There are actually two sources of stochasticity which usefully can be termed homogeneous and inhomogeneous by analogy with similar distinctions made in other branches of nonlinear optics [30]. In the inhomogeneous case, the stochasticity is a feature of the *input pulse* to the fiber, and does *not* derive from any agency in the fiber itself. Then, $F=0$, and the soliton parameters remain fixed. However, the stochastic element present in $q(0,t)$ results in an uncertainty in the value of these fixed parameters, which are known only in terms of a probability distribution [28,29]. In the homogeneous case, the stochasticity originates in some random property of the fiber itself, such as in spontaneous emissions in a fiber amplifier, where

$$F=\sigma(x,t)\ , \tag{3f}$$

or in density fluctuations in the fiber material, where

$$F=i\sigma(x)\left[\beta_0 q-i\beta_1\frac{\partial q}{\partial t}+\beta_2\frac{\partial^2 q}{\partial t^2}\right] . \tag{3g}$$

In either case, σ is an appropriate stochastic term. For purely dispersive fluctuations, β_0, β_1, and β_2 are real, as in $\sigma(x)$. The evolution equations for the soliton parameters are now described by a set of Langevin equations, possibly of multiplicative type [30]. These examples will be discussed in detail in Sec. III.

We now quote two results, which should find useful application in studies on Eq. (1). The first result is *exact* and is true for any form of the perturbation F, which need not be small. It states that, under the action of the perturbation iF, the infinity of conserved quantities C_n, $n=0,1,2,\ldots$, associated with the unperturbed equation, evolve according to

$$\frac{dC_n}{dx}=\int_{-\infty}^{\infty}[F^*,F](2i\mathcal{L})^n\begin{bmatrix} q\\ q^*\end{bmatrix}dt\ . \tag{4}$$

$\mathcal{L}$ is the integro-differential operator associated with the squared eigenfunctions obtained from the linear eigenvalue problem connected with Eq. (1), and is defined in Sec. II. An equivalent form for Eq. (4) was first reported by Karpman and Maslov [4].

The second result describes the generation of the radiation field, and generalizes recent work by Gordon [8]. Introduce an "associate field" f, which evolves according to the equation

$$i\frac{\partial f}{\partial x}=\frac{\partial^2 f}{\partial t^2}-\frac{i}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-2i\xi t}}{\xi^2+\eta_1^2}\{F^*\phi_1\bar{\psi}_1+F\phi_2\bar{\psi}_2\}^*\times dt\,d\xi\ , \tag{5}$$

where the asterisk denotes complex conjugation, and $\phi_i,\psi_i,\bar{\phi}_i,\bar{\psi}_i$ are components of Jost functions which arise in the linear eigenvalue problem associated with Eq. (1) [1,2]. A solution of Eq. (5) will be required subject to $f(x=0,t)=0$. The action of the perturbation iF on the soliton q_s generates a radiation field $\delta q(x,t)$ so that, at any location x,

$$q(x,t)=q_s+\delta q(x,t) \tag{6a}$$

with

$$\delta q(0,t)=0\ . \tag{6b}$$

Then, the required form for $\delta q(x,t)$ is

$$\delta q(x,t)=\frac{\partial^2 f}{\partial t^2}-2\gamma\frac{\partial f}{\partial t}+\gamma^2 f+q_s^2 f^*\ , \tag{7}$$

where $\gamma=-q_s^{-1}\partial q_s/\partial t$. The algorithm for finding δq is first to solve Eq. (5) for the associate field $f(x,t)$, then to find $\delta q(x,t)$ from this, using Eq. (7).

Equation (5) appears intractable at first sight, since analytic forms are not generally known for ϕ_i, etc. However, in the perturbative limit considered here, these Jost function components can be approximated by their "solitonic" expressions, and the integrals can then be evaluated using standard techniques. For example, with F given by Eq. (3a), the last term in Eq. (5) becomes

$$i\epsilon\frac{\partial^3 f}{\partial t^3}+\frac{i\epsilon}{4}\frac{\partial q_s}{dt}\ , \tag{8}$$

leading to a simple linear inhomogeneous evolution equation for f, which can be solved using standard (Fourier) transform techniques.

Equations (4), (5), and (7) are proven in Sec. II and are the main results of this article.

It is interesting to note what is happening in the use of the algorithm described to find $\delta q(x,t)$, since of course one could find an evolution equation for δq directly by substituting Eq. (6) into Eq. (1); on linearizing, this gives

$$i\frac{\partial\delta q}{\partial x}=\frac{\partial^2\delta q}{\partial t^2}+4\delta q|q_s|^2-2q_s^2\delta q^*+iF\ . \tag{9}$$

The difficulty arises when an attempt is made to solve this using standard techniques, such as a Fourier transformation with respect to the t variable, since the presence of q_s results in convolution terms which prove intractable to further analysis. Similar complicating terms are *not* present in the evolution equation for f. Replacing Eq. (9) with the equivalent equations (5) and (7) corresponds to finding $\delta q(x,t)$ by expanding it *not* in terms of the standard Fourier modes $\exp(i\omega t)$ but rather in a basis of distorted continuum modes which take proper account of the existence of the soliton. In the language of quantum optics, the soliton presence leads to a set of "dressed" continuum modes. These modes are precisely the Jost functions which arise in the linear eigenvalue problem associated with Eq. (1). The inner integral in Eq. (5) describes the manner in which F is projected onto these modes.

The salient features of inverse scattering theory required for this article are reviewed briefly in the next section, after which the main results Eqs. (4), (5), and (7) are proven. In Sec. III, the specific examples noted in Eqs. (3) are examined in detail. A few concluding comments are made in Sec. IV.

II. THE EVOLUTION EQUATIONS

A. Background material

A knowledge of the basic techniques of inverse scattering theory is assumed, and is summarized in Appendix A. Here, we identify those features required in connection with Eqs. (4), (5), and (7). The notation used throughout is the one defined in Refs. [1,2], with one modification: the roles of the independent variables x and t to denote space and time are interchanged to conform with Eq. (1). The variable x now describes "time" (distance down the fiber), while t denotes "space" and therefore appears explicitly in the scattering problem associated with Eq. (1) [cf. Eqs. (A1)].

Scattering data are summarized by the coefficients a and b (see Appendix A). Kaup [34] has shown that the evolution equations for these quantities are

$$\frac{\partial a}{\partial x}=\int_{-\infty}^{\infty}\left[\frac{\partial q}{\partial x}\phi_2\psi_2-\frac{\partial r}{\partial x}\phi_1\psi_1\right]dt$$

$$=\int_{-\infty}^{\infty}\left[\frac{\partial r}{\partial x},-\frac{\partial q}{\partial x}\right]\Phi\, dt\ , \tag{10}$$

$$\frac{\partial b}{\partial x}=\int_{-\infty}^{\infty}\left[\frac{\partial q}{\partial x}\phi_2\bar{\psi}_2-\frac{\partial r}{\partial x}\phi_1\bar{\psi}_2\right]dt$$

$$=\int_{-\infty}^{\infty}\left[\frac{\partial r}{\partial x},-\frac{\partial q}{\partial x}\right]\check{\Phi}\, dt\ . \tag{11}$$

Here, $r=-q^*$, as appropriate for Eq. (1), $\phi=[\phi_1,\phi_2]^T$, $\bar{\phi}$, ψ, and $\bar{\psi}$ are two-component spinor Jost functions discussed briefly in Appendix A; and the second forms of Eqs. (10) and (11) serve to introduce the bilinear spinors Φ and $\check{\Phi}$. In either case, we write this as $[g,h]^T$, where $g=-\phi_1\psi_1$, and $h=-\phi_2\psi_2$ for Φ and $g=-\phi_1\bar{\psi}_1$ and $h=-\phi_2\bar{\psi}_2$ for $\check{\Phi}$. Evolution equations for g and h are quoted in Eqs. (A7), with appropriate definitions of the auxiliary function k. A formal solution to these leads to the relation (A8), which is satisfied by both Φ and $\check{\Phi}$ with appropriate choice of k_-. For Φ, $k_-=\frac{1}{2}a$; for $\check{\Phi}$, $k_-=-\frac{1}{2}b$. In other words,

$$\mathcal{L}\Phi=\zeta\Phi+\frac{a}{2i}\begin{bmatrix} q \\ q^* \end{bmatrix}\ , \tag{12}$$

$$\mathcal{L}\check{\Phi}=\zeta\check{\Phi}-\frac{b}{2i}\begin{bmatrix} q \\ q^* \end{bmatrix}\ . \tag{13}$$

In each case, $\mathcal{L}$ is the integro-differential operator defined in Eq. (A9).

Equations (12) and (13) can be solved to express both Φ and $\check{\Phi}$ in terms of a formal asymptotic series; from Eq. (11), it follows that

$$\Phi=-a\sum_{n=0}^{\infty}\frac{1}{(2i\zeta)^{n+1}}(2i\mathcal{L})^n\begin{bmatrix} q \\ q^* \end{bmatrix}\ . \tag{14}$$

We will require also the asymptotic expansion for $\ln a$; this is [2]

$$\ln a=\sum_{n=0}^{\infty}\frac{1}{(2i\zeta)^{n+1}}C_n\ , \tag{15}$$

where the coefficients C_n, the conserved functionals for the unperturbed form of Eq. (1), are given by [2]

$$C_0=\int_{-\infty}^{\infty}|q|^2dt\ , \tag{16a}$$

$$C_1=-\int_{-\infty}^{\infty}q^*\frac{\partial q}{\partial t}dt\ , \tag{16b}$$

$$C_2=\int_{-\infty}^{\infty}\left[q^*\frac{\partial^2 q}{\partial t^2}+|q|^4\right]dt\ , \tag{16c}$$

$$C_3=-\int_{-\infty}^{\infty}\left[q^*\frac{\partial^3 q}{\partial t^3}+3q|q|^2\frac{\partial q^*}{\partial t}\right]dt\ , \tag{16d}$$

and so on. Finally, we state the complementary forms for C_n in terms of the spectral data: these are [2]

$$C_n=\sum_{m=1}^{N}\frac{1}{n+1}[(2i\zeta_m^*)^{n+1}-(2i\zeta_m)^{n+1}]$$

$$-\frac{1}{\pi}\int_{-\infty}^{\infty}(2i\xi)^n\ln[1-|b(\xi,x)|^2]d\xi\ . \tag{17}$$

The discrete sum gives the contribution to C_n from an arbitrary N-soliton state (throughout this article $N=1$), whereas the integral denotes the contribution from the (continuum) radiation modes.

B. Evolution equations for the perturbed system

Consider first the derivation of Eq. (4). Substituting (14) into Eq. (10), differentiating Eq. (15) with respect to x, then comparing coefficients of $(2i\zeta)^{-(n+1)}$ gives

$$\frac{dC_n}{dx}=\int_{-\infty}^{\infty}\left[\frac{\partial q^*}{\partial x},\frac{\partial q}{\partial x}\right](2i\mathcal{L})^n\begin{bmatrix} q \\ q^* \end{bmatrix}dt\ . \tag{18}$$

Next, we substitute for $\partial q/\partial x$ from Eq. (1); this is best done by noting that Eq. (1), together with its conjugate, can be written in the form [2]

$$\frac{\partial}{\partial x}\begin{bmatrix} q^* \\ q \end{bmatrix}-i(2i\mathcal{L}_A)^2\begin{bmatrix} q^* \\ -q \end{bmatrix}=\begin{bmatrix} F^* \\ F \end{bmatrix}\ , \tag{19}$$

where $\mathcal{L}_A$ is the formal adjoint of the operator $\mathcal{L}$. Since

$$(2i\mathcal{L}_A)^m\begin{bmatrix} q^* \\ -q \end{bmatrix}=\begin{bmatrix} \dfrac{\delta C_m}{\delta q} \\ \dfrac{-\delta C_m}{\delta q^*} \end{bmatrix} \tag{20}$$

and

$$(2i\mathcal{L})^n\begin{bmatrix} q \\ q^* \end{bmatrix}=\begin{bmatrix} \dfrac{\delta C_n}{\delta q^*} \\ \dfrac{\delta C_n}{\delta q} \end{bmatrix}\ , \tag{21}$$

where $\delta C_m/\delta q$, etc. are functional derivatives of the C_m, and since all C_n functionals commute [2], we get

$$\frac{dC_n}{dx}=\int_{-\infty}^{\infty}[F^*,F](2i\mathcal{L})^n\begin{bmatrix} q^* \\ q \end{bmatrix}dt \tag{22}$$

as required. This result is *exact* for any choice of the perturbation F, which need not be small, and does not require that the number N of solitons remains fixed.

A generalized form for the evolution equation for the spectral parameter b [which is simply related to the associate field f; see Eq. (25) below] is found in a similar way. Substituting Eq. (19) into Eq. (11) gives [35]

$$\frac{\partial b}{\partial x} = -4i\zeta^2 b + \int_{-\infty}^{\infty} [F^*, F]\check{\Phi}\, dt \ . \tag{23}$$

The first term follows using standard manipulations for the unperturbed system, while $\check{\Phi} = -[\phi_1\bar{\psi}_1, \phi_2\bar{\psi}_2]^T$ satisfies Eq. (13). The complex eigenfunction $\zeta = \xi + i\eta$ is discussed briefly in Appendix A. The spectral parameter $b(\xi, x)$ is a measure of the radiation field present in the pulse; for a pure soliton, $b=0$. For a soliton input to the fiber, $q(0,t) = q_s$ and $b(0,\xi) = 0$. The last term in Eq. (23) is the source which generates the radiation field. We will be interested in the case when $\zeta = \xi$ is real and where $\check{\Phi}$ is approximated by its solitonic expression. Then, the integral in Eq. (23) is the projection of the perturbation F onto these dressed continuum modes, labeled by the parameter ξ, in much the same way that a spectral component of a source F is found by projecting onto a continuum mode $\exp(i\omega t)$ in simple Fourier analysis. Indeed, in the absence of the soliton, $\check{\Phi} = -[\exp(-2i\xi t), 0]$, and the source term is simply the conjugate of the Fourier component of F, with (2ξ) the transform variable. To make this explicit, and for later use, introduce the Fourier operator $\mathcal{F}$ and its inverse by

$$\hat{f} \equiv \mathcal{F}(f) = \int_{-\infty}^{\infty} e^{2i\xi t} f(t)\, dt \ , \tag{24a}$$

$$f \equiv \mathcal{F}^{-1}\hat{f} = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-2i\xi t}\hat{f}(\xi)\, d\xi \ , \tag{24b}$$

for any function f. Then, in the absence of the soliton, the source term in Eq. (23) is simply $\hat{F}^*(\xi, x) \equiv [\mathcal{F}(F(x,t))]^*$. Rather than work with the spectral parameter $b(\xi, x)$, it is more convenient to introduce an associate field $f(x,t)$, such that its transform $\hat{f}(\xi, t) \equiv \mathcal{F}(f(x,t))$ is related to $b(\xi, x)$ by

$$\hat{f}(\xi, x) = \frac{1}{4}\frac{b^*(\xi, x)}{\xi^2 + \eta_1^2} \ , \tag{25}$$

where η_1 is the soliton parameter. Simple manipulation of Eq. (23), together with an inverse Fourier transform, then leads to Eq. (5).

Having derived the evolution equation for the associate field f, it remains to link this to the radiation field δq [cf. Eq. (6)]. This is done by using the squared eigenfunction expansion for $q(x,t)$, which in its most general form states that [2]

$$q = -\frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\frac{\bar{b}(\xi,x)}{a(\xi,x)}\phi_1^2(\xi;x,t) + \frac{b(\xi,x)}{\bar{a}(\xi,x)}\bar{\phi}_1^2(\xi;x,t)\right\} d\xi + 2i\sum_{m=1}^{N}\left[\frac{1}{b_m a_m'}\phi_1^2(\zeta_m;x,t) - \frac{1}{\bar{b}_m\bar{a}_m'}\bar{\phi}_1^2(\zeta_m;x,t)\right] . \tag{26}$$

The integral represents the contribution from the radiation field δq, whereas the discrete sum gives the contribution to $q(x,t)$ from a general N-soliton state. Here $N=1$, so that the latter contribution is q_s, while the former is the required expression for $\delta q(x,t)$.

Equation (1) implies certain symmetry relationships between $\bar{\phi}$ and ϕ, $\bar{b}$ and b, and $\bar{a}$ and a; these are [2]

$$[\bar{\phi}_1, \bar{\phi}_2] = [\phi_2^*, -\phi_1^*] \tag{27a}$$

$$\bar{a}(x) = a^*(\xi), \quad \xi \text{ real} \tag{27b}$$

$$\bar{b}(\xi) = b^*(\xi), \quad \xi \text{ real} \ . \tag{27c}$$

Then,

$$\delta q(x,t) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{b^*(\xi,x)}{a(\xi,x)}\phi_1^2(\xi;x,t) + \frac{b(\xi,x)}{a^*(\xi,x)}\phi_2^{*2}(\xi;x,t)\right] d\xi \ . \tag{28}$$

For a single soliton in an unperturbed fiber, b is zero, and remains zero as the soliton propagates, so that δq is zero. The presence of the perturbation changes b in accordance with Eq. (23), so that b is first order in the perturbing term. To find the leading-order contribution to $\delta q(x,t)$, we now approximate ϕ_1, ϕ_2, and a by their appropriate expressions for the single-soliton state; these are

$$a(\xi) = \frac{\xi - i\eta_1}{\xi + i\eta_1} \ , \tag{29a}$$

$$\phi_1 = \frac{e^{-i\xi t}}{\xi + i\eta_1}(\xi - i\eta_1 \tanh 2\eta_1 t) \ , \tag{29b}$$

$$\phi_2 = -\frac{i\eta_1}{\xi + i\eta_1}e^{-i\xi t + 4i\eta_1^2 x}\operatorname{sech} 2\eta_1 t \ . \tag{29c}$$

To simplify the algebra, and with no loss of generality, we have set the (constant) soliton parameter $\xi_1 = 0$—a Galilean transformation effects this change. Using these results, together with Eq. (25), it is easy to show that

$$\frac{b^*}{a}\phi_1^2 = \hat{f}(\xi, x)e^{-2i\xi t}(4\xi^2 - 2i\xi\gamma - \gamma^2) \ , \tag{30a}$$

$$\frac{b}{a^*}\phi_2^2 = -\hat{f}^*(\xi, x)e^{2i\xi t}q_s^2 \ , \tag{30b}$$

where $\gamma = -q_s^{-1}\partial q_s/\partial t$. Inserting these into Eq. (28) and carrying out the required inverse (Fourier) transform leads directly to Eq. (7).

This completes the required proofs of Eqs. (4)–(7).

III. SOME SPECIFIC FORMS FOR F

A. The integrable case

Consider the choice

$$F = -i\frac{\delta C_m}{\delta q^*} \ , \tag{31}$$

where C_m is any one of the conserved functionals associated with the unperturbed ($F=0$) nonlinear Schrödinger (NLS) equation. The addition of a perturbing term such as Eq. (31) simply generates a new member of the NLS family, which is itself integrable with the *same* conserved densities C_n as the unperturbed equation and for which the evolution equation for b is homogeneous, with the additional term $i(2i\xi)^m b$. We therefore anticipate a null result: all C_n's should remain conserved quantities, and the *inhomogeneous* contribution to the source term in the evolution equation for b should be identically zero.

Substituting Eq. (31) into Eq. (22), and using Eq. (21), produces

$$\frac{dC_n}{dx}=i\{C_m,C_n\}=0 \ , \tag{32}$$

since all functionals commute. Similarly, substituting Eq. (31) into the integral expression in Eq. (23), using Eq. (20), and replacing one of the adjoint operators $\mathcal{L}_A$ with its equivalent action $\mathcal{L}$ on $\check{\Phi}$, gives

$$i\int_{-\infty}^{\infty}\left[\frac{\delta C_{m-1}}{\delta q},-\frac{\delta C_{m-1}}{\delta q^*}\right]2i\mathcal{L}\check{\Phi}\,dt \ . \tag{33a}$$

Now use Eq. (13), noting that in the final term $[q,q^*]^T=[\delta C_0/\delta q^*,\delta C_0/\delta q]^T$. Since C_0 and C_{m-1} commute, the contribution from the final term to the integral is zero, leaving the contribution from the term containing $(2i\xi)$. Continuing this iterative process, the integral reduces to

$$i(2i\xi)^m\int_{-\infty}^{\infty}[q^*,-q]\check{\Phi}\,dt \ . \tag{33b}$$

With g, h, and k defined by Eqs. (A6), the integrand is (gq^*-qh), which according to the first of Eqs. (A7) is equal to $\partial k/\partial t$. The integral in Eq. (33b) then reduces to $k(+\infty)-k(-\infty)$, which, from the definition of k and the definitions of the Jost functions in Appendix A, is equal to the spectral coefficient b. Hence the above becomes

$$i(2i\xi)^m b$$

as required. The inhomogeneous contribution to the source term in Eq. (23) is zero, whereas the homogeneous contribution is precisely that expected from inverse scattering theory.

B. Third-order dispersion

Here, F is given by Eq. (3a). Consider first an investigation of the effects of the perturbation F on the conserved densities C_n, after which the salient features of the generated radiation field will be discussed.

For the cases $n=0$ and 1, the integrals in Eq. (22) are zero, so that C_0 and C_1 are conserved quantities (i.e., independent of x). With $n=2$, the right-hand side of Eq. (22) becomes

$$2\epsilon\int_{-\infty}^{\infty}|q|^2\left[q\frac{\partial^3 q^*}{\partial t}+q^*\frac{\partial^3 q}{\partial t^3}\right]dt \ ,$$

which, after a little rearrangement and judicious use of Eq. (1), can be expressed in the form

$$i\epsilon\frac{\partial}{\partial x}\int_{-\infty}q^*\frac{\partial^3 q}{\partial t^3}dt \ .$$

Hence, using Eq. (16c), we deduce that

$$\hat{C}_2=\int_{-\infty}^{\infty}\left[q^*\frac{\partial^2 q}{\partial t^2}+|q|^2-i\epsilon q^*\frac{\partial^3 q}{\partial t^3}\right]dt \tag{34}$$

is a conserved quantity, as first noted by Malomed [14].

With $n=3$, Eq. (22) produces

$$\frac{\partial}{\partial x}\left\{C_3+\frac{3i\epsilon}{2}\int_{-\infty}^{\infty}\left[\left|\frac{\partial^2 q}{\partial t^2}\right|^2-\left[\frac{\partial}{\partial t}|q|^2\right]^2\right]dt\right\}$$
$$=9i\epsilon^2\int_{-\infty}^{\infty}\left|\frac{\partial q}{\partial t}\right|^2\frac{\partial^3}{\partial t^3}(|q|^2)dt \tag{35}$$

so that, to order ϵ, $\hat{C}_3$, the quantity in parenthesis, is conserved. Similar "quasiconserved" densities—that is, modified forms of C_n which are conserved to $O(\epsilon)$—exist for all further values of $n>3$. The proof of this statement follows from the observation that $b(x,\xi)$ is $O(\epsilon)$, so that the integral contribution to C_n in Eq. (17) is $O(\epsilon^2)$, assuming that ϵ is a suitably small parameter. Since $N=1$ (by assumption, that is, a single soliton), and since C_0 and C_1 are conserved, the change in the soliton parameters ξ_1 and η_1 is $O(\epsilon^2)$; the proof of the statement follows.

Consider next the radiation field. To $O(\epsilon)$, the soliton parameters ξ_1 and η_1 remain constant. To simplify the algebra, and with no loss of generality, ξ_1 can self-consistently be set to zero. The soliton forms for ϕ_1 and ϕ_2 are given in Eqs. (29). Similar expressions for $\bar{\psi}_1$ and $\bar{\psi}_2$ are readily deduced using Eqs. (A4) with (27). However, since b is required to $O(\epsilon)$, we may replace $\bar{\psi}$ with $\bar{a}\phi$, where $\bar{a}(\xi)=a^*(\xi)$ is given by Eq. (29a).

To get the evolution equation for b, it remains to evaluate the integral [cf. Eq. (23)]

$$I=\int_{-\infty}^{\infty}\left[\frac{\partial^3 q_s^*}{\partial t^3},\frac{\partial^3 q_s}{\partial t^3}\right]\begin{bmatrix}-\phi_1\bar{\psi}_1\\-\phi_2\bar{\psi}_2\end{bmatrix}dt \ , \tag{36}$$

using analytic forms for ϕ_i, etc. noted above. Substituting, and evaluating the integral using standard techniques, then gives

$$I=2i\xi(\xi^2+\eta_1^2)\hat{q}_s^* \ , \tag{37}$$

where $\hat{q}_s^*$ is the conjugate of the Fourier transform of the soliton q_s, as defined by Eq. (24a). Explicitly,

$$\hat{q}_s(\xi,x)=\pi e^{-4i\eta_1^2 x}\operatorname{sech}\frac{\pi\xi}{2\eta_1} \ . \tag{38}$$

If Eq. (37) is divided by $4(\xi^2+\eta_1^2)$, conjugated [cf. Eq. (25)], and the inverse Fourier transform taken, the final term in Eq. (8) results. In other words, the contribution (37) is essentially the inhomogeneous source term quoted in Eq. (8). The first term written there is actually $O(\epsilon^2)$ [since b, and hence f, are $O(\epsilon)$] and derives from boundary terms in Eq. (36), obtained when the standard "integration by parts" operation is carried out. A little care needs to be taken with this, or else the appropriate contribution

$$-8i\xi^3 b \tag{39}$$

is easily omitted. The details of the calculation are given in Appendix B.

Substituting these results into Eq. (13) then gives the required evolution equation for b. Rather than quote this, we introduce the equivalent associate field $\hat{f}(\xi,x)$, defined by Eq. (25), whose evolution equation is hence obtained as

$$i\frac{\partial\hat{f}}{\partial x}=-4\xi^2\hat{f}-\epsilon 8\xi^3\hat{f}+\frac{\epsilon}{4}2\xi\hat{q}_s \ . \tag{40}$$

An inverse transform produces the required evolution equation for f,

$$i\frac{\partial f}{\partial x}=\frac{\partial^2 f}{\partial t^2}+i\epsilon\frac{\partial^3 f}{\partial t^3}+\frac{i\epsilon}{4}\frac{\partial q_s}{\partial t} \ . \tag{41}$$

On solving this, the radiation field δq is obtained directly from Eq. (7).

Consider now a discussion of Eq. (40). This remarkably simple equation will be shown to account for all features noted in numerical simulations of the system Eqs. (1) and (3a) [6,7]. The results of such a simulation are shown in Fig. 1, reproduced from Ref. 7. The most prominent feature is the strong resonance peak in the pulse spectrum, which is observed to occur at a frequency displaced from line center by an amount [6,7]

$$\delta\omega\sim\frac{1}{\epsilon} \ . \tag{42}$$

Moreover, these studies reveal that the peak becomes more pronounced as the propagation distance down the fiber increases.

The explicit x-dependent factor $\exp(-4i\zeta_1^2 x)$ contained in $\hat{q}_s$ is removed by introducing

$$\hat{F}=e^{+4i\eta_1^2 x}\hat{f} \ . \tag{43}$$

Then, the evolution equation for $\hat{F}$ is

$$\frac{\partial\hat{F}}{\partial x}=iD(\omega)\hat{F}+\frac{i\epsilon\pi}{4}\omega\,\mathrm{sech}\frac{\pi\omega}{4\eta_1} \ , \tag{44}$$

where the "dispersive function" $D(\omega)$ is

$$D(\omega)=\omega^2+4\eta_1^2-\epsilon\omega^3 \ . \tag{45}$$

For notational convenience, we have introduced the frequency variable $\omega=-2\xi$. Secular growth occurs when $D(\omega)=0$, and is the origin of the observed resonance peak; this occurs when

$$\omega=\frac{1}{\epsilon}+4\eta_1^2\epsilon+O(\epsilon^3) \ , \tag{46}$$

in agreement with the observation, Eq. (42), but note the new feature of dependence on pulse intensity contained in the higher-order second term. The asymmetric source term induces a similar asymmetry in the spectral components $\hat{f}(\omega,x)$, as observed numerically.

For any value of propagation distance x, $\hat{F}$ will be returned to zero at those frequency components satisfying $xD(\omega)=2\pi n$, where n is an integer. This accounts for the secondary oscillations in the spectrum, and for the observed asymmetry in those secondary oscillations [since $D(\omega)$ is asymmetric]. Finally, at line center, the source term vanishes, and $\hat{F}(x,0)$ remains zero, which again conforms with numerical observation.

The addition of further dispersive terms, say

$$F\to\tilde{F}=\epsilon\frac{\partial^3 q}{\partial t^3}+i\mu\frac{\partial^4 q}{\partial t^4}+\nu\frac{\partial^5 q}{\partial t^5} \ , \tag{47}$$

does not alter things in any significant way. The dispersion function $D(\omega)$ is now given by

$$D=\omega^2+4\eta_1^2-\epsilon\omega^3+\mu\omega^4+\nu\omega^5 \ . \tag{48}$$

As before, secular growth occurs when $D(\omega)=0$, but now there may be more than one resonance feature.

C. Periodic amplification

Here, F is given by Eq. (3d). Unlike the previous dispersive case, the perturbation causes C_0 (and all other C_n) to vary with x, imparting a corresponding x dependence to the soliton parameters ξ_1 and η_1. Evaluation of the integral in Eq. (23) is then at best difficult, if not impossible. However, a change of variables to a set of "stretched coordinates," which correctly capture the evolution of the soliton parameters, results in a situation where the redefined functionals C_0 and C_1 are again constants of the motion. In this new coordinate system, the soliton parameters are constant to $O(\Gamma)$.

Introduce

$$\tau=te^{2\Gamma x} \ , \tag{49a}$$

$$z=(e^{4\Gamma x}-1)/4\Gamma \ , \tag{49b}$$

$$p=q\exp[-2\Gamma x-i\tau\beta(z)] \ , \tag{49c}$$

where the phase factor $\beta(z)$ is

$$\beta(z)=2\xi_1[1-(1+4\Gamma z)^{-1/2}] \ . \tag{49d}$$

In terms of the new variables, Eq. (1)—with F given by Eq. (3d)—becomes

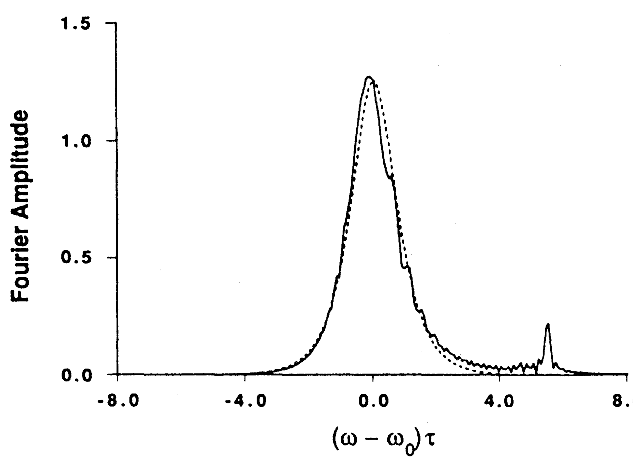

FIG. 1. Soliton spectrum, with perturbative effect of third-harmonic dispersion. Taken from Fig. 3, Ref. [7].

$$i\frac{\partial p}{\partial z}-\frac{\partial^2 p}{\partial \tau^2}-2p|p|^2=-\frac{i\Gamma}{1+4\Gamma z}\left[p+2\tau\frac{\partial p}{\partial \tau}\right]+\frac{4\Gamma\xi_1}{1+4\Gamma z}\tau\cdot p-\beta^2 p+2i\beta\frac{\partial p}{\partial \tau}-\frac{iG}{1+4\Gamma z}\sum_{n=1}^{\infty}p\delta(z-z_n)\ , \tag{50}$$

where

$$z_n=(e^{4n\Gamma x_a}-1)/4\Gamma \tag{51a}$$

$$\simeq n x_a,\quad \Gamma \ll 1$$

$$\equiv n z_a\ . \tag{51b}$$

This generalizes a similar transformation introduced elsewhere where ξ_1 [and hence the phase $\beta(z)$] was set to zero [23]. Later, we too set ξ_1 to zero, but before doing so first demonstrate that this can be done in a self-consistent manner, at least to $O(\Gamma)$.

With F deduced from the right-hand side of Eq. (50), the functions C_0 and C_1—defined now with "new" variables replacing "old" throughout—are independent of z. The soliton solution to the unperturbed form of Eq. (50)

$$p(z,\tau)\equiv p_s=2\eta_1 e^{-2i\xi_1\tau+4i(\xi_1^2-\eta_1^2)z}\operatorname{sech}[2\eta_1(\tau-4\xi_1 z)] \tag{52}$$

is then invariant, in the sense that to $O(\Gamma)$, the parameters ξ_1 and η_1 remain constant. Equation (52) is the "stretched" soliton solution; the solution q_s expressed in terms of the "normal" variables is found using Eqs. (49).

In view of these comments the parameter ξ_1 can self-consistently be set to zero. Then, $\beta(z)$ is zero, and so the second, third, and fourth terms on the right-hand side of Eq. (50) vanish. The assumption $\Gamma \ll 1$ is retained, so that z_n, the positions of discrete loss, are given by Eqs. (51b).

With the perturbation F specified by

$$F=-\frac{\Gamma}{1+4\Gamma z}\left[p+2\tau\frac{\partial p}{\partial \tau}\right]-\frac{G}{1+4\Gamma z}\sum_{n=1}^{\infty}p\delta(z-z_n)\ ,$$

techniques described in Sec. III B can be used to find the evolution equation for b, or equivalently for $\hat{f}$; this is

$$i\frac{\partial \hat{f}}{\partial z}=-4\xi^2\hat{f}-\frac{A(z)}{1+4\Gamma z}\frac{i}{4(\xi^2+\eta_1^2)}\hat{p}_s(\xi,z)\ . \tag{53}$$

Here, $A(z)$ is the periodic function, Eq. (3e), and $\hat{p}_s(\xi,z)$ is the transform of the soliton pulse p_s. An inverse transform of Eq. (53) gives the required evolution equation for f, which is

$$i\frac{\partial f}{\partial z}=\frac{\partial^2 f}{\partial \tau^2}-\frac{i}{4\eta_1^2}\frac{A(z)}{1+4\Gamma z}p_s\otimes h\ . \tag{54}$$

Here

$$h(\tau)=\eta_1 e^{-2\eta_1|\tau|}$$

is the normalized two-sided exponential function, and $\otimes$ denotes (Fourier) convolution. The required δq (or rather, δp) is found in the usual way from Eq. (7), with an obvious replacement of old by new variables. Equations (54) with (7) are equivalent to a similar set recently reported by Gordon [8], obtained using a "guiding-center" approximation [24,25] to describe the evolution of the soliton pulse q_s.

We conclude this section with a discussion of Eq. (53). This remarkably simple equation accounts for all features noted in numerical simulations of Eq. (1) with (3d), Fig. 2. The most prominent feature is a discrete set of primary resonances in the pulse spectrum, which appear at fixed values of frequency $\omega\equiv 2\xi$ determined by the vanishing of a dispersion function

$$D(\omega)\equiv\omega^2+4\eta_1^2-nk_a=0,\quad n=1,2,\ldots, \tag{55}$$

where $k_a=2\pi/z_a$. Moreover, these appear to grow linearly with distance z, and all have the same relative increment above the background spectrum of the soliton pulse.

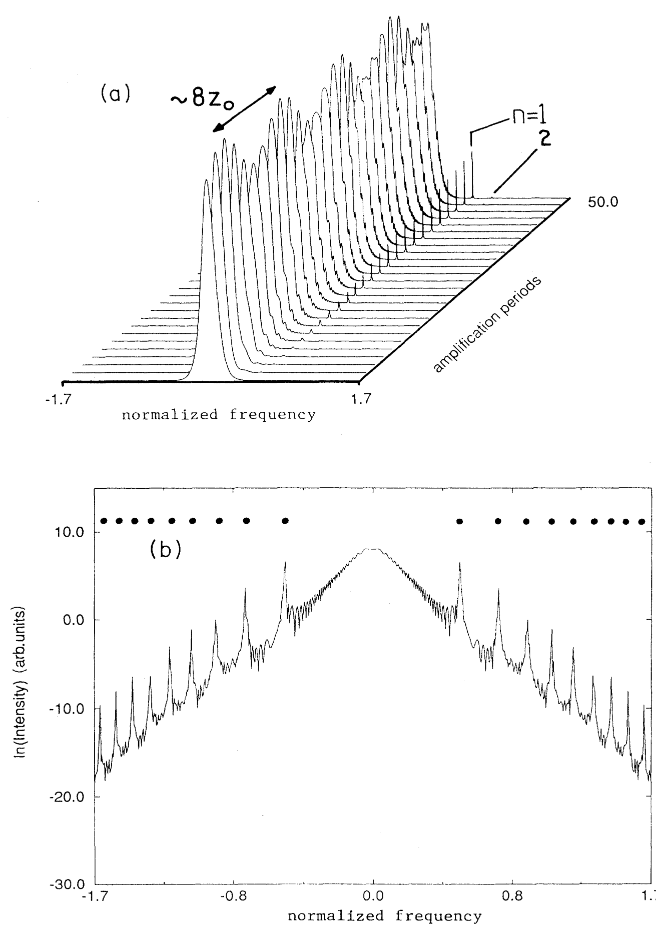

FIG. 2. (a) Spectral evolution of the soliton profile over 50 amplification periods. Depicted is the $n=1$ and 2 resonance modes, the modulation of the soliton peak at $\sim 8z_0$, and secondary resonances. (b) Logarithmic intensity plot of the pulse profile at $50z_a$, showing the higher-order modes $n=2,3,\ldots,12$. Positions of the normalized resonant frequencies ($\tau_p\delta\omega_n$), according to Eq. (59), are indicated by ●.

The first two observations are easily explained, by using

$$\sum_{n=1}^{\infty} \delta(z-nz_a) = \sum_{n=-\infty}^{\infty} e^{ink_a z} , \tag{56}$$

and introducing

$$\hat{F} = e^{-4i\xi^2 z}\hat{f} . \tag{57}$$

Then, Eq. (53) reads

$$i\frac{\partial \hat{F}}{\partial z} = -\frac{\pi}{(\xi^2+\eta_1^2)(1+4\Gamma z)} \times \left\{ \Gamma e^{-4i(\xi^2+\eta_1^2)z} - G \sum_{n=-\infty}^{\infty} e^{-i[4(\xi^2+\eta_1^2)-nk_a]z} \right\} \operatorname{sech}\frac{\pi\xi}{2\eta_1} . \tag{58}$$

A vanishing of the second exponent gives the resonance condition, Eq. (55), with an accompanying secular growth in the spectral component $\hat{F}(\xi=\xi_n)$ over distances z such that $4\Gamma z \ll 1$. In alternative form, Eq. (55) reads

$$\delta\omega_n = \pm\frac{1}{2\pi\tau_p}\sqrt{8n(z_0/z_a)-1}, \quad n=1,2,\ldots , \tag{59}$$

where z_a is the period of $A(x)$, $8z_0$ is the "soliton period," where $z_0 \equiv \pi/(4\eta_1)^2$, and $2\pi\tau_p\delta\omega \equiv \xi/\eta_1$, relates $\delta\omega$ to ξ through the soliton pulse halfwidth τ_p. Equation (59) was first reported by Gordon [8], and independently by Kelly [9], but see also Kaup [13]. The important point to note is that the resonance spikes have the same origin as those discussed in Sec. III B; only their number and locations differ.

With G set to zero (no periodic loss), all primary resonances are removed. A solution of Eq. (57) for $\hat{F}(z)$ then indicates that, as z increases, $|\hat{F}(z)|$ asymptotes in an oscillatory manner to a final steady-state value. Since C_0 is a constant of the motion (in the stretched variables), there will be a complementary evolution for the soliton parameter η_1, consistent with numerical results reported elsewhere [23].

The secondary oscillations in Fig. 2(a) are easily explained by noting that $\hat{F}(z)$ is returned to zero [to $O(\Gamma)$] for those frequency components $\xi \equiv \xi_m$ which satisfy the phase-matching condition

$$4(\xi^2+\eta_1^2)z = 2\pi m \tag{60}$$

(where m is an integer) for *any* value of z. As z increases, the relative spacing between ξ_m and ξ_{m+1} decreases, as is observed in Fig. 2(a).

Finally, a modulation at the soliton period $8z_0$ occurs at the peak of the pulse spectrum, where $\xi=0$. The modulation period $(8z_0)$ has the same value as that found in the modulations of higher-order N-soliton states $N \geq 2$. This modulation is due to the $\exp(-4i\eta_1^2 z)$ term in Eq. (58), which can be expressed in the alternative form $\exp(-i\pi z/4z_0)$.

It remains to explain the feature displayed in Fig. 2(b), where the relative heights of all resonance peaks over the background soliton pulse are the same. Denote by ξ_n, the resonance frequency values satisfying Eq. (55). Let $\delta\hat{p}(\xi=\xi_n,z)$ denote the Fourier transform of $\delta p(\tau,z)$ evaluated at $\xi=\xi_n$, $n=1,2,\ldots$. To first order in Γ, the incremental contribution $\delta\hat{I}(\xi_n,z)=(\hat{p}_s^*\delta\hat{p}+\text{c.c.})$ to the power spectrum is

$$\delta\hat{I}(\xi_n,z) = 2Gz\frac{\xi_n^2-\eta_1^2}{\xi_n^2+\eta_1^2}|\hat{p}_s(\xi_n,z)|^2 \tag{61a}$$

$$\simeq 2Gz|\hat{p}_s(\xi_n,z)|^2 , \tag{61b}$$

where the second form holds when $\xi_n^2 \gg \eta_1^2$. Linearity in z is a consequence of the secular behavior of Eq. (58) at resonance, and Eq. (7) has been used to link δq (or rather, δp) to f. Equation (61b) reveals that the ratio of incremental power at resonance to the background power of the soliton pulse has the same value for all resonances, in agreement with numerical observations [Fig. 2(b)].

D. Soliton self-frequency shift

Here, F is given by Eq. (3b). In the preceding sections, we have established the procedure to be followed to analyze the effects of *any* perturbation on the input soliton, and so need only summarize the results for this next case. The perturbation F leaves C_0 invariant but causes C_1 to change with distance. Consequently, to $O(\epsilon)$, η_1 is invariant, but not ξ_1. For the single soliton input, the appropriate evolution for ξ_1 is

$$\xi_1 = \xi_0 + \mu x, \quad \mu = \frac{64}{15}\epsilon\eta_1^4 , \tag{62}$$

indicating a linear change in soliton velocity. Using similar arguments to those used in Sec. III C, the soliton can be brought to an invariant form (i.e., to rest) by an appropriate change of variables. Introduce

$$\tau = t - 4\mu x^2 , \tag{63a}$$

$$z = x , \tag{63b}$$

$$p = e^{i\beta}q , \tag{63c}$$

where the phase function β is

$$\beta = 2\mu x t - 4\mu x^2(2\xi_0+\mu x) . \tag{63d}$$

In terms of these variables, Eq. (1), with F given by Eq. (3b), becomes

$$i\frac{\partial p}{\partial z} - \frac{\partial^2 p}{\partial \tau^2} - 2p|p|^2 = 4i\mu z\frac{\partial p}{\partial \tau} - 2\mu(\tau - 8\xi_0 x) - \epsilon p\frac{\partial}{\partial \tau}|p|^2 . \tag{64}$$

Identifying the right-hand side here with iF, the functions C_0 and C_1 are independent of z for the restricted case when the input to the fiber is the single soliton p_s. The single soliton is then invariant in the sense that, to $O(\epsilon)$, the parameters ξ_0 and η_1 [note that ξ_0 replaces ξ_1; cf. Eq. (62)] remain constant. The parameter ξ_0 can self-consistently be set to zero, so that F becomes

$$F=4\mu z\frac{\partial p}{\partial\tau}+2i\mu\tau+i\epsilon p\frac{\partial}{\partial\tau}|p|^2 . \tag{65}$$

The radiation field is generated in accordance with Eq. (23). It is easily checked that the contributions from the first, second, and third terms in F to the integral in Eq. (23) are, respectively, $16i\xi\mu zb$, 0, and $-8\epsilon\hat{p}_s^*\xi(3\xi^2+7\eta_1^2)/15$. Converting to an evolution equation for f in the usual way gives

$$i\frac{\partial f}{\partial z}=\frac{\partial^2 f}{\partial\tau^2}+8i\mu z\frac{\partial f}{\partial\tau}+\frac{1}{5}\epsilon\frac{\partial p_s}{\partial\tau}+\frac{4\epsilon}{15}\frac{\partial}{\partial\tau}(h\otimes p_s) . \tag{66}$$

Here, $h\otimes$ denotes the convolution of the two-sided exponential function $h(\tau)$ [cf. Eq. (54)] with the soliton p_s.

Equation (66) has a pleasing interpretation which accords with intuition: the inhomogeneous source terms are in the moving frame (z,τ). Using the transformation equations (63a) and (63b), the homogeneous terms can be written in the form

$$i\frac{\partial f}{\partial x}-\frac{\partial^2 f}{\partial t^2} ,$$

bringing the description of the evolution of f back to the "fixed" coordinate system. In this frame, the source sweeps across the spectrum, generating $\hat{f}$ at the appropriate frequencies as it does so. By virtue of the relative movement, new frequency modes are constantly excited, while a long tail of freely evolving modes are left behind. This feature is observed numerically, and accords with intuition [19]. A more detailed study of this, and of the nonlocal problem [17–19], is in progress and results will be reported in due course.

E. Stochastic perturbations

In a recent article [30], a distinction was made between two types of stochasticity, which were termed homogeneous and inhomogeneous. The latter arises from an indeterminacy associated with the input pulse to the fiber $q(0,t)$, and *not* from any agency in the fiber itself. The evolution equations for the soliton parameters are deterministic, only their initial values are random. An additional perturbation—such as Eq. (3c)—resulting in the presence of an attracting fixed point in these evolution equations necessarily reduces the initial statistical spread, and so diminishes the stoichasticity [30]. The main features were elucidated first in an article by Elgin [28], then later by Gordon and Haus [29]. See [30] for a further discussion.

In the homogeneous case, the stochasticity is associated with some random property of the fiber itself. All fibers contain impurities, imperfections and random density fluctuations, causing small stochastic fluctuations in the fiber refractive index. In a suitable dispersive limit, this leads to the form for F quoted in Eq. (3g), where the (real) random stochastic variable $\sigma(x)$ has the statistical properties

$$\langle\sigma(x)\rangle=0,\quad \langle\sigma(x)\sigma(x')\rangle=2D\delta(x-x') . \tag{67}$$

Angular brackets denote an ensemble average. This type of stochastic term is fairly straightforward to analyze, and results may be summarized as follows (see Ref. [36] for further details).

(i) The term with β_0 generates an overall stochastic phase factor $\phi(x)$, so that

$$q_s\rightarrow e^{i\beta_0\phi}q_s ,$$

where

$$\phi(x)=\int_0^x\sigma(x')dx' . \tag{68}$$

(ii) The term with β_1 produces an uncertainty in the soliton velocity, giving rise to a stochastic velocity increment

$$\delta V=-\frac{1}{x}\beta_1\phi(x) . \tag{69}$$

This has the statistical properties

$$\langle\delta V\rangle=0,\quad \langle(\delta V)^2\rangle=2D\beta_1^2/x . \tag{70}$$

For a single length of fiber, all solitons q_s passing a given point x will have the same value of δV. A statistical spread results only when similar solitons are propagated the same distance down different fibers.

(iii) The final term with β_2 generates a background radiation field $\delta q(x,t)$, which is $O(\beta_2)$ in magnitude, and whose associate field f satisfies the evolution equation

$$i\frac{\partial f}{\partial x}=\frac{\partial^2 f}{\partial t^2}+\sigma(x)\beta_2\frac{\partial^2 f}{\partial t^2}+\frac{1}{2}\sigma(x)\beta_2 q_s . \tag{71}$$

The proof of these statements is given elsewhere [36]. The important point to note is that the system comprising Eq. (1), with F given by Eq. (3g), is tractable because $\sigma(x)$ is independent of the variable t. The stochastic variable σ can then be removed from various integrals, resulting in a set of Langevin equations, such as Eq. (71) above.

More generally, σ is a function of both x and t. If the stochasticity derives from spontaneous emissions in a bandwidth-limited fiber amplifier, the appropriate form for F is

$$F=\Gamma q+\gamma\frac{\partial^2 q}{\partial t^2}+\sigma(x,t) , \tag{72}$$

together with some suitable statement for the statistical properties of $\sigma(x,t)$. Equations (1) and (72) were analyzed in a recent article in the (unphysical) limit where the t dependence in σ was simply ignored, so that $\sigma=\sigma(x)$ [30]. The object was to demonstrate that the resulting Langevin equations for the soliton parameters ξ_1 and η_1 were necessarily of multiplicative type, whose statistical properties need bear no resemblance to those of the corresponding linear Langevin system obtained by ignoring multiplicative terms.

The more important case where σ retains its t dependence has not been analyzed to date. The difficulty, from a mathematical viewpoint, is that the t dependence in σ precludes its removal from the various integral expressions for the C_n. For example, the evolution equation for η_1 (obtained from the evolution equation for C_0) contains the term

$$\int_{-\infty}^{\infty}[\sigma^*(x,t)q_s+\text{c.c.}]dt \ . \tag{73}$$

q_s contains the stochastic variables ξ_1 and η_1. To proceed further, one needs a careful statement defining the statistical properties of $\sigma(x,t)$, which to date, has not been given. An investigation of this problem, within the general mathematical framework described in this article, is in progress and results will be reported in due course.

IV. CONCLUDING COMMENTS

The main results of this article, Eqs. (4), (5), and (7), have been shown to have useful applications to all the example considered in Sec. III, and hence validate the comment expressed in the introduction concerning the use of a "natural mathematical framework." One might object to this claim on the grounds that all the examples considered are in some sense trivial, since they involve the study of perturbations around a single-soliton state. However, similar investigations elsewhere invariably require numerical simulations because attempts to analyze the problem using the *wrong* type of continuum modes (Fourier) quickly become very involved—compare, for example, the analysis in Sec. III B with that presented in Ref. [7] on a similar study of the same problem. The analysis here is trivial *because* the correct framework has been used.

The next step will be to develop the general technique to study perturbations of N-soliton states; aspects of this work is now in progress. Equations (4) hold for this more general case, likewise Eq. (23) for the evolution of the b field, but now a different expression for $\check{\Phi}$ is required, as appropriate for the N-soliton state. The equation connecting δq with f, Eq. (7), is invalid, though the appropriate corrected form can be deduced from the general equation (28) once appropriate expressions are deduced for ϕ_i.

ACKNOWLEDGMENT

It is a pleasure to thank S. M. J. Kelly for many useful discussions concerning this work, and for the use of Fig. 2.

APPENDIX A

The linear eigenvalue problem associated with the unperturbed form (i.e., $F=0$) of Eq. (1) is [1,2]

$$\begin{aligned}&\frac{\partial v_1}{\partial t}+i\zeta v_1=qv_2 \ ,\\ &\frac{\partial v_2}{\partial t}-i\zeta v_2=-q^*v_1 \ ,\end{aligned} \tag{A1}$$

where $\zeta=\xi+i\eta$ is a complex eigenvalue. Jost function solutions ϕ, $\bar{\phi}$, ψ, and $\bar{\psi}$ are defined such that

$$\begin{aligned}&\phi \underset{x\to-\infty}{\sim} \begin{bmatrix}1\\0\end{bmatrix}e^{-i\zeta x} \ ,\\ &\bar{\phi} \underset{x\to-\infty}{\sim} \begin{bmatrix}0\\-1\end{bmatrix}e^{i\zeta x} \ ,\end{aligned} \tag{A2}$$

$$\begin{aligned}&\psi \underset{x\to+\infty}{\sim} \begin{bmatrix}0\\1\end{bmatrix}e^{i\zeta x} \ ,\\ &\bar{\psi} \underset{x\to+\infty}{\sim} \begin{bmatrix}1\\0\end{bmatrix}e^{-i\zeta x} \ .\end{aligned} \tag{A3}$$

The relationship between these defines the scattering data a, $\bar{a}$, b, and $\bar{b}$:

$$\begin{aligned}&\phi=a\bar{\psi}+b\psi, \quad \bar{\phi}=-\bar{a}\psi+\overline{b\psi} \ ,\\ &\psi=-a\bar{\phi}+\bar{b}\phi, \quad \bar{\psi}=\bar{a}\phi+b\bar{\phi} \ ,\end{aligned} \tag{A4}$$

where the latter (inverse) expressions are obtained using

$$a\bar{a}-b\bar{b}=1 \ . \tag{A5}$$

A squared eigenfunction problem can be associated with Eq. (1): for example, defining

$$\begin{aligned}&g=-\phi_1\bar{\psi}_1 \ ,\\ &h=-\phi_2\bar{\psi}_2 \ ,\\ &k=\tfrac{1}{2}(\phi_2\bar{\psi}_1+\phi_1\bar{\psi}_2)\end{aligned} \tag{A6}$$

leads to the evolution equations

$$\begin{aligned}&\frac{\partial k}{\partial t}=q^*g-qh \ ,\\ &\frac{\partial g}{\partial t}=-2i\zeta g-2qk \ ,\\ &\frac{\partial h}{\partial t}=2i\zeta h+2q^*k \ .\end{aligned} \tag{A7}$$

A formal solution of Eqs. (A7) leads to the relationship

$$\zeta\begin{bmatrix}g\\h\end{bmatrix}=\mathcal{L}\begin{bmatrix}g\\h\end{bmatrix}+ik_-\begin{bmatrix}q\\q^*\end{bmatrix} \ , \tag{A8}$$

where k_- denotes $k(t\to-\infty)$, and $\mathcal{L}$ is the integro-differential operator

$$\mathcal{L}=\frac{1}{2i}\begin{bmatrix}-\partial_t+2qI_-[r\cdot\ ] & 2qI_-[q\cdot\ ]\\ -2rI_-[r\cdot\ ] & \partial_t-2rI_-[q\cdot\ ]\end{bmatrix} . \tag{A9}$$

Here, $r=-q^*$ as appropriate for Eq. (1), and I_- denotes the integral operator

$$I_-[r\cdot\ ]f\equiv I_-[r,f]\equiv\int_{-\infty}^{t}r(t')f(t')dt \ , \tag{A10}$$

for any function f. Note that $[g,h]^T$ is an eigenfunction of $\mathcal{L}$ (with eigenvalue ζ) only if k_- is zero.

APPENDIX B

It is required to evaluate the integral

$$I=\int_{-\infty}^{\infty}\left[\frac{\partial^3q_s^*}{\partial t^3},\frac{\partial^3q_s}{\partial t^3}\right]\begin{bmatrix}g\\h\end{bmatrix}dt \ , \tag{B1}$$

where g and h are defined in Eqs. (A6). The quantity k will also be required, as will the evolution Eqs. (A7).

Integrate Eq. (B1) by parts and use Eqs. (A7) to substitute for $\partial g/\partial t$ and $\partial h/\partial t$ to give

$$I=2i\zeta\int_{-\infty}^{\infty}\left[\frac{\partial^2 q_s^*}{\partial t^2},\frac{\partial^2 q_s}{\partial t^2}\right]\left[\begin{matrix} g \\ -h\end{matrix}\right]+\mathcal{T}\ , \tag{B2}$$

where $\mathcal{T}$ denotes "other terms." Repeating this operation twice more, similarly gives

$$I=(2i\zeta)^3\int_{-\infty}^{\infty}(q_s^* g-q_s h)dt+\mathcal{T}\ .$$

However, the integrand is just $\partial k/\partial t$, so that

$$I=(2i\zeta)^3 k\,|_{-\infty}^{+\infty}+\mathcal{T}\ . \tag{B3}$$

Since $k(+\infty)=-k(-\infty)=b/2$ [cf. Eq. (A6) with (A2)–(A4)] the final result, with ζ set to (real) ξ, is

$$I=-8i\xi^3 b+\mathcal{T}\ . \tag{B4}$$

The other terms correspond to the contribution equation (37), which can alternatively be obtained direct from Eq. (36), as discussed in the text.

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