PHYSICAL REVIEW A VOLUME 47, NUMBER 5 MAY 1993

Statistics of difference events in homodyne detection

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(Received 14 December 1992)

The statistics of the difference events in a balanced homodyne-detection scheme is studied without using the standard assumption of a strong, classical local oscillator. Starting from the quantum theory of photon counting, we derive the probability distribution for the difference events in the two detection channels. In the limit of the local oscillator being strong compared with the signal, the difference-number statistics tends to the statistics of the electric-field strength of the signal field. For weak signals, this limit may be obtained already far from a classical behavior of the local oscillator. While changing the local oscillator intensity, a transition of the observable concerning the signal field occurs from the photon-number difference towards the electric-field strength. Such a measurement scheme renders it possible to observe the quantum features of a coherent state or a single-photon state from the point of view of the field-strength statistics, a picture which is closely related to classical optics. The possibility to get some insight in the statistical properties of the phase difference of two microscopic fields is discussed.

PACS number(s): 42.50.Ar, 03.65.−w

I. INTRODUCTION

Homodyne detection is a well-established method for measuring phase-sensitive properties of light. Usually, a signal field is superimposed with a much stronger local oscillator. Consequently, the resulting field is rather strong and can be detected with photodiodes. In such a scheme a photocurrent is produced which may be treated classically. Nevertheless, from the statistical properties of this classical current one may get some insight into the nonclassical statistics of light. Theoretical studies of homodyne detection have been given in a series of papers; see, e.g., [1–5]. In these approaches the mean value and the variance of the photocurrent have been considered and related to the corresponding properties of the field quadratures.

In quantum optics the method of balanced homodyning plays an important role in experiments devoted to the detection [6–8] and application [9–11] of squeezed light. For measuring the squeezing effect a strong local oscillator is applied and thus one may consider the noise properties of the photocurrents produced by the diodes. Since squeezing is related to the variance of the field strength of the light field under study, the measurement of the second moment is sufficient for the demonstration and application of the reduced quantum noise in one field quadrature. Nevertheless, a direct detection of the probability distribution of the recorded signal field and the deriviation of the underlying statistics of the light would be of some interest. Since homodyning yields insight into the field-strength statistics one may expect to observe the corresponding field-strength probability distributions for typical quantum states as first studied by Schubert and Vogel [12]. For example, the vacuum field distribution which is unavoidably connected with a coherent state should be directly measurable. Furthermore, for a single-photon state one might observe a double-peaked probability distribution which is characteristic for the first excited state of the harmonic oscillator. Effects such as higher-order squeezing [13] could directly be derived from the field statistics [14]. Moreover, the field-strength probability distribution yields insight into the phase statistics of light fields in the quantum domain [15].

The measurement of the field-strength distribution is expected to be possible in the limit of a strong, classical local oscillator. From the point of view of quantum optics, however, it is of some interest to study the statistics of homodyne detection for a weak local oscillator. For example, it has been shown that for measuring squeezing in resonance fluorescence the strength of the local oscillator field should be comparable with that of the fluorescence [16]. In such a situation nonclassical anomalous moments of the fluorescence may be observed which disappear with increasing local oscillator strength. Moreover, homodyne detection has been applied recently for measuring statistical properties of the phase difference of two weak coherent fields [17,18]. In homodyne measurements of such weak fields photon-counting techniques must be involved instead of detecting photocurrents.

In the present paper we study the counting statistics of balanced homodyne detection and its relation to the statistics of the light field. The starting point is the quantum theory of light detection [19,20] which is used for calculating the statistics of the difference events in a balanced homodyne-detection scheme. It is shown that the difference statistics of the detected events is related to the probability distribution of the field strength of the signal field, provided the local oscillator is sufficiently strong. Examples for the continuous transition from observing the statistics of the number difference of the signal field to its field-strength statistics are given for a coherent state and a single-photon state; for a first discussion of this behavior see [21]. Moreover, it is demonstrated that the measured probability distributions allow some insight into the statistics of the phase difference between two weak quantum fields.

Our paper is organized as follows. In Sec. II the number-difference statistics is derived from the theory of photon counting and it is related in Sec. III to the field-strength distribution of the light field under study. Section IV is devoted to the properties of the phase difference between two weak fields. A brief summary and some conclusions are given in Sec. V.

II. JOINT PROBABILITY AND DIFFERENCE STATISTICS

Let us consider a typical balanced homodyne-detection scheme as shown in Fig. 1. The signal field under study is superimposed with a local oscillator field by means of a beam splitter. In the two output channels of the device the statistics of the corresponding difference events is observed. Starting from the quantum theory of photon counting [19,20] we will study this statistics for arbitrary ratios of the local oscillator to the signal field strength.

For dealing with the light in the output channels 1 and 2 we have to relate the corresponding photon annihilation operators $\hat{a}_1$ and $\hat{a}_2$, respectively, to the field operators of the input fields. Although we do not restrict our treatment to the usually considered case of a strong local oscillator we will use the standard notations for the two input fields. Combining the signal field (characterized by the annihilation operator $\hat{b}$) with a coherent, local oscillator (annihilation operator $\hat{a}$) by a symmetric (50:50) beam splitter, the operators for the output fields read

$$\hat{a}_1=\frac{1}{\sqrt{2}}(\hat{a}+i\hat{b}) ,\qquad \hat{a}_2=\frac{1}{\sqrt{2}}(\hat{b}+i\hat{a}) , \tag{1}$$

and the corresponding photon-number operators $\hat{n}_{1,2}=\hat{a}^{\dagger}_{1,2}\hat{a}_{1,2}$ are given by

$$\hat{n}_{1,2}=\tfrac{1}{2}(\hat{a}^{\dagger}\mp i\hat{b}^{\dagger})(\hat{a}\pm i\hat{b}) . \tag{2}$$

In the case of idealized detectors the statistics of the difference of counts in the two output channels may be characterized by the operator of the photon-number difference

$$\Delta\hat{n}=\hat{n}_1-\hat{n}_2 . \tag{3}$$

Based on the usual assumption of a strong, coherent local oscillator, we may replace the corresponding field operator by a c number ($\hat{a}\to\alpha$). Therefore the operator for the number difference simplifies according to

$$\Delta\hat{n}=i|\alpha|(\hat{b}e^{-i\varphi_\alpha}-\hat{b}^{\dagger}e^{i\varphi_\alpha}) . \tag{4}$$

This operator is of the same structure as the operator of the electric-field strength of the signal field

$$\hat{E}=i|g|(\hat{b}^{-i\varphi}-\hat{b}^{\dagger}e^{i\varphi}) , \tag{5}$$

where $|g|=[\langle(\Delta\hat{E})^2\rangle_{\text{vac}}]^{1/2}$ characterizes the field-strength noise in the vacuum state. Comparing the

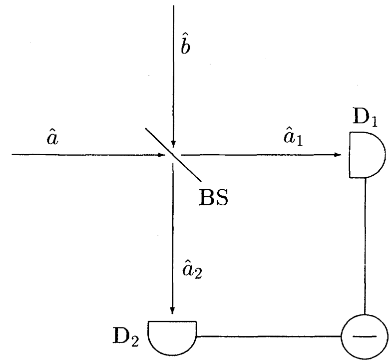

FIG. 1. Experimental scheme for balanced homodyne detection. The signal field (operator $\hat{b}$) and the local oscillator field ($\hat{a}$) are superimposed by a beam splitter (BS). The superimposed light in the two output channels (operators $\hat{a}_{1,2}$) is recorded by two photodetectors ($D_{1,2}$) and a correlator is used to derive the difference statistics of the recorded events.

operators $\Delta\hat{n}$ and $\hat{E}$, the local oscillator amplitude $|\alpha|$ formally corresponds to the vacuum noise:

$$|\alpha|\leftrightarrow|g| \tag{6}$$

and the local oscillator phase φ_α is related to the phase of the signal field φ via

$$\varphi_\alpha\leftrightarrow\varphi . \tag{7}$$

In the following we are interested in the statistics of balanced homodyne detection without making the approximation given in Eq. (4). Thus we have to derive the statistics including the quantum properties of the coherent local oscillator. Since the measurement is performed with two photodetectors we have to start with the joint probability P_{n_1,n_2} for the events in the two output channels. Using the results of the quantum theory of photodetection [19,20] this distribution reads as

$$P_{n_1,n_2}=\left\langle :\frac{(\eta_1\hat{n}_1)^{n_1}}{n_1!}e^{-\eta_1\hat{n}_1}\frac{(\eta_2\hat{n}_2)^{n_2}}{n_2!}e^{-\eta_2\hat{n}_2}:\right\rangle , \tag{8}$$

where η_i ($i=1,2$) is the efficiency of the ith detector. From this quantity we derive the distribution of the difference events $\Delta n=n_1-n_2$ according to (for example, see [22])

$$P_{\Delta n}=\sum_{n_1}P_{n_1,n_1-\Delta n} . \tag{9}$$

We now make use of the fact that the local oscillator is in a coherent state,

$$\hat{a}|\alpha\rangle=\alpha|\alpha\rangle . \tag{10}$$

Moreover, applying the P representation [23]

$$\hat{\rho}=\int d^2\beta\, P(\beta)|\beta\rangle\langle\beta| \tag{11}$$

for the density operator $\hat{\rho}$ of the signal field, we derive from Eq. (8) the following result for the joint probability distribution:

$$P_{n_1,n_2}=\int d^2\beta\, P(\beta)\frac{\left[\frac{\eta_1}{2}\vartheta_1\right]^{n_1}}{n_1!}e^{-(1/2)\eta_1\vartheta_1}\times\frac{\left[\frac{\eta_2}{2}\vartheta_2\right]^{n_2}}{n_2!}e^{-(1/2)\eta_2\vartheta_2} . \quad (12)$$

Based on Eq. (9) we obtain the difference statistics in the form

$$P_{\Delta n}=\int d^2\beta\, P(\beta)[C_{1,2}]^{\Delta n/2}I_{\Delta n}[(\eta_1\eta_2\vartheta_1\vartheta_2)^{1/2}]\times e^{-(1/2)(\eta_1\vartheta_1+\eta_2\vartheta_2)} , \quad (13)$$

where

$$\vartheta_{1,2}=|\alpha|^2+|\beta|^2\pm 2|\alpha||\beta|\sin(\varphi_\alpha-\varphi_\beta) , \quad (14)$$

$I_{\Delta n}$ being the modified Bessel functions and

$$C_{1,2}=\begin{cases}\eta_1\vartheta_1/\eta_2\vartheta_2 & \text{for } \Delta n\geq 0\\ \eta_2\vartheta_2/\eta_1\vartheta_1 & \text{for } \Delta n<0 .\end{cases} \quad (15)$$

This result for the distribution $P_{\Delta n}$ is valid for any ratio of the local oscillator field strength to the signal field strength, for arbitrary detection efficiencies and for any quantum state of the signal field. In practice, however, for some nonclassical states of light such as squeezed states it may be troublesome to use this expression with the P representation of the density operator for practical calculations. In such situations it may be more advantageous to return to the basic equations (8) and (9). However, the results derived in this section using the P representation will turn out to be helpful for deriving some general relations. An example is the proof that the measured difference statistics tends to the field-strength statistics of the signal field when the local oscillator becomes sufficiently strong.

III. DIFFERENCE-COUNTING STATISTICS VERSUS FIELD-STRENGTH STATISTICS

Remembering Eqs. (4) and (5) we expect that the distribution of the measured difference events is closely related to the probability distribution of the electric-field strength of the signal field provided the local oscillator is sufficiently strong. We will derive this relation based on the photon-counting theory, including the nonunity efficiencies of the detectors. For this purpose we consider the joint probability according to Eq. (12) in the limits $|\alpha|\gg|\beta|$ and $\eta_i|\alpha|^2\gg 1$ $(i=1,2)$. Provided the distribution $P(\beta)$ is not too pathological, this condition is sufficient for approximating the Poisson distributions by Gaussian distributions. Replacing the discrete numbers of events (which become rather large) by continuous variables

$$(n_1,n_2)\rightarrow(x,y) , \quad (16)$$

we arrive at

$$p(x,y)=\int d^2\beta\, P(\beta)\frac{1}{(\pi\eta_1\vartheta_1)^{1/2}}\times\exp\left\{-\frac{\left[x-\frac{\eta_1}{2}\vartheta_1\right]^2}{\eta_1\vartheta_1}\right\}\frac{1}{(\pi\eta_2\vartheta_2)^{1/2}}\times\exp\left\{-\frac{\left[y-\frac{\eta_2}{2}\vartheta_2\right]^2}{\eta_2\vartheta_2}\right\} . \quad (17)$$

The difference statistics according to Eq. (9) is now rewritten as

$$p(\Delta x)=\int dx\, p(x,x-\Delta x) . \quad (18)$$

Inserting into this expression the joint probability density as given in Eq. (17) the x integration is easily performed, which yields

$$p(\Delta x)=\int d^2\beta\, P(\beta)\frac{1}{[\pi(\eta_1\vartheta_1+\eta_2\vartheta_2)]^{1/2}}\times\exp\left\{-\frac{[\Delta x-\frac{1}{2}(\eta_1\vartheta_1-\eta_2\vartheta_2)]^2}{\eta_1\vartheta_1+\eta_2\vartheta_2}\right\} . \quad (19)$$

For simplicity we confine ourselves to equal efficiencies of the two detectors $(\eta_1=\eta_2=\eta)$. In practice this situation is realized by balancing the detectors. Applying in Eq. (19) once more the condition that the local oscillator is strong compared with the signal field ($|\alpha|\gg|\beta|$) we find

$$p(\Delta x)=\int d^2\beta\, P(\beta)\frac{1}{(2\pi\eta|\alpha|^2)^{1/2}}\times\exp\left\{-\frac{[\Delta x-2\eta|\alpha|\,|\beta|\sin(\varphi_\alpha-\varphi_\beta)]^2}{2\eta|\alpha|^2}\right\} . \quad (20)$$

Let us compare now the measured difference statistics with the field-strength probability distribution $p(E,\varphi)$ for the field strength E at the phase φ. For this purpose it is advantageous to apply the eigenkets of the field-strength operator [Eq. (5)], which are defined by

$$\hat{E}(\varphi)|E(\varphi)\rangle=E(\varphi)|E(\varphi)\rangle . \quad (21)$$

Based on the number-state representation the solution reads [12,14]

$$|E(\varphi)\rangle=(2\pi)^{-1/4}\sum_{n=0}^{\infty}(2^n n!)^{-1/2}H_n\left[\frac{E(\varphi)}{\sqrt{2}|g|}\right]\times\exp\left\{\frac{-E^2(\varphi)}{4|g|^2}\right\}\times\exp\left\{in\left[\varphi-\frac{\pi}{2}\right]\right\}|n\rangle . \quad (22)$$

The field-strength probability distribution is readily derived using the relation

$$p(E,\varphi)=\mathrm{Tr}\{\hat{\rho}|E(\varphi)\rangle\langle E(\varphi)|\} . \quad (23)$$

Applying now the P representation of the density operator [Eq. (11)] we may write

$$p(E,\varphi)=\int d^2\beta\, P(\beta)p(E,\varphi;|\beta\rangle)\ , \tag{24}$$

that is, the field-strength distribution $p(E,\varphi)$ for an arbitrary field state described by the density operator $\hat{\rho}$ may be expressed in terms of the field-strength distribution $p(E,\varphi;|\beta\rangle)=|\langle\beta|E(\varphi)\rangle|^2$ for the coherent state $|\beta\rangle$, which reads

$$p(E,\varphi;|\beta\rangle)=\frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left\{-\frac{(E-\langle\hat{E}\rangle)^2}{2\sigma^2}\right\}, \tag{25}$$

where

$$\langle\hat{E}\rangle=\langle\beta|\hat{E}(\varphi)|\beta\rangle=2|g|\,|\beta|\sin(\varphi-\varphi_\beta) \tag{26}$$

and

$$\sigma^2=\langle\beta|[\Delta\hat{E}(\varphi)]^2|\beta\rangle=\langle(\Delta\hat{E})^2\rangle_{\rm vac}=|g|^2\ . \tag{27}$$

Finally we arrive at

$$p(E,\varphi)=\int d^2\beta\, P(\beta)\frac{1}{(2\pi|g|^2)^{1/2}}\times\exp\left\{-\frac{[E-2|g|\,|\beta|\sin(\varphi-\varphi_\beta)]^2}{2|g|^2}\right\}. \tag{28}$$

Comparing Eq. (28) with the statistics of difference events as given in Eq. (20) together with Eqs. (6) and (7) agreement of both expressions is found for $\eta=1$.

In the general case of nonideal detectors the distributions $p(\Delta x)$ and $p(E,\varphi)$ are related to each other according to

$$p(\Delta x)\leftrightarrow\tilde{p}(E,\varphi)=\int dE'p(E',\varphi)p_\eta(E-E')\ , \tag{29}$$

where the scaling given in Eq. (6) is now replaced by

$$\eta|\alpha|\leftrightarrow|g|\ . \tag{30}$$

It is seen that the measured distribution $\tilde{p}(E,\varphi)$ represents a convolution of the field-strength distribution $p(E)=\mathrm{Tr}\{\hat{\rho}|E\rangle\langle E|\}$ with the quantity

$$p_\eta(E)=\frac{1}{(2\pi\sigma_\eta^2)^{1/2}}\exp\left\{-\frac{E^2}{2\sigma_\eta^2}\right\} \tag{31}$$

representing the noise contribution due to a nonunity detection efficiency. The variance of this distribution is given by

$$\sigma_\eta^2=|g|^2\left[\frac{1-\eta}{\eta}\right]. \tag{32}$$

For ideal detectors ($\eta=1$) the quantity $p_\eta(E)$ reduces to a δ distribution and the field-strength probability distribution is directly observed.

Let us consider the dependence of the distribution for the measured difference events on the local oscillator amplitude for special states of the signal field. Consider the case of a single-photon state as the input in the signal channel. Clearly, a single-photon state shows nonclassical properties. For this reason the P representation does not exist in the sense of $P(\beta)$ having the properties of a probability density. Nevertheless, we may apply the result derived above for the difference statistics as given in Eq. (13). For a single-photon state we have

$$P(\beta)=\sum_{k=0}^{1}\frac{1}{k!}\begin{pmatrix}1\\k\end{pmatrix}\left[\frac{\partial}{\partial\beta}\frac{\partial}{\partial\beta^*}\right]^k\delta^{(2)}(\beta)\ , \tag{33}$$

where $\delta^{(2)}(\beta)=\delta(\mathrm{Re}\{\beta\})\delta(\mathrm{Im}\{\beta\})$. Inserting this expression into Eq. (13) and integrating by parts we derive an explicit expression for the difference-counting statistics in the form

$$P_{\Delta n}^{(1)}=\left\{1-\frac{(\eta_1+\eta_2)}{2}+\left[\frac{\Delta n}{|\alpha|}-|\alpha|\frac{(\eta_1-\eta_2)}{2}\right]^2\right\}[B_{1,2}]^{\Delta n/2}I_{\Delta n}[(\eta_1\eta_2)^{1/2}|\alpha|^2]\exp\left\{-\frac{|\alpha|^2}{2}(\eta_1+\eta_2)\right\}, \tag{34}$$

where

$$B_{1,2}=\begin{cases}\eta_1/\eta_2 & \text{for } \Delta n\ge 0\\ \eta_2/\eta_1 & \text{for } \Delta n<0\ .\end{cases} \tag{35}$$

This result is valid for arbitrary mean photon numbers of the local oscillator. In the limit of a strong local oscillator we derive from Eq. (19)

$$p^{(1)}(\Delta x)=\frac{1}{[\pi|\alpha|^2(\eta_1+\eta_2)]^{1/2}}\left\{1-\frac{(\eta_1+\eta_2)}{2}+\left[\frac{\Delta x}{|\alpha|}-|\alpha|\frac{(\eta_1-\eta_2)}{2}\right]^2\right\}\exp\left\{-\frac{[\Delta x-\frac{1}{2}|\alpha|^2(\eta_1-\eta_2)]^2}{|\alpha|^2(\eta_1+\eta_2)}\right\}. \tag{36}$$

It should be noted that the distributions corresponding to the limit of a classical local oscillator have already been studied (for $\eta_1=\eta_2$) by Yurke and Stoler [24]. In this approach the authors start from the field-strength distribution rather than the photon-counting theory. The nonunity detection efficiency is modeled by an additional beam splitter in front of the ideal detectors, which partly introduces vacuum noise. Such a model for nonideal detectors is consistent with the result derived above from the photon-counting theory.

In Fig. 2 the difference statistics $P_{\Delta n}^{(1)}$ is shown for the case of ideal detectors ($\eta_1=\eta_2=\eta=1$) and for various mean photon numbers of the local oscillator. For an extremely weak local oscillator ($|\alpha|^2\ll 1$) the difference statistics shows the features expected for a single photon which is divided into two channels. Since the photon is observed with equal probabilities in one of the channels the observed number difference may become $\Delta n = \pm 1$ with a probability of $\frac{1}{2}$. With increasing strength of the local oscillator the difference statistics approaches the probability distribution $p(E,\varphi;|1\rangle)$ for the electric-field strength in a single-photon state. This tendency is in agreement with the general proof given above. Moreover, it is seen that the requirement of a strong local oscillator is already fulfilled for about five local oscillator photons compared with the single photon in the signal channel. In this case the envelope of the discrete difference-counting statistics is in suitable agreement with the field-strength distribution. Thus one may observe the field-strength statistics already far from the case of a classical local oscillator. In Fig. 3 the situation is shown for nonideal detectors ($\eta = 0.75$). The structures of the field-strength distribution are seen to be smoothed out and the convolution of the field-strength distribution with a Gaussian noise distribution as given in Eqs. (29) and (31) is a good approximation already when the mean number of local oscillator photons is about 5.

Consider now the difference statistics for a signal field in a coherent state $|\beta\rangle$. Inserting $P(\beta') = \delta^{(2)}(\beta - \beta')$ into Eq. (13) we arrive at

$$P_{\Delta n}^{(\beta)} = [C_{1,2}]^{\Delta n/2} I_{\Delta n}(\eta_1\eta_2\vartheta_1\vartheta_2)^{1/2} \times \exp\{-\tfrac{1}{2}(\eta_1\vartheta_1 + \eta_2\vartheta_2)\} , \quad (37)$$

where $\vartheta_{1,2}$ and $C_{1,2}$, respectively, are defined in Eqs. (14) and (15). The limit of a strong local oscillator is easily derived from Eq. (19). In Fig. 4 the behavior of the difference distribution is shown for a coherent signal field. Although the mean number of signal photons is assumed to be 1 ($|\beta|^2 = 1$), the distributions significantly differ from that of a single-photon state. When the local oscillator is blocked the measured number difference is zero with large probability. Moreover, values of $\Delta n = \pm 1, \pm 2$ are seen to contribute significantly. This is due to the fact that the coherent state represents a Poissonian distribution of photons. For increasing numbers of local oscillator photons the difference distribution again ap-

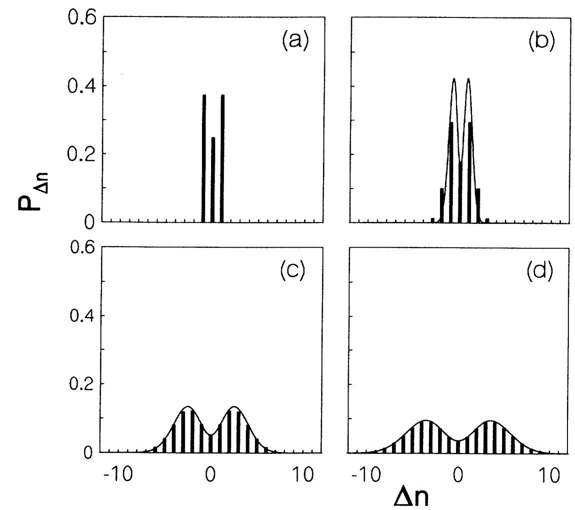

FIG. 3. Difference statistics $P_{\Delta n}$ for a single-photon state for nonideal photodetectors ($\eta_1 = \eta_2 = 0.75$) and for various mean photon numbers of the local oscillator: $|\alpha|^2 = 0$ (a), 0.5 (b), 5 (c), 10 (d). The smooth curves represent the appropriately scaled probability distributions $\tilde{p}(E,\varphi)$, which are the convolution of the field-strength distributions $p(E,\varphi)$ with the noise distribution $p_\eta(E)$ describing the influence of the nonideal detectors.

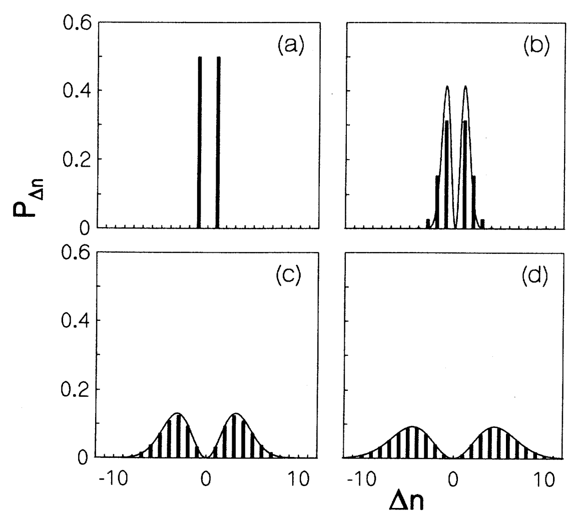

FIG. 2. Difference statistics $P_{\Delta n}$ for a single-photon state for ideal photodetectors ($\eta_1 = \eta_2 = 1$) and for various mean photon numbers of the local oscillator: $|\alpha|^2 = 0$ (a), 0.5 (b), 5 (c), 10 (d). The smooth curves represent the appropriately scaled probability distributions $p(E,\varphi)$ of the electric-field strength.

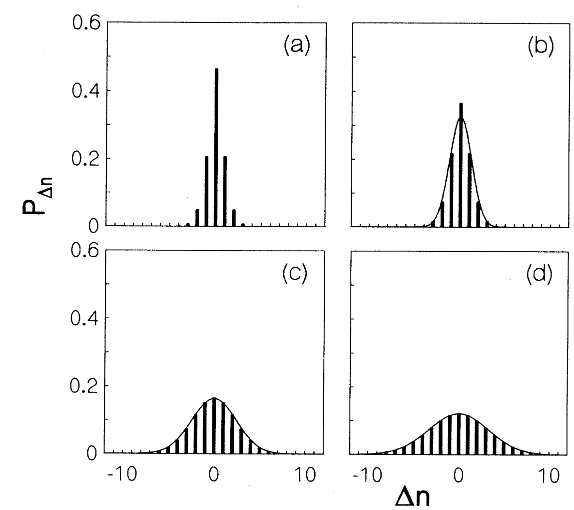

FIG. 4. Difference statistics $P_{\Delta n}$ for a coherent state of mean photon number equal to 1 for ideal photodetectors ($\eta_1 = \eta_2 = 1$) and for various mean photon numbers of the local oscillator: $|\alpha|^2 = 0$ (a), 0.5 (b), 5 (c), 10 (d). The phase difference is given by $\Delta\varphi = \varphi_\alpha - \varphi_\beta = 0$. The smooth curves represent the appropriately scaled probability distributions $p(E,\varphi)$ of the electric-field strength.

proaches the probability distribution for the electric-field strength in the coherent state under study. Changing the phase of the local oscillator, the Gaussian field-strength distribution is shifted (see Fig. 5). Remembering the field-strength distribution according to Eqs. (25)–(27), the observation of this behavior would be a demonstration of the quantum physical counterpart of a classical wave. The quantum physical coherent state behaves like a classical wave which is superimposed by the vacuum noise, the latter scales in the homodyne-observation scheme under study with the local oscillator amplitude; see Eq. (6). Note that we need not consider the situation for nonunity detection efficiency ($\eta \neq 1$) for a coherent signal field. In such a case the distribution remains unchanged when we use the mean numbers of detected counts $\eta|\alpha|^2$ and $\eta|\beta|^2$ instead of the mean photon numbers $|\alpha|^2$ and $|\beta|^2$, respectively [see Eq. (37) together with Eq. (14)].

To our knowledge neither the homodyne statistics $P_{\Delta n}$ in the case of two microscopic fields nor the field-strength distribution $p(E,\varphi)$ for a strong local oscillator have been observed so far. As discussed above the measurement of the latter would give some insight into the behavior of quantized light fields from the point of view of the electric-field strength. The field-strength representation, which was introduced to quantum optics by Schubert and Vogel [12], enables us to directly compare some quantum features with the corresponding classical behavior of the electric-field strength we are familiar with from classical electrodynamics. Moreover, the homodyne measurement of the difference statistics shows (with increasing strength of the local oscillator) a transition from the observable $\Delta\hat{n}$ for the number difference of the signal field to its field strength $\hat{E}$. It should be rather simple to observe these distributions for a coherent signal field. After splitting a laser field into two parts they can be used as the local oscillator and signal fields. In this manner the phases are correlated and the phase-sensitive distributions are easily measured by varying the phase difference between the input beams. For detecting the homodyne difference statistics and the field-strength statistics of a single-photon state, the field state could be prepared similarly to the experimental demonstration of localized single-photon states by Hong and Mandel [25]. In this case the phase diffusion of the local oscillator is meaningless since the corresponding distributions of the single-photon state are phase insensitive.

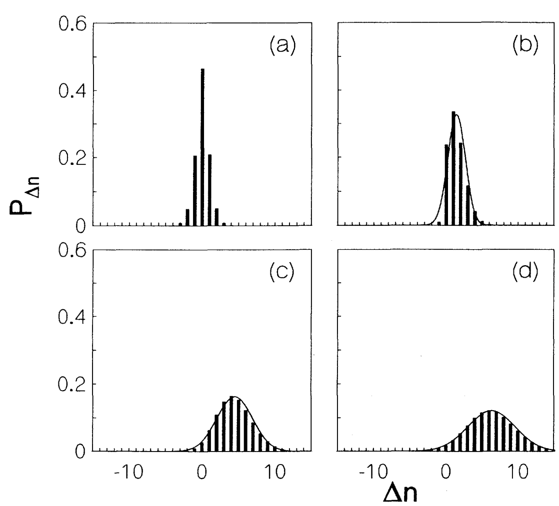

FIG. 5. The same as in Fig. 4 but for the phase difference $\Delta\varphi = \varphi_\alpha - \varphi_\beta = \pi/2$.

We would like to emphasize that a first measurement of the statistics of difference events has been performed for twin pulses of light [26]. In this experiment the photon numbers are rather large and correlated pulses are used instead of the local oscillator and the signal field. Nevertheless, the measured distributions could be derived from the counting theory in a similar manner as done above for the balanced homodyne-detection scheme.

IV. PHASE STATISTICS AND HOMODYNE DETECTION

The phase probability distribution of a quantum state $|\psi\rangle$ of the radiation field may be introduced on the basis of the phase state

$$|\varphi\rangle = (2\pi)^{-1/2} \sum_{m=0}^{\infty} e^{im\varphi}|m\rangle \qquad (38)$$

by

$$p(\varphi;|\psi\rangle) = |\langle\varphi|\psi\rangle|^2 . \qquad (39)$$

Recently a truncated version of these phase states has been used to define a Hermitian phase operator [27]. Since we only deal with physical states $|\psi\rangle$ containing a finite energy the truncation of the Hilbert space is superfluous. The problem connected with the distribution $p(\varphi;|\psi\rangle)$ consists of the fact that there exists no scheme for the measurement of this quantity. Standard approaches to measure phase properties are based on interference techniques, such as homodyne detection. Measured operators for the sine and cosine of the phase difference of the two input fields have recently been defined by Noh, Fougéres, and Mandel [17,18]. Their operator definitions are directly related to a homodyne-measurement scheme and consequently the agreement between theory and experiment is very good.

A different way to introduce phase distributions which are related to homodyne detection was proposed by Vogel and Schleich [15]. This concept is based on the probability distribution of the electric-field strength [12]. The basic idea is closely related to the classical understanding of the phase of a wave, which may be characterized by the electric-field strength $E(\varphi)$ considered as a function of the phase φ. In quantum physics the corresponding statistical information is contained in the field-strength probability distribution $p(E,\varphi)$ as defined in Eq. (23) together with Eq. (22). When the field-strength statistics has been observed via homodyne detection (see Sec. III) a measurable phase distribution can be introduced in the following manner [15]:

$$p_E(\varphi;|\psi\rangle) = \mathcal{N} p(E=0,\varphi;|\psi\rangle) = \mathcal{N} |\langle E(\varphi)=0|\psi\rangle|^2 , \qquad (40)$$

that is, the field-strength distribution is considered for the field value $E=0$ as a function of the parameter-valued phase φ. The normalization constant $\mathcal{N}$ is given by

$$\mathcal{N}=\left\{\int_0^\pi d\varphi|\langle E(\varphi)=0|\psi\rangle|^2\right\}^{-1}. \tag{41}$$

In this approach no phase operator is required, the fundamental operator being the well-defined field operator $\hat{E}$. The conditions for a suitable agreement of the phase distribution based on the measurable field strength with the phase distribution given in Eq. (39) have been studied in [15]. For example, consider the situation for a coherent state $|\psi\rangle=|\beta\rangle$. According to the scaling given in Eqs. (6) and (7) we derive from Eqs. (25)–(27) together with (40)

$$p_E(\varphi;|\beta\rangle)=\mathcal{N}\frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left\{-\frac{\langle\hat{E}\rangle^2}{2\sigma^2}\right\}, \tag{42}$$

where

$$\langle\hat{E}\rangle=2|\alpha|\,|\beta|\sin(\varphi-\varphi_\beta) \tag{43}$$

and

$$\sigma=|\alpha|,\quad \varphi=\varphi_\alpha. \tag{44}$$

It should be noted that the normalized distribution $p_E(\varphi;|\beta\rangle)$ as given by Eqs. (41)–(44) is independent of the local oscillator amplitude. On the other hand, we obtain from Eqs. (38) and (39)

$$p(\varphi,|\beta\rangle)=\frac{1}{2\pi}e^{-|\beta|^2}\left|\sum_{m=0}^{\infty}\frac{|\beta|^m}{\sqrt{m!}}\exp\{-im(\varphi-\varphi_\beta)\}\right|^2. \tag{45}$$

In Fig. 6 the phase distributions given in Eqs. (42) and (45) are compared for various mean photon numbers $|\beta|^2$ of the signal field. In agreement with the general conclusions of [15] it is seen that both distributions become close together when the photon number increases. We find that already for $|\beta|^2=4$ both kinds of phase distributions show an excellent agreement.

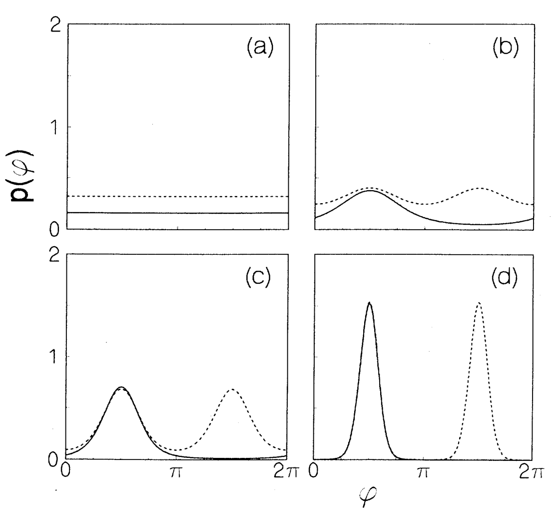

FIG. 6. Comparison of the phase distributions $p(\varphi)=p(\varphi;|\beta\rangle)$ (solid lines) and $p(\varphi)=p_E(\varphi;|\beta\rangle)$ (dashed lines) for $\varphi_\beta=\pi/2$ and for various mean photon numbers $|\beta|^2$ of the signal field: $|\beta|^2=0$ (a), 0.25 (b), 1 (c), 4 (d).

For a squeezed state $|\beta\rangle_s$ the phase distribution given in Eqs. (42)–(44) remains valid when we replace σ by σ_s with

$$\sigma_s^2=|\alpha|^2[1+2|\nu|^2+2|\nu|(1+|\nu|^2)^{1/2}\cos(2\varphi-\varphi_\nu)]. \tag{46}$$

The quantities $|\beta|$ and $|\nu|$, respectively, characterize the coherent amplitude and the strength of squeezing of the signal field, $|\nu|=0$ corresponds to a coherent state. In Fig. 7 examples for this phase distribution are given. For different orientations of the squeezing ellipse narrow, broad, or asymmetric phase distributions may be observed.

Let us now return to the more general situation of the difference counting statistics for an arbitrarily weak local oscillator. We may generalize the concept of the phase distribution $p_E(\varphi;|\psi\rangle)$ for the quantum state $|\psi\rangle$ in the limit of a strong local oscillator to the case of the phase difference of two microscopic light fields. The latter problem is closely related to that studied by Noh, Fougéres, and Mandel [17,18]. Generalizing the approach of [15], an operational distribution $p_{\Delta n}(\Delta\varphi)$ for the phase difference can be introduced by considering the statistics $P_{\Delta n=0}$ and as a function of the phase difference $\Delta\varphi=\varphi_\alpha-\varphi_\beta$:

$$p_{\Delta n}(\Delta\varphi)=\mathcal{M}P_{\Delta n=0}, \tag{47}$$

with the normalization constant

$$\mathcal{M}=\left\{\int_0^\pi d(\Delta\varphi)P_{\Delta N=0}\right\}^{-1}. \tag{48}$$

For a strong, coherent local oscillator this quantity tends to the operational phase distribution $p_E(\varphi;|\psi\rangle)$. Consider now the more general situation for two microscopic fields which are in coherent states $|\alpha\rangle$ and $|\beta\rangle$. In this

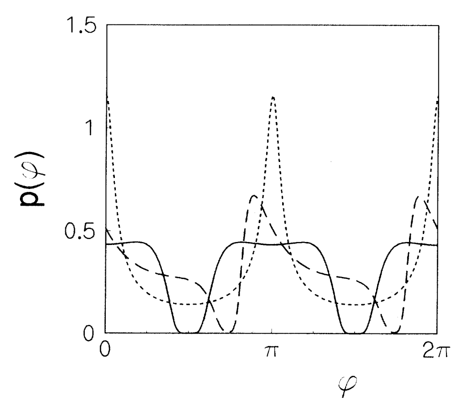

FIG. 7. Phase statistics $p(\varphi)=p_E(\varphi;|\beta\rangle_s)$ for a squeezed state (with $|\beta|^2=1$, $|\nu|^2=1$, $\varphi_\beta=0$) and for various orientations of the noise ellipse: $\varphi_\nu=0$ (solid line), $\varphi_\nu=\pi/2$ (long-dashed line), and $\varphi_\nu=\pi$ (short-dashed line).

case we easily obtain from Eq. (37)

$$P^{(\beta)}_{\Delta n=0}=I_{\Delta n=0}(\eta_1\eta_2\vartheta_1\vartheta_2)^{1/2}\exp\{-\tfrac{1}{2}(\eta_1\vartheta_1+\eta_2\vartheta_2)\} \ . \tag{49}$$

In Fig. 8 the distribution $p_{\Delta n}(\Delta\varphi)$ is shown for a ratio of the mean photon numbers $|\alpha|^2/|\beta|^2=2$, which is far from the situation of a strong local oscillator. It is seen that, although the distribution $P_{\Delta n}$ of the difference events is discrete we arrive at a continuous distribution $p_{\Delta n}(\Delta\varphi)$ for the phase difference $\Delta\varphi$ which becomes sharper with increasing mean photon numbers.

A quantitative comparison of the phase statistics based on the distribution $p_E(\varphi,|\psi\rangle)$ with the measured data and the corresponding measured phase operators of Noh, Fougéres, and Mandel [17,18] was presented by Lynch [28]. For the data recorded for a strong local oscillator agreement of both concepts was found for photon numbers of the signal field larger than 1. Based on the distribution $p_{\Delta n}(\Delta\varphi)$ we may now compare the corresponding variances of the phase quantities with the data for a weak local oscillator. For this reason we calculate the moments of the phase quantity F [where $F=S=\sin(\Delta\varphi)$ and $F=C=\cos(\Delta\varphi)$, respectively, being the sine and the cosine of the phase difference] according to

$$\langle F^k\rangle=\int_{\pm\pi/2} d(\Delta\varphi)F^k(\Delta\varphi)p_{\Delta n}(\Delta\varphi) \ , \tag{50}$$

where $\int_{\pm\pi/2}$ means integration over a π interval around the peak of the distribution $p_{\Delta n}(\Delta\varphi)$. This approach generalizes that of Lynch to arbitrarily weak local oscillator fields. In Fig. 9 we compare the moments calculated from Eq. (50) with the data presented in [17]. Data points for (signal) photon numbers below 1 are not given since in this case one cannot expect a suitable agreement already for a strong local oscillator; see [15,28]. For a ratio of the photon numbers of the local oscillator to the signal field of 7 the agreement is rather good. Thus we may conclude that this ratio is already large enough to be considered as a strong local oscillator situation, which is in agreement with our findings for the difference statistics in Sec. III. For a ratio of 1 some deviations are found to appear. However, compared with the deviation between the measured data and other phase concepts that already appear for a strong local oscillator (see [17]) the agreement is still surprisingly good. The origin of the similarities may be seen in the same type of measurement scheme which is used for defining the phase properties. On the other hand, the two concepts are quite different from each other. Whereas Noh, Fougéres, and Mandel define phase operators, a c-number phase appears in our approach. Consequently, the data needed for deriving the curves (according to Fig. 9) from the homodyne measurement are quite different for the two kinds of phase quantities.

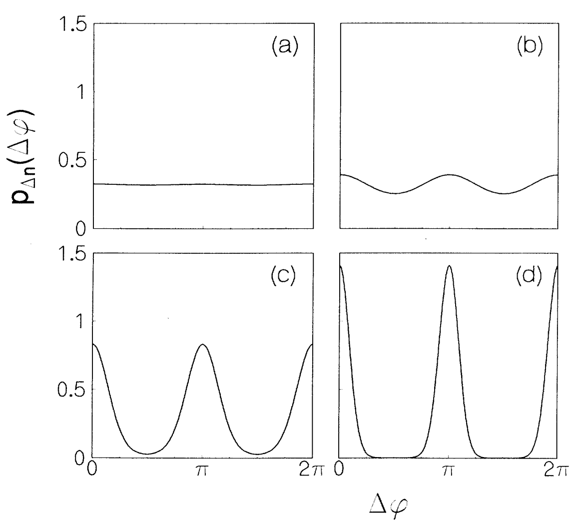

FIG. 8. Distribution $p_{\Delta n}(\Delta\varphi)$ of the phase difference of two coherent states for the ratio of the mean photon numbers $|\alpha|^2/|\beta|^2=2$ and for various mean photon numbers of the local oscillator: $|\alpha|^2=0.2$ (a), 1 (b), 4 (c), 10 (d).

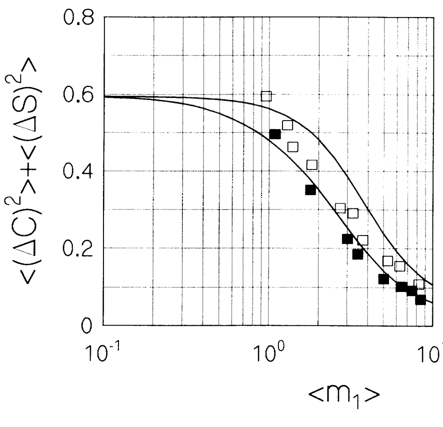

FIG. 9. Comparison of $\langle(\Delta C)^2\rangle+\langle(\Delta S)^2\rangle$ as calculated from the operational distribution $p_{\Delta n}(\Delta\varphi)$ (solid lines) with the experimental data (markers) of Noh, Fougéres, and Mandel [17]. For comparing with the measurement we have introduced the quantity $\langle m_1\rangle=2\eta|\beta|^2$ as used in [17]. The upper curve (and markers) corresponds to a ratio of mean photon numbers $|\alpha|^2/|\beta|^2=1$; for the lower curve (and markers) $|\alpha|^2/|\beta|^2=7$.

V. SUMMARY AND CONCLUSIONS

In the present paper we have studied the statistics of the difference events in a balanced homodyne-detection scheme. The starting point of our consideration was the quantum theory of photon detection. From the joint probability of the events measured by the two detectors involved in the detection scheme we derived the distribution for the difference events of both counters. From this quantum statistical point of view the usual case of homodyne detection is considered, that is, the local oscillator is strong compared with the signal field. In this limit and for ideal detectors the difference statistics represents the probability distribution of the electric-field strength of the signal field. Nonideal detectors yield a convolution of the field-strength distribution with a noise distribution characterizing the nonunity detection efficiency. Experiments which use the balanced detection scheme in the regime of photon counting have recently been performed in connection with the phase statistics of quantized light fields. The determination of the difference statistics in such an experiment would render it possible to demonstrate a change of observables from the difference of the photon numbers of the signal field in the two detection ports to the electric-field strength by simply increasing the intensity of the local oscillator. Using two coherent input fields such an experiment would allow one to directly observe the quantum counterpart of a classical wave, including the vacuum noise distribution for the electric field, which is displaced in a phase-sensitive manner when the local oscillator phase is shifted with respect to the signal phase. Moreover, using a localized single-photon state as the signal field one may demonstrate the double-peaked field-strength distribution of a single photon, which represents the quantum optical analog of the spatial probability distribution for the harmonic oscillator known from any quantum-mechanics textbook. For the case of weak signal fields (for example, a single photon) we have found that the difference statistics of the superimposed light already is close to the field-strength distribution when the mean number of local oscillator photons is about 5, that is in a regime far from a classical description of the local oscillator.

It has been shown recently that from the field-strength probability distribution of a quantum field one may also get some insight into its phase statistics. In the present paper we have generalized this operational concept in order to consider the properties of the phase difference of two quantum fields of low mean photon numbers. The phase-difference distribution is derived from the distribution for the difference events in the homodyne experiment. The latter quantity is considered for a difference of events equal to zero as a function of the phase difference of the input fields. In this approach the phases play the role of parameters. Nevertheless, the phase information available for two microscopic input fields may be rather close to that based on recently defined measured phase operators.

Note added. We would like to thank M. G. Raymer for bringing a paper of S. L. Braunstein [Phys. Rev. A **42**, 474 (1990)] to our attention in which homodyne detection is also considered for weak local oscillators. Compared with our approach Braunstein starts with the definition of a characteristic function for the operator of the photon-number difference rather than from the theory of light detection. For the special case of unity detection efficiency some of our results reduce to that found by Braunstein.

ACKNOWLEDGMENT

One of us (W.V.) gratefully acknowledges valuable discussions with H. J. Kimble and L. Mandel.

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