Quantum optical master equations: The use of damping bases

Hans-Jürgen Briegel and Berthold-Georg Englert*
Sektion Physik, Universität München, Am Coulombwall 1, D-8046 Garching, Germany
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We present the complete analytical solution of the cavity QED of a two-level atom and a field mode at zero temperature. It includes both dissipation of the field due to a finite Q value of the cavity and incoherent decay mechanisms for the atom. This analytical solution is provided by a powerful method for treating general master equations that appear in quantum optics. As distinct from the usual approaches we first deal with that part of the master equation which describes the dissipative coupling of the field and the atom to their reservoirs. Rather than using number-state or dressed-state bases we expand the density operator into the eigenstates of the nonunitary parts of the master equation which model the dissipative part of the dynamics. The set of these eigenstates is the damping basis. With the aid of this expansion we find the eigenvalues and eigenstates of the total Liouville operator. The evolution of an arbitrary initial state is then known. We employ these results to give an exact solution of the dynamics of the photon field in realistic experiments with one-atom masers at very low temperatures. It includes detuning, cavity leakage effects, spontaneous decay mechanisms for the atoms, a Fizeau-type velocity distribution for the atomic beam, and a statistical parameter for the probability of the excitation of incoming atoms, covering the limits of Poissonian pumping and of regular pumping. On the same grounds one can treat the one-atom laser, consisting of a single atom which stays in permanent interaction with the field mode and which is continuously pumped by external heat baths.

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I. INTRODUCTION

A. The problem

The dynamical evolution of a two-level atom interacting with one mode of the quantized photon field in a cavity or in a laser is well described by a master equation of the structure

$$\frac{\partial}{\partial t}P=\frac{1}{i\hbar}[H,P]+L_aP+L_\sigma P \tag{1.1}$$

in which the symbol P denotes the density operator that specifies the state of the system. The right-hand side consists of two parts, a unitary one and a nonunitary one. The unitary part involves the commutator with the Hermitian Hamilton operator

$$H=\hbar\omega a^\dagger a+\tfrac{1}{2}\hbar\Omega\sigma_z-\tfrac{1}{2}\hbar g(a^\dagger\sigma_-+a\sigma_+) \tag{1.2}$$

for the interaction between the photon mode with frequency ω and the atom with level spacing $\hbar\Omega$. The dynamical variables are the ladder operators $a^\dagger, a$ for the field and

$$\sigma_\pm=\sigma_x\pm i\sigma_y=(\sigma_\mp)^\dagger \tag{1.3}$$

for the atom, where σ_x, σ_y, and also σ_z are Pauli's spin operators. The constant g is the Rabi frequency that measures the coupling strength of this interaction.

The nonunitary part

$$LP=L_aP+L_\sigma P \tag{1.4}$$

accounts for losses to and gains from reservoirs. The first term

$$L_aP=-\frac{A}{2}(\nu+1)[a^\dagger aP+Pa^\dagger a-2aPa^\dagger]-\frac{A}{2}\nu[aa^\dagger P+Paa^\dagger-2a^\dagger Pa]\,, \tag{1.5}$$

with constants $A,\nu\geq 0$, describes the coupling of the field to a thermal reservoir at a temperature which corresponds to a mean number ν of thermal photons in the cavity or the laser. The second term reads

$$L_\sigma P=-\frac{B}{8}(1-s)[\sigma_+\sigma_-P+P\sigma_+\sigma_--2\sigma_-P\sigma_+]-\frac{B}{8}s[\sigma_-\sigma_+P+P\sigma_-\sigma_+-2\sigma_+P\sigma_-]-\frac{2C-B}{4}[P-\sigma_zP\sigma_z]\,, \tag{1.6}$$

with constants $2C\geq B\geq 0$ and $0\leq s\leq 1$; it models incoherent pumping and decay processes of the atom.

When $A,B,C=0$, (1.1) is the master equation of the familiar Jaynes-Cummings model [1] whose eigenvalues are well known.

When $g=0$, on the other hand, (1.1) describes pure relaxation of the oscillating mode and the atomic system, which then are decoupled. In this situation (and only then) the constant A gives the decay rate of the mean number $\langle a^\dagger a\rangle$ of photons in the cavity or in the laser towards the thermal-state value $\langle a^\dagger a\rangle_\infty=\nu$. The constants B and C are the corresponding atomic decay rates for the inversion measure $\langle\sigma_z\rangle$ and the polarization $\langle\sigma_\pm\rangle$ towards their equilibrium values $\langle\sigma_z\rangle_\infty=2s-1$ and $\langle\sigma_\pm\rangle_\infty=0$. Note that the third term in (1.6), involving the constant C, accounts for processes, which are

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sometimes termed *virtual,* such as atomic collisions that lead to a loss of atomic coherence without changing the level population. The parameter s, ranging from 0 to 1, further characterizes the atomic reservoir. If $s=0$, (1.6) only effects transitions from the upper to the lower level, modeling processes such as incoherent transitions due to spontaneous emission of photons into modes other than the privileged cavity (or laser) mode, or transitions induced by atomic collisions and electric scattering fields in a cavity. When $s>0$, there are also transitions from the lower to the upper level. The resulting equilibrium value of the inversion measure, $\langle \sigma_z \rangle_\infty = 2s-1$, grows with increasing s. In particular, when $s>\frac{1}{2}$, the atomic bath effectively acts like a (incoherent) pumping reservoir under whose action the atomic population becomes inverted (i.e., the inversion becomes positive). So s can be called a pumping parameter, which is to be explained in more detail later on.

So far the total system under consideration consists of a single atom (Ω) interacting with a single field mode (ω), both being coupled to their specific reservoir. The characterizing constants are the system parameters ω,Ω,ν,s and the coupling constants g, A, B, C. To model a real laser, whose active medium consists of N two-level atoms, one could think of employing a well-established method given by Weidlich, Risken, and Haken [2] and by Risken [3] who, roughly speaking, replace the operators of the microscopic inversion, σ_z, and polarization, $\sigma_\pm$, by the macroscopic sums

$$S_z=\sum_{j=1}^{N}\sigma_z^j, \quad S_\pm=\sum_{j=1}^{N}\sigma_\pm^j \tag{1.7}$$

and discard any correlations $\langle \sigma^j\sigma^k\sigma^1\cdots\rangle$, $j\neq k\neq l\neq\cdots$, between the observables of different atoms. We shall return to this in a forthcoming paper with Ginzel and Schenzle [4].

As it stands, the master equation (1.1), together with (1.2), (1.5), and (1.6), is the basic equation of the quantum theory of the interaction between atoms and electromagnetic fields which are part of a thermodynamically open system. For $s=0$ it describes the cavity QED of an atom in a cavity with finite Q value and provides, in particular, the exact description of the dynamics in a one-atom maser. There, alternating periods of atomic interaction with the cavity field and of the pure decay of the latter obey Eq. (1.1) with $g\neq 0$ and $g=0$, respectively.

When $s>0$, an additional continuously pumping reservoir for the atom is included which leads to a positive atomic inversion when $s>\frac{1}{2}$ and thus describes a one-atom laser, which could be generalized to the many-atom laser by the standard method described in [2] and [3].

In the present paper we present a powerful approach for solving the master equation (1.1). In contrast to the standard methods which resort to approximate Fokker-Planck-type equations we employ an algebraic method that transforms (1.1) into coupled systems of 4×4-matrix equations. In the important case when $\nu=0$ and $s=0$, which is realized in the ongoing experiments with the one-atom maser [5], these can be solved *analytically.* In the most general situation, one is eventually led to 4×4-matrix continued fractions which can be solved numerically (with little effort) for any given set of parameters $\omega,\Omega,\nu,s,g,A,B,C$ without the need of additional structural approximations.

B. The method

In order to obtain all dynamical information about the atom-and-field coupled system described by the master equation (1.1) it is expedient to solve the corresponding eigenvalue equation

$$\mathcal{L}P\equiv\frac{1}{i\hbar}[H,P]+LP=\Gamma P\ , \tag{1.8}$$

where $\mathcal{L}$ denotes the total Liouville operator, L is the nonunitary part (1.4), and Γ is a, possibly complex, eigenvalue of $\mathcal{L}$. A note on the terminology is in order. In this paper we will consistently call P the *state* of the system, and L and the commutator with H are linear *operators* acting on this state. Of course, P itself is a linear operator acting on the Hilbert space *vectors,* and L or $\mathcal{L}$ are therefore called superoperators by some authors. We hope there will be no confusion.

When the system is closed, $L=0$, (1.8) refers to the Jaynes-Cummings model whose solutions are well known. The eigenstates of the commutator in (1.8) are then given by dyadic products of eigenvectors of H, popularly called "dressed" vectors.

When the system is open, $L\neq 0$, the nonunitary part that models the system's coupling to external reservoirs makes things more difficult. Although the master equation (1.1) does not describe a unitary evolution of the state P, the solution of the eigenvalue equation (1.8) nevertheless provides us with the solution of the time-dependent problem (1.1).

Suppose we have solved (1.8) and found all eigenvalues Γ and the corresponding eigenstates P_Γ. Then, if we can write the expansion

$$P(0)=\sum_{\Gamma}\check{c}_\Gamma P_\Gamma \tag{1.9}$$

for the initial state given at time $t=0$, we know the state of the system at any later time, namely,

$$P(t)=\sum_{\Gamma}\check{c}_\Gamma e^{\Gamma t}P_\Gamma\ . \tag{1.10}$$

Here and in the sequel the symbolic sum over Γ also properly takes account of the different eigenstates belonging to a degenerate eigenvalue. Thus, once (1.8) is solved, the remaining problem is, how does one find the expansion coefficients $\check{c}_\Gamma$ appearing in (1.9)? The answer is

$$\check{c}_\Gamma=\mathrm{Tr}\{\check{P}_\Gamma P(0)\}\ , \tag{1.11}$$

where Tr, the "total trace," symbolizes the injunction to trace over both the photon and the atom variables, and the states $\check{P}_\Gamma$ satisfy the duality relation

$$\mathrm{Tr}\{\check{P}_\Gamma P_{\Gamma'}\}=\delta_{\Gamma\Gamma'}\ . \tag{1.12}$$

Accordingly, we call the states $\check{P}_\Gamma$ *dual* to the eigenstates P_Γ.

In order to find these dual states, consider the time evolution of the expectation value of an observable $\mathcal{O}$.

An immediate consequence of the master equation (1.1) is

$$\frac{d}{dt}\langle \mathcal{O} \rangle = \frac{d}{dt}\mathrm{Tr}\{\mathcal{O}P\} = \mathrm{Tr}\left\{\mathcal{O}\frac{\partial}{\partial t}P\right\} = \mathrm{Tr}\{\mathcal{O}\mathcal{L}P\} \ , \quad (1.13)$$

provided that $\mathcal{O}$ is not explicitly time dependent itself. The dual conjugate $\check{\mathcal{L}}$ to $\mathcal{L}$ is the operator that obeys

$$\mathrm{Tr}\{\mathcal{O}\mathcal{L}P\} = \mathrm{Tr}\{(\check{\mathcal{L}}\mathcal{O})P\} = \mathrm{Tr}\{P\check{\mathcal{L}}\mathcal{O}\} \quad (1.14)$$

for all states P and all observables $\mathcal{O}$ [6]. Now, $\check{\mathcal{L}}$ has eigenstates, too, and these are just the states which we have denoted by $\check{P}_\Gamma$,

$$\check{\mathcal{L}}\check{P}_\Gamma = \Gamma\check{P}_\Gamma \ , \quad (1.15)$$

where the Γ are identical with the eigenvalues of $\mathcal{L}$. This can be seen immediately by inserting $P_{\Gamma'}$ and $\check{P}_\Gamma$, instead of P and $\mathcal{O}$, into (1.14), thereby confirming the duality relation (1.12).

One can read (1.14) as if $\mathcal{L}$ were acting to the left on $\mathcal{O}$ according to

$$\mathcal{O}\mathcal{L} = \check{\mathcal{L}}\mathcal{O} \ . \quad (1.16)$$

Then $\check{P}_\Gamma$ is simply the left eigenstate (or, more frivolously, "eigen-bra") to $\mathcal{L}$with eigenvalue Γ,

$$\check{P}_\Gamma\mathcal{L} = \Gamma\check{P}_\Gamma \ , \quad (1.17)$$

whereas P_Γ is the corresponding right eigenstate (or "eigen-ket"),

$$\mathcal{L}P_\Gamma = P_\Gamma\Gamma \ . \quad (1.18)$$

In this context, the notion of duality is quite familiar.

The right eigenstates will be employed to expand arbitrary states P with the aid of the completeness relation

$$P = \sum_\Gamma P_\Gamma \mathrm{Tr}\{\check{P}_\Gamma P\} \ . \quad (1.19)$$

Likewise, the left eigenstates are used to expand arbitrary observables $\mathcal{O}$,

$$\mathcal{O} = \sum_\Gamma \check{P}_\Gamma \mathrm{Tr}\{\mathcal{O}P_\Gamma\} \ . \quad (1.20)$$

The duality relation (1.12) then supplies the "coordinate sum"

$$\langle \mathcal{O} \rangle = \mathrm{Tr}\{\mathcal{O}P\} = \sum_\Gamma \mathrm{Tr}\{\mathcal{O}P_\Gamma\}\mathrm{Tr}\{\check{P}_\Gamma P\} \quad (1.21)$$

for the expectation value $\langle \mathcal{O} \rangle$. Of course, for a given $\mathcal{L}$ the completeness of either set of eigenstates has to be established.

In summary, the master equation (1.1) is solved in general, if together with their eigenvalues all the eigenstates P_Γ and $\check{P}_\Gamma$ of the Liouville operator $\mathcal{L}$ and its dual $\check{\mathcal{L}}$, respectively, are found. Then, for a given initial state $P(0)$ the state at a later time t reads

$$P(t) = e^{\mathcal{L}t}P(0) \equiv \sum_\Gamma \mathrm{Tr}\{\check{P}_\Gamma P(0)\}e^{\Gamma t}P_\Gamma = \sum_\Gamma \check{c}_\Gamma e^{\Gamma t}P_\Gamma \ . \quad (1.22)$$

The temporal evolution of the expectation value of an observable

$$\mathcal{O} = \sum_\Gamma c_\Gamma \check{P}_\Gamma \quad (1.23)$$

is correspondingly computed according to

$$\langle \mathcal{O} \rangle_t = \mathrm{Tr}\{\mathcal{O}e^{\mathcal{L}t}P(0)\} \equiv \sum_\Gamma c_\Gamma e^{\Gamma t}\check{c}_\Gamma \ . \quad (1.24)$$

Now let us return to the question of how one, actually, solves the eigenvalue problem (1.8). As a first step one would expand the equation into eigenstates of an operator which is only a part of the total operator $\mathcal{L}$ and whose eigenstates are known. Conventionally this is the commutator with the free part of H, i.e., the atomic inversion σ_z and the photon number $a^\dagger a$, which would lead to a number-state representation of (1.8), or one could use the total Hamiltonian H, leading to an expansion of (1.8) into a dressed-state basis. In any case, the resulting systems of coupled differential equations for the expansion coefficients become rather involved.

In this paper, we will pursue a different approach, which consists in systematically using bases provided by the eigenstates of the *nonunitary* parts (summarized in L) of the total Liouville operator $\mathcal{L}$. This is a straightforward and powerful ansatz. It eventually supplies the complete analytic solution of the master equation in the important special case when $\nu,s=0$, and substantially simplifies the numerical treatment of the master equation (1.1) in its most general form by reducing the problem to the calculation of 4×4-matrix continued fractions without any further structural approximations.

So the strategy is the following. First we investigate the eigenvalue problem posed by the nonunitary part (1.4)–(1.6) of the master equation:

$$LP_\lambda = \lambda P_\lambda \ , \quad (1.25)$$

where the symbols λ for the eigenvalues and P_λ for the eigenstates are used to distinguish them from the solutions of the total Liouville operator. Similar to (1.12) there are left eigenstates $\check{P}_\lambda$ of L, too, the right eigenstates of the dual conjugate $\check{L}$. We find that both the eigenstates P_λ and their duals $\check{P}_\lambda$ form complete sets into which any observable of the system, and in particular its state $P(t)$, can be expanded. It turns out that such an expansion is a very natural one, since all essential statistical information about the field is comprised in its first few coefficients only. In the language of phase-space functions this method removes, in a certain sense, the terms involving second-order derivatives in the corresponding Fokker-Planck-type equations from the very beginning and leaves one with differential equations of first order only.

C. Outline

In Sec. II we investigate the nonunitary parts L_a and L_σ of the master equation in detail. We find that L_a and L_σ together possess a complete set of eigenstates into which any function of the dynamical variables $a^\dagger, a, \sigma_+, \sigma_-$ can be expanded. These eigenstates and their eigenvalues are derived explicitly and their properties discussed. In Sec. III these results are used to attack

the actual problem posed by Eq. (1.1). Upon expanding the state

$$P=P(a^{\dagger},a,\sigma_{+},\sigma_{-},t) \tag{1.26}$$

into said eigenstates of $L=L_a+L_\sigma$ the master equation transforms into coupled systems of ordinary differential equations of first order for the expansion coefficients. Conveniently arranged, these equations form groups that do not mix. Finally one is left with separate systems of 4×4-matrix recurrence relations which can be solved by continued fractions or even analytically. In the following sections this general frame is further developed and employed for a rigorous treatment of cavity quantum electrodynamics and laser physics.

First, in Sec. IV the analytic solutions are presented. They supply a complete and exact description of the interaction of a two-level atom and a photon mode in a cavity at zero temperature, including leakage effects for the photons out of the cavity and incoherent transitions for the atom.

This is of particular interest for an exact description of the dynamics in the one-atom maser which is the subject of Sec. V. As the damping constants enter the system's eigenfrequencies, which are generalized Rabi frequencies, the trapping-state condition is modified, too. In this approach both parts of a one-atom-maser cycle are treated on almost equal footing: the interaction period and the period of pure decay of the cavity field are both described by the same equations with values $g\neq0$ and $g=0$, respectively, the latter being elementary. In particular, future experiments with highly regular pumping, where a second atom enters the cavity immediately after the preceding one has left it, cannot be described by current one-atom-maser theories which ignore cavity damping while the atom is present.

Finally, Sec. VI gives an outlook to the situation at finite temperature and when the atom is coupled to a heat bath which serves as a pumping reservoir. This is to be called the "theory of a one-atom *laser*." Even in this situation, when pumping of the atom is included, the eigenvalues determining the intensity dynamics are found to be mere solutions of an ordinary continued fraction equation. The whole spectrum of eigenvalues and the general dynamical behavior of the system can be found by solving 4×4-matrix continued fractions numerically, the results of which are reported in a paper with Ginzel, Martini, and Schenzle [7].

II. DAMPING BASES

In the following we will call L_a and L_σ as in (1.5) and (1.6), respectively, *damping operators*. Note, however, that L_σ can also describe incoherent pumping, which is a mechanism structurally identical with damping. The sets of eigenstates of these damping operators will consequently be referred to as *damping bases*.

A. Atomic damping

The eigenstates $P_\lambda(\sigma_+,\sigma_-)$ of L_σ depend on the dynamical variables σ_+,σ_- of the atom and satisfy the equation

$$L_\sigma P_\lambda=\lambda P_\lambda\ , \tag{2.1}$$

with L_σ defined in (1.6). Apart from the state corresponding to $\lambda=0$ in (2.1) these are simply the spin variables themselves. One immediately finds

$$\begin{aligned} &L_\sigma\sigma_0=0\ ,\\ &L_\sigma\sigma_z=-B\sigma_z\ ,\\ &L_\sigma\sigma_\pm=-C\sigma_\pm\ , \end{aligned} \tag{2.2}$$

where

$$\sigma_0=\tfrac{1}{2}[1+(2s-1)\sigma_z)]\ . \tag{2.3}$$

So the eigenvalues are zero and the negatives of the coupling constants B and C for the atom and its reservoir. Note that they are independent of s, which is just a parameter characterizing the equilibrium state of the atomic reservoir. This will be similar for the photon damping whose eigenvalues are also independent of the mean number ν of thermal photons in the cavity, i.e., of the reservoir temperature.

The dual operator $\check{L}_\sigma$ corresponding to (1.6) reads

$$\begin{aligned} \check{L}_\sigma\check{P}=\check{P}L_\sigma=&-\frac{B}{8}(1-s)\{\sigma_+\sigma_-\check{P}+\check{P}\sigma_+\sigma_--2\sigma_+\check{P}\sigma_-\}\\ &-\frac{B}{8}s\{\sigma_-\sigma_+\check{P}+\check{P}\sigma_-\sigma_+-2\sigma_-\check{P}\sigma_+\}\\ &-\frac{2C-B}{4}\{\check{P}-\sigma_z\check{P}\sigma_z\}\ . \end{aligned} \tag{2.4}$$

As discussed in Sec. I A it has the same eigenvalues as L_σ and its eigenstates are dual to those of L_σ:

$$\begin{aligned} &\check{\sigma}_0L_\sigma=\check{L}_\sigma\check{\sigma}_0=0\ ,\\ &\check{\sigma}_zL_\sigma=\check{L}_\sigma\check{\sigma}_z=-B\check{\sigma}_z\ ,\\ &\check{\sigma}_\pm L_\sigma=\check{L}_\sigma\check{\sigma}_\pm=-C\check{\sigma}_\pm\ , \end{aligned} \tag{2.5}$$

where

$$\check{\sigma}_0=1,\quad \check{\sigma}_z=\tfrac{1}{2}[\sigma_z-(2s-1)],\quad \check{\sigma}_\pm=\tfrac{1}{4}\sigma_\mp\ . \tag{2.6}$$

One immediately verifies the duality relations (1.12), here

$$\mathrm{tr}\{\check{\sigma}_j\sigma_k\}=\delta_{jk},\quad j,k\in\{0,z,\pm\}\ . \tag{2.7}$$

If we write symbolically

$$\left.\frac{\partial P}{\partial t}\right|_\sigma=L_\sigma P \tag{2.8}$$

for that part of the evolution of the state which originates in the coupling to the atomic reservoir, the eigenstates of the dual $\check{L}_\sigma$ obey

$$\left.\frac{d}{dt}\right|_\sigma\langle\check{\sigma}_0\rangle=0\ , \tag{2.9a}$$

$$\left.\frac{d}{dt}\right|_\sigma\langle\check{\sigma}_z\rangle=-B\langle\check{\sigma}_z\rangle\ , \tag{2.9b}$$

$$\left.\frac{d}{dt}\right|_{\sigma}\langle\check{\sigma}_{\pm}\rangle=-C\langle\check{\sigma}_{\pm}\rangle\ . \tag{2.9c}$$

Equation (2.9a) simply states that the atomic damping keeps the normalization of the state constant. In view of (2.6), (2.9b) and (2.9c) say that the atomic inversion and polarization decay with mean inverse lifetimes B and C, respectively, which motivates the parametrization of L_σ in (1.6). The expectation values in the steady state σ_0, reached in the limit $t\to\infty$, are

$$\langle\sigma_z\rangle_\infty=2s-1 \quad\text{and}\quad \langle\sigma_\pm\rangle_\infty=0\ . \tag{2.10}$$

This means that under the influence of the atomic reservoir the atomic coherence gets lost, independent of the bath parameter $s=\langle(1+\sigma_z)/2\rangle_\infty$, as it should be. The inversion approaches a steady-state value which is between -1 and 1, depending on the value of s which ranges from 0 to 1. If $s>\frac{1}{2}$, then $\langle\sigma_z\rangle_\infty>0$ and the atomic population becomes inverted. For these reasons we call s the *pumping parameter* for the atom.

Naturally, any state or observable $\mathcal{O}=\mathcal{O}(\sigma_+,\sigma_-)$ depending solely on the dynamical variables of the atom can uniquely be expanded into the eigenstates of both L_σ and $\check{L}_\sigma$:

$$\mathcal{O}(\sigma_+,\sigma_-)=\check{c}_0\sigma_0+\check{c}_z\sigma_z+\check{c}_-\sigma_-+\check{c}_+\sigma_+\ , \tag{2.11}$$

where

$$\check{c}_0=\mathrm{tr}\{\check{\sigma}_0\mathcal{O}\},\quad \check{c}_z=\mathrm{tr}\{\check{\sigma}_z\mathcal{O}\},\quad \check{c}_\pm=\mathrm{tr}\{\check{\sigma}_\pm\mathcal{O}\}\ . \tag{2.12}$$

Similarly,

$$\mathcal{O}(\sigma_+,\sigma_-)=c_0\check{\sigma}_0+c_z\check{\sigma}_z+c_-\check{\sigma}_-+c_+\check{\sigma}_+\ , \tag{2.13}$$

with

$$c_0=\mathrm{tr}\{\mathcal{O}\sigma_0\},\quad c_z=\mathrm{tr}\{\mathcal{O}\sigma_z\},\quad c_\pm=\mathrm{tr}\{\mathcal{O}\sigma_\pm\}\ . \tag{2.14}$$

When $\mathcal{O}=\mathcal{O}(a^\dagger,a,\sigma_+,\sigma_-)$ depends on the dynamical variables of the photon, too, then the expansions (2.11) and (2.13) hold similarly if in (2.12) and (2.14) the trace tr is taken only over the atomic variables and then the expansion coefficients c_j and $\check{c}_j$ are still functions of $a^\dagger$ and a.

B. Photon damping

The solution of the eigenvalue problem for the photon damping operator L_a in (1.5) is somewhat more involved than for L_σ but still a straightforward business. The eigenstates ρ_λ of L_a and their eigenvalues are determined by

$$L_a\rho_\lambda=\lambda\rho_\lambda \tag{2.15}$$

subject to the constraint that

$$\mathrm{tr}\{B(a^\dagger,a)\rho_\lambda(a^\dagger,a)\}<\infty \tag{2.16}$$

holds for any bound observable $B(a^\dagger,a)$. Pure damping does not couple different diagonals of a state in the number-vector representation, so that the ansatz

$$\rho_\lambda(a^\dagger,a)=a^{\dagger\kappa}f(a^\dagger a) \tag{2.17a}$$

or

$$\rho_\lambda(a^\dagger,a)=f(a^\dagger a)a^k \tag{2.17b}$$

for $k=0,1,2,\ldots$, is invited, where f is a function of the photon number $a^\dagger a$. Inserted into (2.15), this ansatz yields a recursion for $f(a^\dagger a)$. Considerably more convenient is the use of a normally ordered function $:f(a^\dagger a):$ rather than $f(a^\dagger a)$ itself. Then (2.15) implies an ordinary second-order differential equation for f, viz.,

$$(\nu+1)zf''(z)+[z+(\nu+1)(k+1)]f'(z)+\left[\frac{k}{2}+1-\lambda/A\right]f(z)=0\ , \tag{2.18}$$

where $z=a^\dagger a$. Its derivation uses the identities

$$\begin{aligned}&:f(a^\dagger a):a^\dagger=a^\dagger:f(a^\dagger a)+f'(a^\dagger a):\ ,\\ &a^\dagger:f(a^\dagger a):a=:a^\dagger af(a^\dagger a):\ .\end{aligned} \tag{2.19}$$

Equation (2.18) turns into Kummer's differential equation if $-z/(\nu+1)$ is regarded as the variable. Its analytical solutions obey the constraint (2.16) only if

$$-\frac{\lambda}{A}-\frac{k}{2}=n=0,1,2,\ldots \tag{2.20}$$

for any given k. For these values of λ the solutions of (2.18) are

$$f(z)=\exp\left[-\frac{z}{1+\nu}\right]L_n^{(k)}\left[\frac{z}{1+\nu}\right]\ , \tag{2.21}$$

with the generalized Laguerre polynomials $L_n^{(k)}$. So the result is

$$L_a\rho_n^{(k)}=-A\left[n+\frac{|k|}{2}\right]\rho_n^{(k)}\ ,\qquad n=0,1,2,\ldots,\ k=0,\pm1,\pm2,\ldots, \tag{2.22}$$

with

$$\rho_n^{(k)}=\begin{cases}a^{\dagger k}\dfrac{(-1)^n}{(1+\nu)^{k+1}}:L_n^{(k)}\left[\dfrac{a^\dagger a}{1+\nu}\right]\exp\left[-\dfrac{a^\dagger a}{1+\nu}\right]: & \text{for } k\geq0\ ,\\ \dfrac{(-1)^n}{(1+\nu)^{|k|+1}}:L_n^{(|k|)}\left[\dfrac{a^\dagger a}{1+\nu}\right]\exp\left[-\dfrac{a^\dagger a}{1+\nu}\right]:a^{|k|} & \text{for } k\leq0\ .\end{cases} \tag{2.23}$$

The two possibilities of (2.17) are now distinguished by the sign of k. Reversing the sign of k is equivalent to taking the adjoint; the eigenvalues $-A(n+|k|/2)$ involve the magnitude of k only. The $a^\dagger a$-independent factors in (2.23) provide us with a convenient normalization.

The eigenstates $\rho_n^{(k)}$ are linearly independent and complete, as for any given k the $L_n^{(k)}$ are polynomials of order n and thus independent for different values of n; for different values of k the corresponding $\rho_n^{(k)}$ involve different powers of the variables $a^\dagger$ or a, so they are independent in the first place.

Note that, although the eigenstates of L_a depend on ν, its eigenvalues are independent of this parameter which is related to the bath temperature. This is similar to the situation of atomic damping, where the pumping parameter s only enters the eigenstates of L_σ, but not its eigenvalues.

For the dual damping operator $\check{L}_a$, given by

$$\check{\rho}L_a=\check{L}_a\check{\rho}=-\frac{A}{2}(\nu+1)\{a^\dagger a\check{\rho}+\check{\rho}a^\dagger a-2a^\dagger\check{\rho}a\}$$
$$-\frac{A}{2}\nu\{aa^\dagger\check{\rho}+\check{\rho}aa^\dagger-2a\check{\rho}a^\dagger\}\ , \qquad (2.24)$$

which is to be compared with (1.5), the derivation of its eigenstates is done analogously. The result is

$$\check{\rho}_n^{(k)}L_a=\check{L}_a\check{\rho}_n^{(k)}=-A\left[n+\frac{|k|}{2}\right]\check{\rho}_n^{(k)}\ ,$$
$$n=0,1,2,\ldots,\quad k=0,\pm1,\pm2,\ldots, \qquad (2.25)$$

with

$$\check{\rho}_n^{(k)}=\begin{cases}\left[\dfrac{-\nu}{1+\nu}\right]^n\dfrac{n!}{(n+k)!}:L_n^{(k)}\left[\dfrac{a^\dagger a}{\nu}\right]:a^k & \text{for } k\geq 0\\ \left[\dfrac{-\nu}{1+\nu}\right]^n\dfrac{n!}{(n+|k|)!}a^{\dagger|k|}:L_n^{(|k|)}\left[\dfrac{a^\dagger a}{\nu}\right]: & \text{for } k\leq 0.\end{cases} \qquad (2.26)$$

After some algebra, which is reported in the Appendix, one can convince oneself that these $\check{\rho}_n^{(k)}$ are indeed dual to the $\rho_n^{(k)}$ of (2.23),

$$\mathrm{tr}\{\check{\rho}_n^{(k)}\rho_{n'}^{(k')}\}=\delta_{nn'}\delta_{kk'}$$
$$\text{for } n,n'=0,1,2,\ldots,\ k,k'=0,\pm1,\pm2,\ldots\ . \qquad (2.27)$$

The states $\rho_n^{(k)}$ are all traceless, except for $n=0$ and $k=0$:

$$\mathrm{tr}\{\rho_n^{(k)}\}=\delta_{n0}\delta_{k0}\ . \qquad (2.28)$$

Note that the ground state $\rho_0^{(0)}$, which is the steady state of L_a, is the thermal state:

$$\rho_0^{(0)}=\frac{1}{1+\nu}:\exp\left[-\frac{a^\dagger a}{1+\nu}\right]:=\frac{1}{1+\nu}\left[\frac{\nu}{1+\nu}\right]^{a^\dagger a}\ , \qquad (2.29)$$

if we recognize that the number of thermal photons, ν, is related to the temperature T by $\nu/(1+\nu)=\exp(-\hbar\omega/k_BT)$.

Now, any state or observable $\mathcal{O}=\mathcal{O}(a^\dagger,a)$ depending only on the photon variables can be expanded uniquely into the eigenstates of both L_a and $\check{L}_a$:

$$\mathcal{O}(a^\dagger,a)=\sum_{n,k}\check{\alpha}_{nk}\rho_n^{(k)}\ , \qquad (2.30)$$

with

$$\check{\alpha}_{nk}=\mathrm{tr}\{\check{\rho}_n^{(k)}\mathcal{O}\}\ , \qquad (2.31)$$

and similarly

$$\mathcal{O}(a^\dagger,a)=\sum_{n,k}\alpha_{nk}\check{\rho}_n^{(k)}\ , \qquad (2.32)$$

where

$$\alpha_{nk}=\mathrm{tr}\{\mathcal{O}\rho_n^{(k)}\}\ . \qquad (2.33)$$

In particular, $\mathcal{O}$ can be the reduced state of the joint atom-photon system, the state of the electromagnetic field alone:

$$\rho(a^\dagger,a,t)=\mathrm{tr}_\sigma P(a^\dagger,a,\sigma_+,\sigma_-,t)\ , \qquad (2.34)$$

where tr_σ traces over the atom variables only. Equation (2.30) then reads

$$\rho(t)=\sum_{n,k}\check{\alpha}_{nk}(t)\rho_n^{(k)}\ , \qquad (2.35)$$

with the expansion coefficients

$$\check{\alpha}_{nk}(t)=\langle\check{\rho}_n^{(k)}\rangle_t=\left[\frac{-\nu}{1+\nu}\right]^n\frac{n!}{(n+k)!}$$
$$\times\left\langle:L_n^{(k)}\left[\frac{a^\dagger a}{\nu}\right]:a^k\right\rangle_t \qquad (2.36)$$

for $k\geq 0$, and the complex conjugates for $k\leq 0$.

Note that, although the expectation values in (2.36) might look rather involved on first sight, the expansion (2.35) is actually a very natural one, as the first few coefficients already contain the basic statistical information about the field. For example,

$$\langle a^\dagger a\rangle=\nu+(1+\nu)\check{\alpha}_{10} \qquad (2.37a)$$

is the mean number of photons, and

$$\langle a\rangle=\check{a}_{01} \qquad (2.37b)$$

is the mean electric field, and similarly for higher powers of $a,a^\dagger$ and $a^\dagger a$. In contrast to expansions into a number-vector basis where one needs *all* diagonal expansion coefficients to calculate just the mean photon number $\langle a^\dagger a\rangle$, the most essential statistical quantities of the field are provided by a few coefficients only in (2.35).

We close this section with a comment on the special case $\nu=0$. For $\nu=0$, the right eigenstates of L_a acquire the form

$$\rho_n^{(k)}=\begin{cases}a^{\dagger k}(-1)^{a^\dagger a+n}\begin{bmatrix}n+k\\ a^\dagger a+k\end{bmatrix}, & k\geq 0\\ (-1)^{a^\dagger a+n}\begin{bmatrix}n+|k|\\ a^\dagger a+|k|\end{bmatrix}a^{|k|}, & k\leq 0\ ,\end{cases} \qquad (2.38)$$

and the dual states are

$$\check{\rho}_n^{(k)}=\begin{cases}\dfrac{n!}{(n+k)!}\begin{pmatrix}a^\dagger a\\ n\end{pmatrix}a^k=a^{\dagger n}a^{n+k}/(n+k)! & \text{for } k\geq 0\\ a^{\dagger|k|}\begin{pmatrix}a^\dagger a\\ n\end{pmatrix}\dfrac{n!}{(n+|k|)!}=a^{\dagger n+|k|}a^n/(n+|k|)! & \text{for } k\leq 0\ .\end{cases} \tag{2.39}$$

The statements (2.22), (2.25) and (2.27) can be verified easily. The corresponding eigenvalues are, of course, still the same, as they are independent of ν.

In this situation, the coefficients of a corresponding expansion (2.35) of the state ρ of the field have an even more immediate significance, for instance,

$$\begin{aligned}&\langle a^\dagger a\rangle=\check{\alpha}_{10}\ ,\\ &\langle a\rangle=\check{\alpha}_{01}\end{aligned} \tag{2.40}$$

for the mean number of photons and the mean electric field, and

$$\begin{aligned}&\langle (a^\dagger a)^2\rangle-\langle a^\dagger a\rangle^2=2\check{\alpha}_{20}+\check{\alpha}_{10}-\check{\alpha}_{10}^2\ ,\\ &\langle a^\dagger a\rangle-\langle a^\dagger\rangle\langle a\rangle-|\langle a^2\rangle-\langle a\rangle^2|\\ &\qquad=\check{\alpha}_{10}-|\check{\alpha}_{01}|^2-|2\check{\alpha}_{02}-\check{\alpha}_{01}^2|\end{aligned} \tag{2.41}$$

for the variance of $\langle a^\dagger a\rangle$ and the squeezing measure of the field.

III. EXPANSIONS OF THE MASTER EQUATION

In the following we will apply the results of Sec. II and successively expand the state P into eigenstates of L_σ and L_a. The master equation then transforms into groups of matrix equations for the expansion coefficients.

A. Expansion into eigenstates of L_σ

The state P in Eq. (1.1) is a function of the dynamical variables $a^\dagger, a$ and, in particular, σ_+, σ_-. We can therefore employ the expansion (2.11) and write

$$P=\rho\sigma_0+\rho_z\sigma_z+\rho_-\sigma_-+\rho_+\sigma_+\ . \tag{3.1}$$

The coefficients appearing in (3.1) are functions of the photon variables:

$$\begin{aligned}&\rho=\mathrm{tr}_\sigma\{P\}\ ,\\ &\rho_z=\mathrm{tr}_\sigma\{\check{\sigma}_z P\}\ ,\\ &\rho_\pm=\mathrm{tr}_\sigma\{\check{\sigma}_\pm P\}\ .\end{aligned} \tag{3.2}$$

When inserted into the master equation (1.1), (3.1) yields a coupled system of differential equations for the photon functions $\rho, \rho_z, \rho_\pm$:

$$\begin{aligned}&\left(\frac{\partial}{\partial t}-L_a\right)\rho=-i\omega[a^\dagger a,\rho]\\ &\qquad+2ig([a,\rho_-]+[a^\dagger,\rho_+])\ ,\\ &\left(\frac{\partial}{\partial t}-L_a+B\right)\rho_z=-i\omega[a^\dagger a,\rho_z]\\ &\qquad+ig(\{a,\rho_-\}-\{a^\dagger,\rho_+\}\\ &\qquad-(2s-1)[a,\rho_-]\\ &\qquad-(2s-1)[a^\dagger,\rho_+])\ ,\\ &\left(\frac{\partial}{\partial t}-L_a+C\right)\rho_-=-i\omega[a^\dagger a,\rho_-]+i\Omega\rho_-\\ &\qquad+i\frac{g}{4}([a^\dagger,\rho]+2\{a^\dagger,\rho_z\}\\ &\qquad+(2s-1)\{a^\dagger,\rho\})\ ,\\ &\left(\frac{\partial}{\partial t}-L_a+C\right)\rho_+=-i\omega[a^\dagger a,\rho_+]-i\Omega\rho_+\\ &\qquad+i\frac{g}{4}([a,\rho]-2\{a,\rho_z\}\\ &\qquad-(2s-1)\{a,\rho\})\ ,\end{aligned} \tag{3.3}$$

where $\{\ ,\ \}$ denotes the anticommutator and L_a is the operator of (1.5). If one is only interested in the reduced state of the field then ρ_z and $\rho_\pm$ in (3.3) are merely auxiliary functions for the calculation of ρ. This system of equations could now be treated further by using "quasiprobability functions" [3] to get a c-number representation of (3.3).

In this paper we will, however, stick to our strategy and first take care of the nonunitary damping operators.

B. Expansions into eigenstates of L_a

Since the objects of interest ρ, ρ_z, and $\rho_\pm$ are all functions of the photon variables $a^\dagger$ and a we can expand them into eigenstates $\rho_n^{(k)}$ of the photon damping operator L_a. Employing (2.35) for $\rho(t)$ and similar expansions for ρ_z and $\rho_\pm$ we write

$$\begin{aligned}&\rho(t)=\sum_{n,k}\check{\alpha}_{nk}(t)\rho_n^{(k)},\quad \check{\alpha}_{n,-k}=\check{\alpha}_{n,k}^*\ ,\\ &\rho_z(t)=\sum_{n,k}\check{\beta}_{nk}(t)\rho_n^{(k)},\quad \check{\beta}_{n,-k}=\check{\beta}_{n,k}^*\ ,\\ &\rho_-(t)=\sum_{n,k}\check{\gamma}_{n,k-1}(t)\rho_n^{(k)},\quad \check{\gamma}_{n,-k}=\check{\eta}_{n,k}^*\ ,\\ &\rho_+(t)=\sum_{n,k}\check{\eta}_{n,k+1}(t)\rho_n^{(k)},\quad \check{\eta}_{n,-k}=\check{\gamma}_{n,k}^*\ ,\end{aligned} \tag{3.4}$$

where the restrictions on the coefficients are due to the fact that ρ and ρ_z are Hermitian whereas ρ_+ and ρ_- are Hermitian conjugates of each other.

Now we use the following commutation and anticommutation relations:

$$[a^\dagger,\rho_n^{(k)}]=\rho_n^{(k+1)},\quad [a,\rho_n^{(k+1)}]=-\frac{n+1}{\nu+1}\rho_{n+1}^{(k)}\ ,$$
$$\{a^\dagger,\rho_n^{(k)}\}=2(\nu+\tfrac{1}{2})\rho_n^{(k+1)}+2(\nu+1)\rho_{n-1}^{(k+1)},\quad \{a,\rho_n^{(k+1)}\}=2(n+k+1)\rho_n^{(k)}+2(\nu+\tfrac{1}{2})\frac{n+1}{\nu+1}\rho_{n+1}^{(k)} \tag{3.5}$$

for $n,k=0,1,2,\ldots$, and the adjoint relations for $k=-1,-2,\ldots$, and insert the expansions (3.4) into (3.3). After conveniently arranging the coefficients we eventually arrive at a system of differential equations which is equivalent to the master equation (1.1):

$$\frac{d}{dt}\begin{pmatrix}\check\alpha_{n+1,k}\\ \check\beta_{n,k}\\ \frac{1}{i}\check\gamma_{n,k}\\ \frac{1}{i}\check\eta_{n+1,k}\end{pmatrix}=\begin{pmatrix}-A\left(n+\frac{k}{2}+1\right)-i\omega k & 0 & 2g\frac{n+1}{\nu+1} & -2g\\ 0 & -A\left(n+\frac{k}{2}\right)-B-i\omega k & -2g(n+k+1) & 2g(\nu+1)\\ g(\nu+1)\left(s-\frac{1}{2}\right) & g\left(\nu+\frac{1}{2}\right) & -A\left(n+\frac{k}{2}+\frac{1}{2}\right)-C-i\omega k-i\Delta & 0\\ -g(n+k+1)\left(s-\frac{1}{2}\right) & -g\left(\nu+\frac{1}{2}\right)\frac{n+1}{\nu+1} & 0 & -A\left(n+\frac{k}{2}+\frac{1}{2}\right)-C-i\omega k+i\Delta\end{pmatrix}\begin{pmatrix}\check\alpha_{n+1,k}\\ \check\beta_{n,k}\\ \frac{1}{i}\check\gamma_{n,k}\\ \frac{1}{i}\check\eta_{n+1,k}\end{pmatrix}$$
$$+\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&g(\nu+1)&0&0\\0&-g(n+k+1)&0&0\end{pmatrix}\begin{pmatrix}\check\alpha_{n+2,k}\\ \check\beta_{n+1,k}\\ \frac{1}{i}\check\gamma_{n+1,k}\\ \frac{1}{i}\check\eta_{n+2,k}\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&0&-2g(\nu+s)\frac{n}{\nu+1}&2g(\nu+s)\\ \frac{g}{4}[1+(2\nu+1)(2s-1)]&0&0&0\\ -\frac{g}{4}[1+(2\nu+1)(2s-1)]\frac{n+1}{\nu+1}&0&0&0\end{pmatrix}\begin{pmatrix}\check\alpha_{n,k}\\ \check\beta_{n-1,k}\\ \frac{1}{i}\check\gamma_{n-1,k}\\ \frac{1}{i}\check\eta_{n,k}\end{pmatrix}$$

for $n=0,1,2,3,\ldots$, $k=1,2,3,\ldots$; (3.6a)

$$\frac{d}{dt}\begin{pmatrix}\check\alpha_{n+1,0}\\ \check\beta_{n,0}\\ \frac{1}{i}\check\gamma_{n,0}\\ \frac{1}{i}\check\eta_{n,0}\end{pmatrix}=\begin{pmatrix}-A(n+1) & 0 & 2g\frac{n+1}{\nu+1} & -2g\frac{n+1}{\nu+1}\\ 0 & -An-B & -2g(n+1) & 2g(n+1)\\ g(\nu+1)(s-\frac{1}{2}) & g(\nu+\frac{1}{2}) & -A(n+\frac{1}{2})-C-i\Delta & 0\\ -g(\nu+1)(s-\frac{1}{2}) & -g(\nu+\frac{1}{2}) & 0 & -A(n+\frac{1}{2})-C+i\Delta\end{pmatrix}\begin{pmatrix}\alpha_{n+1,0}\\ \check\beta_{n,0}\\ \frac{1}{i}\check\gamma_{n,0}\\ \frac{1}{i}\check\eta_{n,0}\end{pmatrix}$$
$$+\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&g(\nu+1)&0&0\\0&-g(\nu+1)&0&0\end{pmatrix}\begin{pmatrix}\check\alpha_{n+2,0}\\ \check\beta_{n+1,0}\\ \frac{1}{i}\check\gamma_{n+1,0}\\ \frac{1}{i}\check\eta_{n+1,0}\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&0&-2g(\nu+s)\frac{n}{\nu+1}&2g(\nu+s)\frac{n}{\nu+1}\\ \frac{g}{4}[1+(2\nu+1)(2s-1)]&0&0&0\\ -\frac{g}{4}[1+(2\nu+1)(2s-1)]&0&0&0\end{pmatrix}\begin{pmatrix}\check\alpha_{n,0}\\ \check\beta_{n-1,0}\\ \frac{1}{i}\check\gamma_{n-1,0}\\ \frac{1}{i}\check\eta_{n-1,0}\end{pmatrix}$$

for $n=0,1,2,\ldots$, $k=0$; (3.6b)

$$\frac{d}{dt}\begin{pmatrix}\check\alpha_{0,k}\\ \frac{1}{i}\check\eta_{0,k}\end{pmatrix}=\begin{pmatrix}-A\frac{k}{2}-i\omega k & -2g\\ -gk\left(s-\frac{1}{2}\right) & -A\left(\frac{k}{2}-\frac{1}{2}\right)-C-i\omega k+i\Delta\end{pmatrix}\begin{pmatrix}\check\alpha_{0,k}\\ \frac{1}{i}\check\eta_{0,k}\end{pmatrix}-gk\begin{pmatrix}0\\ \check\beta_{0,k}\end{pmatrix}\quad \text{for } n=-1,\ k=1,2,3,\ldots; \tag{3.6c}$$

$$\frac{d}{dt}\check\alpha_{00}=0\quad \text{for } n=-1,\ k=0\ . \tag{3.6d}$$

This is supplemented by the complex conjugate equations. Note that in the system (3.6) coefficients with different indices k do not couple. Therefore we obtain closed groups of equations with respect to the index k. With respect to the index n, every group has the form of a three-term recurrence relation.

For the further discussion we write (3.6) compactly in the form

$$\frac{d}{dt}\check{X}_{nk}=M_n^{(k)}\check{X}_{nk}+G_n^{(k)}\check{X}_{n+1,k}+F_n\check{X}_{n-1,k}\ , \tag{3.7}$$

with $n=-1,0,1,2,\ldots$ and $k=0,\pm1,\pm2,\ldots$, where $\check{X}_{nk}$ denotes a column of four coefficients and $M_n^{(k)}$, $G_n^{(k)}$, and F_n are the corresponding 4×4 matrices involved [8]. The system of differential equations is the "coordinate representation" of the master equation in the damping bases. An eigenvalue of the master equation is then a simultaneous eigenvalue of a group of equations with common index k,

$$\mathcal{L}P=\Gamma P \Longleftrightarrow \Gamma\check{X}_{nk}=M_n^{(k)}\check{X}_{nk}+G_n^{(k)}\check{X}_{n+1,k}+F_n\check{X}_{n-1,k}\ . \tag{3.8}$$

Note that the matrices $G_n^{(k)}$ and F_n couple the dynamics of the nth column $\check{X}_{nk}$ to the columns with neighboring indices n, $\check{X}_{n+1,k}$ and $\check{X}_{n-1,k}$, respectively. Most generally, the recursion in (3.8) can be solved by matrix continued fractions, which is further discussed in [7].

For a complete solution of (1.1) we also have to calculate the left eigenstates of $\mathcal{L}$ which are the right-eigenstates of its dual $\check{\mathcal{L}}$ according to (1.15) as was pointed out in Sec. I B. To do this, we proceed as above, except that we now employ the expansions into the eigenstates of the dual damping operators. That is,

$$\check{P}=\check{\rho}\check{\sigma}_0+\check{\rho}_z\check{\sigma}_z+\check{\rho}_-\check{\sigma}_-+\check{\rho}_+\check{\sigma}_+\ , \tag{3.9}$$

where

$$\check{\rho}=\mathrm{tr}_\sigma\{\check{P}\sigma_0\}\ ,\quad \check{\rho}_z=\mathrm{tr}_\sigma\{\check{P}\sigma_z\}\ ,$$
$$\check{\rho}_\pm=\mathrm{tr}_\sigma\{\check{P}\sigma_\pm\}\ , \tag{3.10}$$

in analogy to (2.13). Similarly, we apply (2.32) to further expand the photon functions (3.10):

$$\check{\rho}=\sum_{n,k}\alpha_{nk}\check{\rho}_n^{(k)}\ ,\quad \check{\rho}_z=\sum_{n,k}\beta_{nk}\check{\rho}_n^{(k)}\ ,$$
$$\check{\rho}_-=\sum_{n,k}\gamma_{n,k-1}\check{\rho}_n^{(k)}\ ,\quad \check{\rho}_+=\sum_{n,k}\eta_{n,k+1}\check{\rho}_n^{(k)}\ , \tag{3.11}$$

as in (3.4). Using the following commutation relations for the dual eigenstates $\check{\rho}_n^{(k)}$:

$$[a^\dagger,\check{\rho}_n^{(k+1)}]=-\check{\rho}_n^{(k)}\ ,$$
$$[a,\check{\rho}_n^{(k)}]=\frac{n}{1+\nu}\check{\rho}_{n-1}^{(k+1)}\ ,$$
$$\{a^\dagger,\check{\rho}_n^{(k+1)}\}=2(\nu+\tfrac{1}{2})\check{\rho}_n^{(k)}+2(\nu+1)\check{\rho}_{n+1}^{(k)}\ , \tag{3.12}$$
$$\{\alpha,\check{\rho}_n^{(k)}\}=2(n+k+1)\check{\rho}_n^{(k+1)}+2(\nu+\tfrac{1}{2})\frac{n}{1+\nu}\check{\rho}_{n-1}^{(k+1)}$$

for $n,k=0,1,2,\ldots$, and the adjoint relations for $k=-1,-2,\ldots$, we finally arrive at a system of equations which is equivalent to the eigenvalue equation (1.15) for the dual Liouville operator:

$$\check{\mathcal{L}}\check{P}=\Gamma\check{P}\Longleftrightarrow \Gamma X_{nk}=X_{nk}M_n^{(k)}+X_{n-1,k}G_{n-1}^{(k)}+X_{n+1,k}F_{n+1}\ , \tag{3.13}$$

with $n=-1,0,1,2,\ldots$ and $k=0,\pm1,\pm2,\ldots$. This is to be compared with the corresponding equation (3.8). Now, the X_{nk} denote *rows,* containing the coefficients appearing in (3.11), namely,

$$X_{nk}=(\alpha_{n+1,k},\beta_{nk},i\gamma_{nk},i\eta_{n+1,k})\quad \text{for } n=0,1,2,\ldots,\ k=1,2,\ldots,$$
$$X_{n0}=(\alpha_{n+1,0},\beta_{n0},i\gamma_{n0},i\eta_{n0})\quad \text{for } n=0,1,2,\ldots,\ k=0\ , \tag{3.14}$$
$$X_{-1,k}=(\alpha_{0k},0,0,i\eta_{0k})\quad \text{for } n=-1,\ k=1,2,\ldots,$$
$$X_{-1,0}=(\alpha_{00},0,0,0)\quad \text{for } n=-1,\ k=0\ ,$$

in contrast to the $\check{X}_{nk}$ in (3.8) which denote *columns,* comprising the coefficients of the expansion (3.4). Note that the matrices M,G,F appearing in (3.6) or (3.7) are the *same* as for the dual coefficients in (3.13).

We can summarize these results compactly by writing

$$X_k=(X_{-1,k},X_{0,k},X_{1,k},\ldots),\quad \check{X}_k=\begin{pmatrix}\check{X}_{-1,k}\\ \check{X}_{0,k}\\ \check{X}_{1,k}\\ \vdots\end{pmatrix}\ ,$$

and

$$\Lambda_k=\begin{pmatrix} M_{-1}^{(k)} & G_{-1}^{(k)} & & \\ F_0 & M_0^{(k)} & G_0^{(k)} & 0 \\ & F_1 & M_1^{(k)} & G_1^{(k)} \\ 0 & \ddots & \ddots & \ddots \end{pmatrix}\ , \tag{3.15}$$

which comprises all the coefficients of (3.4) and (3.11) for a fixed k in a (super) column $\check{X}_k$ and in a row X_k and all the matrices in (3.6) or (3.7) in the single tridiagonal (super) matrix Λ_k.

In this notation the master equation (1.1) and the corresponding eigenvalue equations have a clear coordinate counterpart

$$\frac{\partial}{\partial t}P=\mathcal{L}P\Longleftrightarrow\frac{d}{dt}\check{X}_k=\Lambda_k\check{X}_k,\quad k=0,\pm1,\pm2,\ldots,$$
$$\left.\begin{aligned}\mathcal{L}P_\Gamma&=\Gamma P_\Gamma\\ \check{\mathcal{L}}\check{P}_\Gamma&=\Gamma\check{P}_\Gamma\end{aligned}\right\}\Longleftrightarrow\begin{cases}\Gamma\check{X}_k^{(\Gamma)}=\Lambda_k\check{X}_k^{(\Gamma)}\\ \Gamma X_k^{(\Gamma)}=X_k^{(\Gamma)}\Lambda_k,\quad k=0,\pm1,\pm2,\ldots.\end{cases} \tag{3.16}$$

So the eigenvalues and (right) eigenstates of the Liouville operator $\mathcal{L}$ and its dual conjugate $\check{\mathcal{L}}$ are given by the eigenvalues and eigencolumns and eigenrows, respectively,

of the matrices Λ_k as in (3.15). Please note how naturally the right eigenstates of $\check{\mathcal{L}}$ appear as left eigenstates of $\mathcal{L}$ in (3.16).

The subsequent sections are devoted to the solutions of (3.16).

IV. ANALYTICAL SOLUTIONS

In this section we concentrate on the special situation in which $\nu=0$ and $s=0$. The master equation (1.1) then reads

$$\frac{\partial}{\partial t}P=\frac{1}{i\hbar}[H,P]-\frac{A}{2}(a^\dagger aP+Pa^\dagger a-2aPa^\dagger)$$
$$-\frac{B}{8}\{\sigma_+\sigma_-P+P\sigma_+\sigma_--2\sigma_-P\sigma_+\}$$
$$-\frac{2C-B}{4}\{P-\sigma_zP\sigma_z\}\ , \tag{4.1}$$

with H as in (1.2). It describes a two-level atom with level spacing $\hbar\Omega$ interacting with a photon mode of frequency ω in a cavity with a vanishing number of thermal photons; the nonunitary terms account for photon leakage out of the cavity [9] and incoherent transitions of the atoms. These can be either real transitions from the upper to the lower level or virtual transitions [the last term in (4.1)] that only destroy atomic coherence without actually changing the inversion [cf. Eq. (2.9c)].

As we have seen in Sec. III, the master equation (4.1) is equivalent to the system (3.7), or (3.6) for $\nu=0=s$. For these parameters, however, the matrices F_n that couple the coefficients $\check{X}_{nk}(t)$ in (3.7) to the coefficients $\check{X}_{n-1,k}(t)$ *vanish.*

Let us first consider initial states $P(t=0)$ whose coefficients $\check{X}_{nk}(0)$ vanish for all k except for $k=0$; then they will do so for all times,

$$\check{X}_{nk}(0)=0,\quad k=1,2,3,\ldots\Longrightarrow\check{X}_{nk}(t)\equiv0\ ,$$
$$k=1,2,3,\ldots \tag{4.2}$$

because in (3.7) coefficients with different indices k are decoupled. If additionally $\check{X}_{n0}(t=0)=0$ for $n>N$, which implies that at time $t=0$ there are not more than $N+1$ photons in the cavity, these coefficients remain zero as well, as their dynamics only depends on coefficients with higher n, which all vanish initially:

$$\check{X}_{n0}(0)\equiv\check{X}_n(0)=0,\quad n=N+1,N+2,\ldots\Longrightarrow\check{X}_n(t)\equiv0\ ,$$
$$n=N+1,N+2,\ldots\ . \tag{4.3}$$

In this situation the master equation corresponds to a finite number of coupled differential equations in the form of two-term recursions:

$$\frac{\partial}{\partial t}P=\mathcal{L}P\Longleftrightarrow\begin{cases}\dfrac{d}{dt}\check{X}_n=M_n\check{X}_n+G\check{X}_{n+1},\\ \qquad n=0,1,2,\ldots,\ N-1,\\ \dfrac{d}{dt}\check{X}_N=M_N\check{X}_N\ ,\end{cases} \tag{4.4}$$

where we have temporarily omitted the subscript $k=0$, for notational simplicity. Accordingly, the corresponding eigenvalue problem in this finite-dimensional situation reads

$$\Gamma P=\mathcal{L}P\Longleftrightarrow\begin{cases}\Gamma\check{X}_n=M_n\check{X}_n+G\check{X}_{n+1},\\ \qquad n=0,1,2,\ldots,\ N-1,\\ \Gamma\check{X}_N=M_N\check{X}_N\ .\end{cases} \tag{4.5}$$

Therefore the eigenvalues of the matrix M_N are also eigenvalues of the Liouville operator $\mathcal{L}$. The same reasoning applies to the groups of equations with $k\neq0$, and for arbitrary N.

So we have the result that the eigenvalues of the master equation (4.1) are identical with the eigenvalues of the matrices $M_n^{(k)}$ appearing in (3.6) with $\nu=0,s=0$:

$$\det(M_n^{(k)}-\Gamma)=0\ . \tag{4.6}$$

In general, the solutions of (4.6) are the roots of a fourth-order polynomial in Γ, namely,

$$\Gamma'^4+(2C-B)\Gamma'^3+\left[4g^2\left[n+\frac{k}{2}+1\right]-\left[\frac{A-B}{2}\right]^2+\left[\frac{2C-B}{2}\right]^2+\Delta^2\right]\Gamma'^2$$
$$+(2C-B)\left[4g^2\left[n+\frac{k}{2}+1\right]-\left[\frac{A-B}{2}\right]^2\right]\Gamma'+\left[kg^2-\frac{A-B}{2}i\Delta\right]^2-\left[\frac{2C-B}{2}\right]^2\left[\frac{A-B}{2}\right]^2=0 \tag{4.7a}$$

for $n=0,1,2,\ldots$, $k=0,1,2,\ldots$, with $\Gamma'=\Gamma+A(n+k/2+\frac{1}{2})+B/2+i\omega k$ and, for $n=-1$, the residual equations

$$\left[\Gamma+i\omega k+A\frac{k}{2}\right][\Gamma+i\omega k+A(k-1)/2+C-i\Delta]+kg^2=0\quad\text{for } n=-1,\ k=1,2,\ldots \tag{4.7b}$$

and

$$\Gamma=0\quad\text{for } n=-1,\ k=0\ . \tag{4.7c}$$

The eigenstate to $\Gamma=0$ is, of course, the steady-state solution of (4.1). Equations (4.7b) similarly give the "ground-state values" for the groups of equations with different indices k. For instance, (4.7b) with $k=1$ determines the long-time decay constant for an initial perturbation $\delta\langle a\rangle$ of the electric field in the cavity [10]:

$$\Gamma^{\pm}_{-1,1}=-\frac{A}{4}-\frac{C}{2}-i\omega+i\frac{\Delta}{2}\pm i\left[g^2-\left[\frac{A}{4}-\frac{C}{2}+i\frac{\Delta}{2}\right]^2\right]^{1/2}. \tag{4.8}$$

The quartic equation (4.7a) cannot be brought into a simpler form and we will not bore the reader by giving solutions explicitly. For $C=B/2$, which means a vanishing last term in the master equation (i.e., no dephasing collisions for the atom), (4.7a), however, simplifies to a biquadratic equation. The roots of (4.6) are then explicitly given by the following expressions [11]:

$$\Gamma^{\pm\pm}_{nk}=-A\left[n+\frac{k}{2}+\frac{1}{2}\right]-\frac{B}{2}-i\omega k\pm 2i\Omega^{\pm}_{nk},$$

where

$$\Omega^{\pm}_{nk}=\frac{1}{\sqrt{2}}\left[\left[g^2\left[n+\frac{k}{2}+1\right]-\frac{(A-B)^2}{16}+\frac{\Delta^2}{4}\right]\pm\left\{\left[g^2\left[n+\frac{k}{2}+1\right]-\frac{(A-B)^2}{16}+\frac{\Delta^2}{4}\right]^2-\left[\frac{k}{2}g^2-i\Delta\frac{A-B}{4}\right]^2\right\}^{1/2}\right]^{1/2}$$

$$\text{for } n=0,1,2,\ldots,\ k=0,1,2,\ldots, \tag{4.9a}$$

$$\Gamma^{\pm}_{-1,k}=-A\left[\frac{k}{2}-\frac{1}{4}\right]-\frac{B}{4}-i\omega k+i\frac{\Delta}{2}\pm i\left[kg^2-\left[\frac{A-B}{4}+i\frac{\Delta}{2}\right]^2\right]^{1/2}\quad\text{for } n=-1,\ k=1,2,3,\ldots, \tag{4.9b}$$

and

$$\Gamma_{-1,0}=0\quad\text{for } n=-1,\ k=0\,. \tag{4.9c}$$

Here the signs in the labeling of $\Gamma^{\pm\pm}_{nk}$ refer to their order of appearance on the right-hand side and $\Delta=\omega-\Omega$ denotes the detuning. Together with the complex conjugate values, Eqs. (4.9) give the complete set of eigenvalues for the master equation (4.1) if $C=B/2$.

To get familiar with these expressions let us first look at the special case of vanishing damping constants and detuning, $A,B,\Delta=0$. Then, on the main diagonal ($k=0$) we have

$$\Gamma^{\pm\pm}_{n0}=\pm(1\pm1)i\Omega_n=\begin{cases}0, & \text{twofold}\\ \pm 2i\Omega_n, & \Omega_n=g\sqrt{n+1}\,,\end{cases} \tag{4.10}$$

which are the eigenvalues of the Jaynes-Cummings commutator $(1/i\hbar)[H,\cdot]$ associated with the projector eigenstates $|\gamma^{\pm}_n\rangle\langle\gamma^{\pm}_n|$, where the Hilbert space vectors $|\gamma^{\pm}_n\rangle$ obey the eigenvalue equation

$$H|\gamma^{\pm}_n\rangle=|\gamma^{\pm}_n\rangle\{\hbar\omega(n+\tfrac{1}{2})\pm\hbar\Omega_n\}\,,\quad\text{with } n=0,1,2,\ldots, \tag{4.11}$$

and Ω_n is the Rabi frequency. Similarly, for off-diagonal eigenvalues with $k>0$

$$\Gamma^{\pm\pm}_{nk}=-i\omega k\pm ig(\sqrt{n+k+1}\pm\sqrt{n+1})=-i\omega k\pm i(\Omega_{n+k}\pm\Omega_n)\,, \tag{4.12}$$

which are eigenvalues corresponding to dyadic products $|\gamma^{\pm}_{n+k}\rangle\langle\gamma^{\pm}_n|$. The "ground-state" values in (4.9) reduce to

$$\Gamma^{\pm}_{-1,k}=-i\omega k\pm ig\sqrt{k}\quad\text{for } k=1,2,3,\ldots \tag{4.13}$$

and $\Gamma_{-1,0}=0$, of course, corresponding to products $|\gamma^{\pm}_{k-1}\rangle\langle\gamma_{-1}|$ and $|\gamma_{-1}\rangle\langle\gamma_{-1}|$, respectively, where $|\gamma_{-1}\rangle=|(a^{\dagger}a)'=0,\sigma'_z=-1\rangle$ obeys

$$H|\gamma_{-1}\rangle=|\gamma_{-1}\rangle(-\tfrac{1}{2}\hbar\omega)\,. \tag{4.14}$$

When the damping A,B is switched on, (4.10), for example, changes into

$$\Gamma^{\pm\pm}_{n0}=-A\left[n+\frac{1}{2}\right]-\frac{B}{2}\pm 2i\Omega^{\pm}_n\,,\qquad \Omega^{\pm}_n=\frac{1\pm1}{2}\left[g^2(n+1)-\frac{(A-B)^2}{16}\right]^{1/2}, \tag{4.15}$$

where Ω^{+}_n is the modified Rabi frequency. More generally, for $k=0$, (4.9) involves two frequencies Ω^{+}_n and Ω^{-}_n, one of which is purely imaginary, thus attenuating or enhancing the (real) damping part $-A(n+\frac{1}{2})-B/2$. The second one is positive and real and defines the generalized Rabi frequencies (which are the imaginary parts of the eigenvalues).

As an immediate application consider the time dependence of the mean number of photons $\langle a^{\dagger}a\rangle$ in the cavity, when an atom is present. The general form of that expectation value reads (for simplicity we choose $\omega=\Omega$, $B=0$)

$$\langle a^{\dagger}a\rangle_t=\sum_{n=0}^{\infty}e^{-A(n+1/2)t}\left\{a_n+b_n\cos\left[2\left[g^2(n+1)-\frac{A^2}{16}\right]^{1/2}t\right]+c_n\sin\left[2\left[g^2(n+1)-\frac{A^2}{16}\right]^{1/2}t\right]\right\}, \tag{4.16}$$

where the constants a_n, b_n, c_n are determined by the initial state of the field. For a large initial photon number $\langle a^\dagger a \rangle_0$, the terms oscillating at different frequencies, each being damped differently, add up to produce oscillations around the classical exponential decay $\langle a^\dagger a \rangle_t \sim e^{-At}$, since the atom will now and then absorb and emit one of the photons. As long as their mean number is large the modifications are mere fluctuations of the relative order $1/\langle a^\dagger a \rangle$. After the elapse of a sufficiently long time, however, the only surviving term in (4.16) is the one damped by the factor $e^{-At/2}$:

$$\langle a^\dagger a \rangle_t \cong e^{-At/2}\left\{a_0 + b_0 \cos\left[2\left[g^2 - \frac{A^2}{16}\right]^{1/2} t\right] + c_0 \sin\left[2\left[g^2 - \frac{A^2}{16}\right]^{1/2} t\right]\right\} \tag{4.17}$$

as $t \to \infty$ [13]. Now the presence of the atom significantly slows the decay of the remaining photon which can either be lost irreversibly to the photon reservoir or be reversibly absorbed by the atom. Roughly speaking, the photon spends about half of its time inside the atom which shields it from the reservoir and so effectively doubles the photon lifetime in the cavity. This effect, of course, only occurs when the coupling g is large compared to the damping rate A. In Figs. 1 and 2 we discuss the decay of an initial number state of photons in more detail.

Note also that in the classical limit of a negligibly small coupling to the atom, $g \to 0$, all oscillating terms in (4.16) except for the one with the frequency $\Omega_0 = (g^2 - A^2/16)^{1/2}$ vanish, and since $\Omega_0 \to iA/4$, this remaining term gives an additional damping factor $e^{-At/2}$ which combines with the factor already present to the correct classical exponential decay:

$$\left\{a_m + b_m \cos\left[2\left[g^2(n+1) - \frac{A^2}{16}\right]^{1/2} t\right] + c_m \sin\left[2\left[g^2(n+1) - \frac{A^2}{16}\right]^{1/2} t\right]\right\} \to \text{const} \times \delta_{m0} e^{-At/2} \tag{4.18}$$

so that $\langle a^\dagger a \rangle_t \to \langle a^\dagger a \rangle_0 e^{-At}$, as $g \to 0$.

So far we have been concerned with the eigenvalues of Eq. (4.1). To determine the corresponding eigenstates let us return to (4.5) and restrict the discussion, for notational transparence, to $k = 0$ again. Let Γ be an eigenvalue of the master equation. Then the coordinate representation of the associated eigenstate P_Γ is given by simultaneous eigencolumns of the coupled system of matrix equations on the right-hand side of (4.5). These columns $\check{X}_n$ are iteratively computed from $\check{X}_N$, which is an eigencolumn of the matrix M_N. Thus

$$\begin{aligned} &M_N \check{X}_N = \Gamma \check{X}_N \ , \\ &\check{X}_{N-1} = (\Gamma - M_{N-1})^{-1} G \check{X}_N \ , \\ &\check{X}_n = (\Gamma - M_n)^{-1} G (\Gamma - M_{n+1})^{-1} G \cdots \\ &\qquad \times (\Gamma - M_{N-1})^{-1} G \check{X}_N \ . \end{aligned} \tag{4.19}$$

As the matrices M_n do have different eigenvalues for different n the inverse matrices appearing in (4.19) surely exist. The matrix G has only two nonzero entries, so that one can actually reduce (4.19) to the form

$$\check{X}_n = (\Gamma - M_n)^{-1} \tilde{G}_{n,N-1} \check{X}_N \quad \text{for } n \le N-2 \ , \tag{4.20}$$

where $\tilde{G}_{n,N-1}$ is, apart from a multiplicative factor, identical with G:

$$\check{G}_{n,N-1} = g^{N-n-1} G \prod_{j=n+1}^{N-1} [(\Gamma - M_j)^{-1}_{23} + (\Gamma + M_j)^{-1}_{24}] \ . \tag{4.21}$$

The subdeterminants of $\Gamma - M_j$ can be used to evaluate the 23 and 24 matrix elements of the inverse matrices $(\Gamma - M_j)^{-1}$, and so one can give quite explicit expressions for the eigencolumns for any given eigenvalue [14].

After this discussion of right eigenstates of the Liouville operator $\mathcal{L}$ of (4.1) let us now turn to the left ones. For the left eigenstates the equations corresponding to (4.5) are deduced from (3.13) and read, for $k = 0$,

$$\Gamma \check{P} = \check{\mathcal{L}} \check{P} = \check{P} \mathcal{L} \Longleftrightarrow \begin{cases} \Gamma X_n = X_n M_n + X_{n-1} G \ , \\ \qquad n = N+1, N+2, \ldots \\ \Gamma X_N = X_N M_N \ , \end{cases} \tag{4.22}$$

and similarly for $k \neq 0$. The rows X_n corresponding to the eigenstate $\check{P}_\Gamma$ for a given eigenvalue Γ are again iteratively calculated from X_N,

$$\begin{aligned} &X_N M_N = \Gamma X_N \ , \\ &X_{N+1} = X_N G (\Gamma - M_{N+1})^{-1} \\ &\vdots \\ &X_n = X_N G (\Gamma - M_{N+1})^{-1} \cdots G (\Gamma - M_n)^{-1} \\ &\quad = X_N \tilde{G}_{N,n-1} (\Gamma - M_n)^{-1} \ , \end{aligned} \tag{4.23}$$

where the matrices $\tilde{G}_{N,n-1}$ are defined in (4.21). Note, however, that $n > N$ in (4.23) and the iteration does not terminate as in (4.20). Thus, in contrast to P_Γ, the coordinate representation of $\check{P}_\Gamma$ involves infinitely many columns.

We have thus given the analytical solution of cavity QED as described by the master equation (4.1). We now turn to the important and interesting application of this theory to the one-atom maser [15].

V. THE ONE-ATOM MASER ($\nu = 0, s = 0$)

In the situation of the one-atom maser the losses of the resonant photon field are balanced by excited atoms

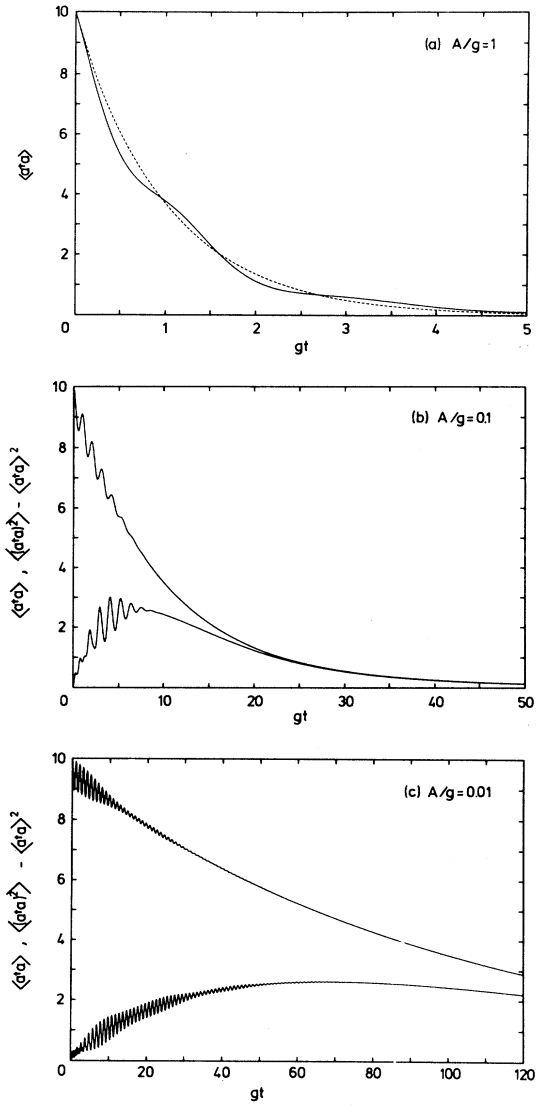

FIG. 1. Decay of the mean number of photons as a function of time for an initial number state with $N=10$ photons. (a) $A/g=1$. There is an equal time scale for the exchange of energy between the atom and the field and for the decay of the field. The dashed line shows the pure exponential decay ($g=0$). (b) $A/g=0.1$. One can see several oscillations which vanish as the damping of the system involves more and more lower frequencies of the system. The lower curve shows the squared variance $\langle (a^\dagger a)^2\rangle - \langle a^\dagger a\rangle^2$. (c) $A/g=0.01$. Here the oscillations are more rapid and exhibit beats. In all three plots (a)–(c) one has $\Delta/g=1$, $B=0$, $C=0$. Note that the oscillations die away once the damping has mixed states with photon numbers less than 10. For large t one can write $\langle a^\dagger a\rangle_t = \sum_{n=0}^{N-1}\sum_{\pm\pm} e^{\Gamma_n^{\pm\pm} t} c_n^{\pm\pm} \sim \sum_{\pm\pm} e^{\Gamma_0^{\pm\pm} t} c_0^{\pm\pm}$ where $\Gamma_0^{\pm\pm}$ are the smallest nonvanishing eigenvalues in (4.9) and $c_n^{\pm\pm}$ are some coefficients [cf. also (4.16) and (4.17) for $\Delta=0$]. Moreover, the coefficients c_0^{++} which multiply the oscillating terms are negligibly small compared to c_0^{+-}. Thus, an initial state with ten photons shows a very different behavior for large times than an initial state with one photon, although in both situations the relevant eigenvalues, $\Gamma_0^{\pm\pm}$, are the same (cf. Fig. 2). Similar curves are given in Refs. [12] and [18].

which pass through the cavity, depositing a photon in it with a certain probability that depends on the state of the field when the atom enters the cavity. Usually the atomic flux is so small that not more than a single atom is in the cavity at a time.

The dynamics of the photon field is described by cycles, see Fig. 3. A cycle begins, say, when an atom enters the cavity. The state of the composite system is then given by

$$P(0)=\rho_\sigma(0)\rho(0)\ , \tag{5.1}$$

where ρ_σ and ρ are the reduced states of the atom and the field, respectively. The atom will then interact for a certain time τ with the field. This is described by the master equation (4.1) or formally by

$$P(\tau)=e^{\mathcal{L}\tau}P(0)\ , \tag{5.2}$$

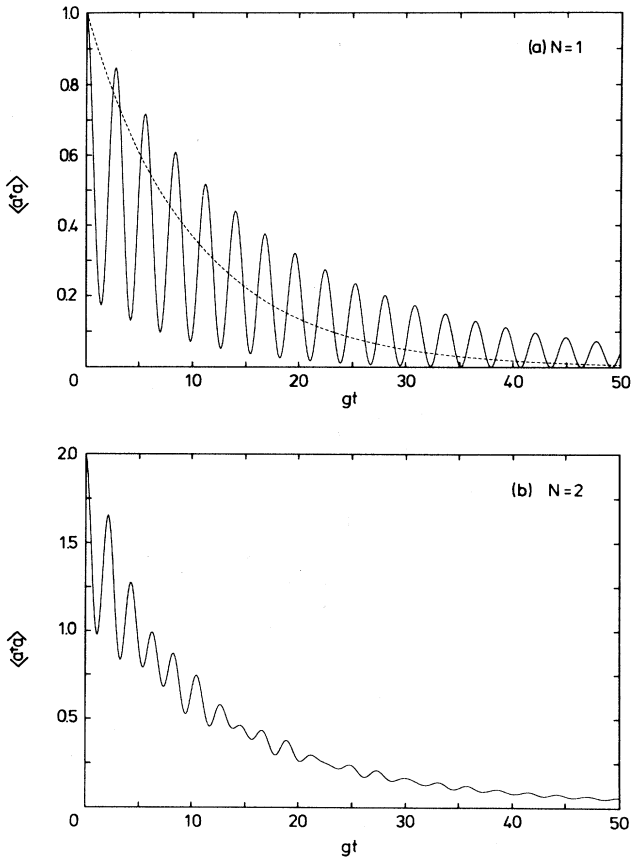

FIG. 2. Decay of an initial number state with one or two photons for $A/g=0.1$, $\Delta/g=1$, $B=0$, $C=0$. (a) $N=1$. The coefficients c_0^{++} and c_0^{+-} in the expansion of $\langle a^\dagger a\rangle$ (cf. Fig. 1) are of the same order of magnitude, so the oscillations are maintained. They are damped by a factor that is smaller than half of the classical exponential decay, which is indicated by the dashed line [13]. (b) $N=2$. The coefficients c_0^{+-} and c_0^{++} are of relative order 1:100, so, compared to (a), only very small residual oscillations survive. For $N=10$, Fig. 1(b) shows how the different frequencies involved interfere to give a smooth curve. Its large-time constant of decay is, however, still less than half of the classical value.

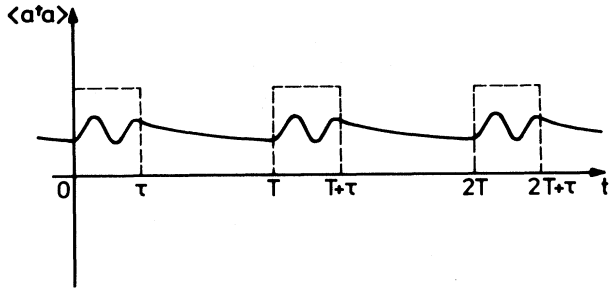

FIG. 3. Cyclically pumped one-atom maser. T is the pumping period, τ the time of interaction between the atom and the field, and $T-\tau$ is the duration of pure decay. To be more realistic we later include a finite probability of excitation for the incoming atoms and a fluctuating interaction time τ owing to a Fizeau-type velocity distribution for the atomic beam.

where $\mathcal{L}$ is the Liouville operator given by the right-hand side of Eq. (4.1). At the time $t=\tau$ the atom leaves the cavity and the reduced state of the photon field is then

$$\rho(\tau)=\mathrm{tr}_\sigma P(\tau) \ . \tag{5.3}$$

Subsequently, the field is subject to pure decay which is described by the master equation (4.1) for $g=0$ (and the trace taken over the atomic variables) or formally by

$$\rho(T)=e^{\mathcal{L}_0(T-\tau)}\rho(\tau) \ , \tag{5.4}$$

where

$$\mathcal{L}_0\rho=-i\omega[a^\dagger a,\rho]+L_a\rho \tag{5.5}$$

and $t=T\geq\tau$ is the time immediately before the next atom enters the cavity. For the following cycle $\rho(T)$ takes on the role of $\rho(0)$ in (5.1), and so on.

So the dynamics of a cycle is split into two parts, which are described by (5.2) and (5.4), respectively. The first part involves the time evolution operator $e^{\mathcal{L}\tau}$, whose precise meaning is given in (1.22) together with the coordinate representations (4.5) and (4.22) for the eigenstates of $\mathcal{L}$ and $\check{\mathcal{L}}$. The second part of the cycle follows (5.4). But this is, in our approach, a trivial special case of the dynamics described by the total $\mathcal{L}$, as the damping-basis states are eigenstates of $\mathcal{L}_0$, too.

Let us now apply the solutions of the eigenvalue problem (4.5) and (4.22) to the one-atom-maser dynamics. Equation (5.2) explicitly reads

$$P(\tau)=\sum_\Gamma e^{\Gamma\tau}P_\Gamma\mathrm{Tr}\{\check{P}_\Gamma P(0)\} \ , \tag{5.6}$$

which is Eq. (1.22). The coordinate representation of the eigenstates P_Γ is

$$\begin{aligned}P_\Gamma&=\rho^{(\Gamma)}\sigma_0+\rho_z^{(\Gamma)}\sigma_z+\rho_-^{(\Gamma)}\sigma_-+\rho_+^{(\Gamma)}\sigma_+\\&=\sum_n(\check{\alpha}_{nk}^{(\Gamma)}\sigma_0+\check{\beta}_{nk}^{(\Gamma)}\sigma_z+\check{\gamma}_{nk-1}^{(\Gamma)}\sigma_-+\check{\eta}_{nk+1}^{(\Gamma)}\sigma_+)\rho_n^{(k)} \ ,\end{aligned} \tag{5.7}$$

where the coefficients $\check{\alpha}_{nk}^{(\Gamma)},\check{\beta}_{nk}^{(\Gamma)},\check{\gamma}_{nk}^{(\Gamma)},\check{\eta}_{nk}^{(\Gamma)}$ make up the columns $\check{X}_{nk}^{(\Gamma)}$ associated with an eigencolumn $\check{X}_{Nk}^{(\Gamma)}$ of the matrix $M_N^{(k)}$, as in Eq. (4.19). Here we explicitly indicate the eigenvalue Γ which also serves as the index of summation in (5.6). Similarly the coordinates of the dual eigenstates are given by

$$\begin{aligned}\check{P}_\Gamma&=\check{\rho}^{(\Gamma)}\check{\sigma}_0+\check{\rho}_z^{(\Gamma)}\check{\sigma}_z+\check{\rho}_-^{(\Gamma)}\check{\sigma}_-+\check{\rho}_+^{(\Gamma)}\check{\sigma}_+\\&=\sum_n(\alpha_{nk}^{(\Gamma)}\check{\sigma}_0+\beta_{nk}^{(\Gamma)}\check{\sigma}_z+\gamma_{nk-1}^{(\Gamma)}\check{\sigma}_-+\eta_{nk+1}^{(\Gamma)}\check{\sigma}_+)\check{\rho}_n^{(k)} \ ,\end{aligned} \tag{5.8}$$

where the coefficients $\alpha_{nk}^{(\Gamma)}$, $\beta_{nk}^{(\Gamma)}$, $\gamma_{nk}^{(\Gamma)}$, $\eta_{nk}^{(\Gamma)}$ make up the rows $X_{nk}^{(\Gamma)}$ associated with an eigenrow $X_{Nk}^{(\Gamma)}$ of the matrix $M_N^{(k)}$. The values of N and k are fixed by the given value of Γ in (5.7) and (5.8).

If we finally write the coordinate representation of the reduced state of the photon field,

$$\rho(t)=\sum_{n,k}\check{\alpha}_{nk}(t)\rho_n^{(k)} \ , \tag{5.9}$$

the first part (5.2) and (5.3) of the one-atom-maser cycle reads

$$\begin{aligned}\rho(\tau)&=\sum_\Gamma e^{\Gamma\tau}\rho^{(\Gamma)}\mathrm{Tr}\{\check{P}_\Gamma P(0)\}\\&=\sum_n\sum_\Gamma e^{\Gamma\tau}\check{\alpha}_{nk}^{(\Gamma)}\mathrm{Tr}\{\check{P}_\Gamma\rho_\sigma(0)\rho(0)\}\rho_n^{(k)} \ .\end{aligned} \tag{5.10}$$

That is,

$$\check{\alpha}_{nk}(\tau)=\sum_\Gamma e^{\Gamma\tau}\check{\alpha}_{nk}^{(\Gamma)}\mathrm{Tr}\{\check{P}_\Gamma\rho_\sigma(0)\rho(0)\} \ , \tag{5.11}$$

which we proceed to express as a mapping of the coefficients $\check{\alpha}_{nk}(0)$ only. Now,

$$\mathrm{Tr}\{\check{P}_\Gamma\rho_\sigma(0)\rho(0)\}=\sum_{n',k'}\check{\alpha}_{n'k'}(0)\mathrm{Tr}\{\check{P}_\Gamma\rho_\sigma(0)\rho_{n'}^{(k')}\} \tag{5.12}$$

and, using (5.8) and the duality relations (2.27) and (2.7),

$$\begin{aligned}\mathrm{Tr}\{\check{P}_\Gamma\rho_\sigma(0)\rho_{n'}^{(k')}\}&=\alpha_{n'k'}^{(\Gamma)}\mathrm{tr}_\sigma\{\rho_\sigma(0)\check{\sigma}_0\}+\beta_{n'k'}^{(\Gamma)}\mathrm{tr}_\sigma\{\rho_\sigma(0)\check{\sigma}_z\}+\gamma_{n'k'-1}^{(\Gamma)}\mathrm{tr}_\sigma\{\rho_\sigma(0)\check{\sigma}_-\}+\eta_{n'k'+1}^{(\Gamma)}\mathrm{tr}_\sigma\{\rho_\sigma(0)\check{\sigma}_+\}\\&\equiv\check{c}_0\alpha_{n'k'}^{(\Gamma)}+\check{c}_z\beta_{n'k'}^{(\Gamma)}+\check{c}_-\gamma_{n'k'-1}^{(\Gamma)}+\check{c}_+\eta_{n'k'+1}^{(\Gamma)}\end{aligned} \tag{5.13}$$

if

$$\rho_\sigma(0)=\check{c}_0\sigma_0+\check{c}_z\sigma_z+\check{c}_-\sigma_-+\check{c}_+\sigma_+ \ . \tag{5.14}$$

In this context, the coefficient $\check{c}_0$ always equals unity, of course. After inserting (5.12)–(5.14) into (5.11) we have

$$\check{\alpha}_{nk}(\tau)=\sum_{\Gamma,n',k'}e^{\Gamma\tau}\check{\alpha}_{nk}^{(\Gamma)}\{\check{c}_0\alpha_{n'k'}^{(\Gamma)}+\check{c}_z\beta_{n'k'}^{(\Gamma)}+\check{c}_-\gamma_{n'k'-1}^{(\Gamma)}+\check{c}_+\eta_{n'k'+1}^{(\Gamma)}\}\check{\alpha}_{n'k'}(0) \ . \tag{5.15}$$

The second part of the cycle, following (5.4), simply produces

$$\check{\alpha}_{nk}(T)=e^{-[i\omega k+A(n+|k|/2)](T-\tau)}\check{\alpha}_{nk}(\tau)\ , \tag{5.16}$$

where $-A(n+|k|/2)$ are the eigenvalues of the damping operator L_a.

So the state of the field at the end of a complete cycle,

$$\rho(T)=\sum_{n,k}\check{\alpha}_{nk}(T)\rho_n^{(k)}\ , \tag{5.17}$$

is given by the mapping [16]

$$\check{\alpha}_{nk}(T)=e^{-[i\omega k+A(n+|k|/2)]T}\sum_{\Gamma,n',k'}e^{[\Gamma+i\omega k+A(n+|k|/2)]\tau}\check{\alpha}_{nk}^{(\Gamma)}\{\check{c}_0\alpha_{n'k'}^{(\Gamma)}+\check{c}_z\beta_{n'k'}^{(\Gamma)}+\check{c}_-\gamma_{n'k'-1}^{(\Gamma)}+\check{c}_+\eta_{n'k'+1}^{(\Gamma)}\}\check{\alpha}_{n',k'}(0)\ . \tag{5.18}$$

All quantities in (5.18) are known, so it provides us with the analytical solution of the dynamics of the field in a one-atom maser including damping mechanisms for the atom and cavity leakage during the passage of the atom through the cavity.

Consider now the special case when all entering atoms are in the upper state,

$$\rho_\sigma(0)=\rho_\sigma(T)=\rho_\sigma(2T)=\cdots=\tfrac{1}{2}(1+\sigma_z)=\sigma_0+\sigma_z\ , \tag{5.19}$$

that is, $\check{c}_0=\check{c}_z=1$, $\check{c}_+=\check{c}_-=0$ in (5.14). In this situation a field that is successively loaded by passing atoms, starting from the vacuum state, remains diagonal. If we set, for the sake of additional notational simplicity, $C=B/2$, (5.18) takes on the form

$$\check{\alpha}_n(T)=e^{-AnT}\sum_{n'}\sum_{m}\sum_{\pm\pm}e^{\{-A(m-n+1/2)-B/2\pm 2i\Omega_m^{\pm}\}\tau}\check{\alpha}_n^{(m,\pm,\pm)}(\alpha_{n'}^{(m,\pm,\pm)}+\beta_{n'}^{(m,\pm,\pm)})\check{\alpha}_{n'}(0)\ , \tag{5.20}$$

with $\Omega_m^{\pm}=\Omega_{m0}^{\pm}$ from (4.9), and the coefficients $\check{\alpha}_n^{(m,\pm\pm)}$, $\alpha_{n'}^{(m,\pm\pm)}$, $\beta_{n'}^{(m,\pm\pm)}$ are the components of the eigencolumns and eigenrows of the matrix $M_m^{(0)}$, associated with the eigenvalues $\Gamma_{m0}^{\pm\pm}$ given in (4.9). Note that, according to (2.40) and (2.41), the mean number of photons and its variance are simply determined by the coefficients $\check{\alpha}_1(T)$ and $\check{\alpha}_2(T)$.

In the cyclically steady state one has $\check{\alpha}_n(T)=\check{\alpha}_n(0)=\check{\alpha}_n^{SS}$ for $n=0,1,2,\ldots$. So these coefficients $\check{\alpha}_n^{SS}$ are mapped onto zero by the matrix with the elements

$$M_{nn'}=e^{AnT}\sum_{m}\sum_{\pm\pm}e^{\{-A(m-n+1/2)-B/2\pm 2i\Omega_m^{\pm}\}\tau}\check{\alpha}_n^{(m,\pm\pm)}\times(\alpha_{n'}^{(m,\pm\pm)}+\beta_{n'}^{(m,\pm\pm)})-\delta_{nn'}\ , \tag{5.21}$$

with $n,n'=0,1,2,\ldots$. Because $\check{\alpha}_{n+1}^{(m,\pm\pm)}=0$ if $n>m$ and similarly $\beta_{n'}^{(m,\pm\pm)}=0$ and $\alpha_{n'+1}^{(m,\pm\pm)}=0$ if $n'<m$, the m summation actually runs only from $n-1$ to n' in (5.21) or (5.20) and the coefficients $\check{\alpha}_n(T)$ depend only on initial coefficients $\check{\alpha}_{n'}(0)$ with $n'\geq n-1$.

So far we have assumed that all atoms are excited and have the same velocity when they enter the cavity. A more realistic description of the one-atom maser must take into account a spread in the velocity of the incoming atoms. This is done by integrating (5.18) over a (narrow) time interval $[\bar{\tau}-\delta\tau/2,\bar{\tau}+\delta\tau/2]$ as the Fizeau-type velocity selectors used in these experiments give approximately equal probabilities for atomic velocities v in a corresponding interval $[\bar{v}-\delta v/2,\bar{v}+\delta v/2]$. This results in replacing the factors

$$e^{[\Gamma+i\omega k+A(n+|k|/2)]\tau} \tag{5.22}$$

in (5.18) by

$$\frac{\sinh\{[\Gamma+i\omega k+A(n+|k|/2)](\delta\tau/2)\}}{[\Gamma+i\omega k+A(n+|k|/2)](\delta\tau/2)}\times e^{[\Gamma+i\omega k+A(n+|k|/2)]\bar{\tau}}\ . \tag{5.23}$$

Further, one must take into account that the incoming atoms are only excited with a certain probability p. If they are excited, the dynamics follows (5.18). If they are *not* excited, the cavity field will be subject to a whole period of pure decay, which happens with the probability $1-p$. The inclusion of this pumping statistical parameter p into the above description modifies (5.18) to

$$\check{\alpha}_{nk}(T)=e^{-[i\omega k+A(n+|k|/2)]T}\times\sum_{n',k'}\left\{p\sum_{\Gamma}e^{[\Gamma+i\omega k+A(n+|k|/2)]\tau}\check{\alpha}_{nk}^{(\Gamma)}(\check{c}_0\alpha_{n'k'}^{(\Gamma)}+\check{c}_z\beta_{n'k'}^{(\Gamma)}+\check{c}_-\gamma_{n'k'-1}^{(\Gamma)}+\check{c}_+\eta_{n'k'+1}^{(\Gamma)})+(1-p)\delta_{nn'}\delta_{kk'}\right\}\check{\alpha}_{n'k'}(0)\ . \tag{5.24}$$

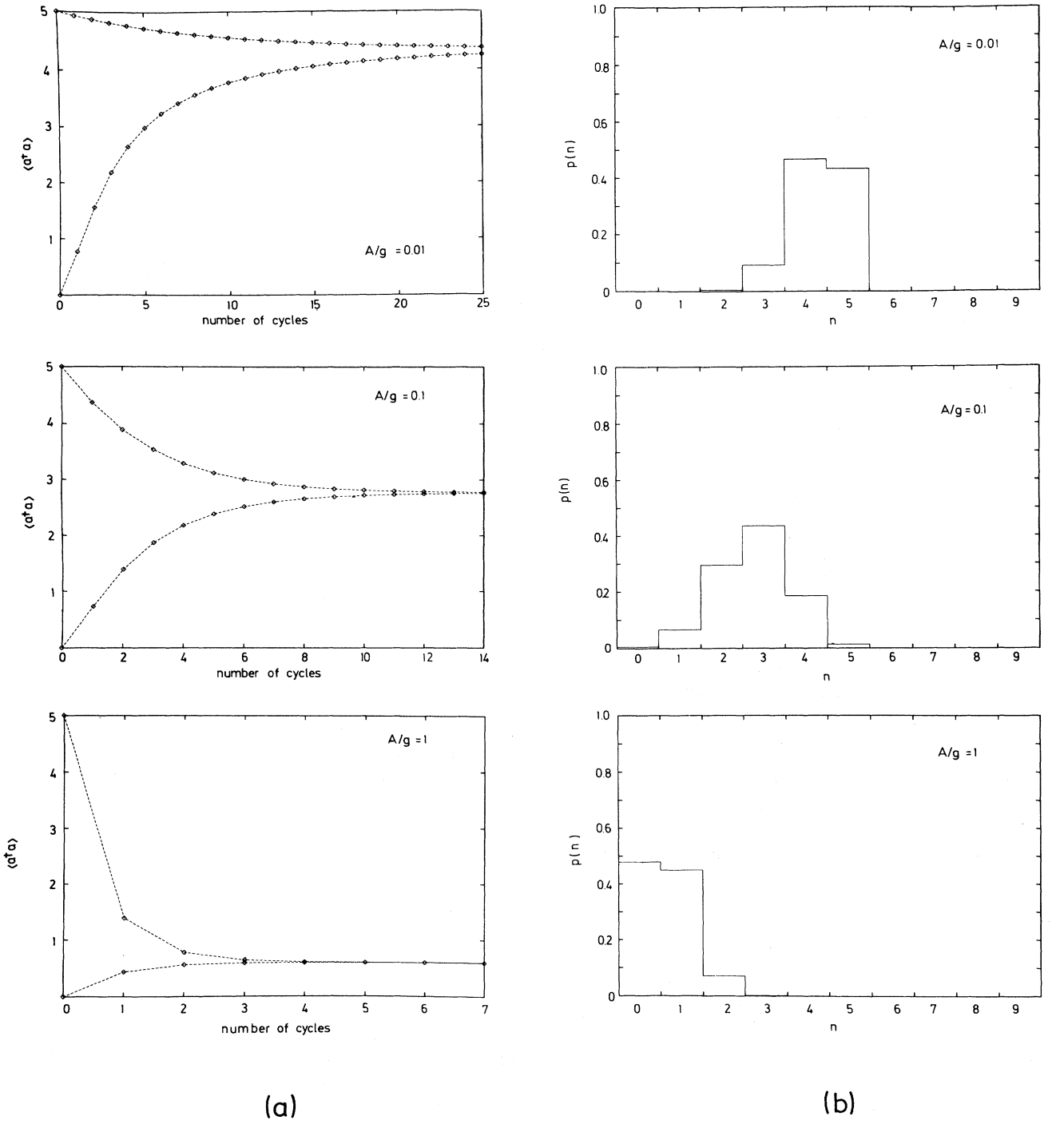

FIG. 4. (a) Stroboscopic evolution of the field in a one-atom maser towards the trapped state (5,1), given by $\Omega_5^+\tau=\pi$, for different values of A. The atoms enter the cavity at equidistant times $t=0,\tau,2\tau,3\tau,\ldots$, which means that there is always exactly one atom in the cavity so that there are no periods of pure relaxation. Both the vacuum and an initial state with five photons evolve towards a (cyclically) steady state with *not more* than five photons. The stroboscopic evolution is described by the quantum map (5.20) with $t=T=\pi/\Omega_5^+$. (b) In the (cyclically) steady state, (5.20) maps the coefficients $\check{\alpha}_n(0)$ onto themselves. Note that, for this trapping value for t the map (5.20) involves only the 6×6 matrix $M_{nn'}$ of (5.21) with n,n' running from 0 to 5. The probability distribution $p(n)$ for the final photon number is plotted for different values of A corresponding to (a). One can see that the final state depends very sensitively on the damping constant and only for very small damping can an approximate number state with $N=5$ photons be obtained.

Together with the replacement (5.22)→(5.23), the mapping (5.24) gives the analytical solution of the dynamics of the photon field in realistic one-atom-maser experiments. It includes detuning, cavity leakage effects, spontaneous decay mechanisms for the atoms, a Fizeau-type velocity distribution for the atomic beam, and a statistical parameter p for the excitation of the incoming atoms covering the limits $p \to 0$ for Poissonian pumping and $p \to 1$ for regular pumping. Incidentally, pumping with atomic coherence is also contained in (5.18) and (5.24) through the coefficients $\check{c}_+$ and $\check{c}_-$ of (5.14).

Numerical results of elementary one-atom-maser calculations are presented in Figs. 4 and 5. More details, including computations of the linewidth and the properties of the steady state with the aid of the quantum map (5.24), are planned to be reported shortly elsewhere.

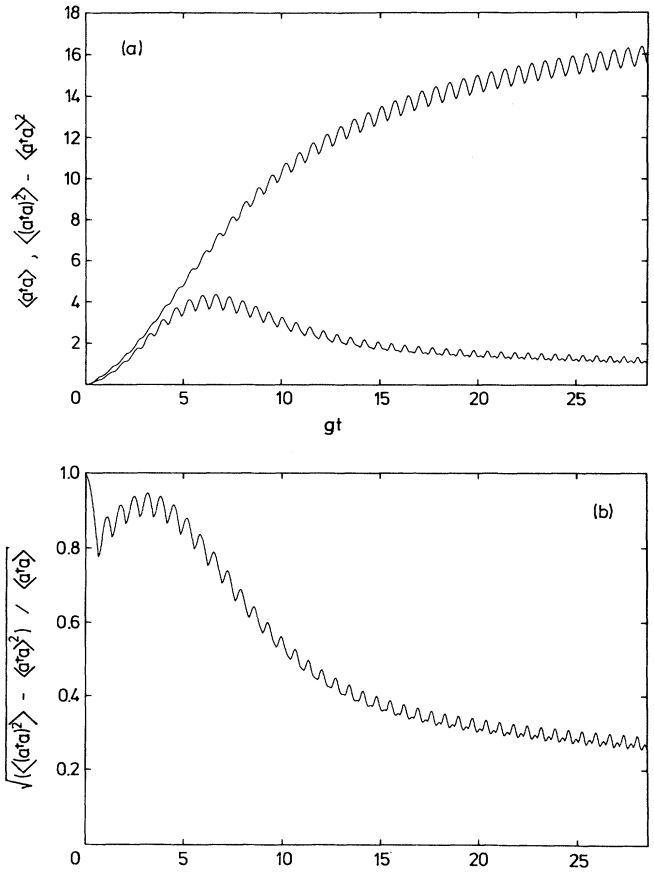

FIG. 5. Explicit evolution of the mean number of photons in a one-atom maser. The initial state is the vacuum and the system evolves towards the trapping state (19,1), which is given by $\Omega_{19}^{+}\tau=\pi$. Note that there are no periods of pure relaxation of the field since $\tau=T$, so an atom enters the cavity immediately after the preceding one has left it. The upper and the lower curves in (a) show the mean and the squared variance of the photon number, respectively. The curve in (b) gives the normalized variance. It shows that the field remains sub-Poissonian all the time while approaching the steady state. The parameters in (a) and (b) are $A/g=0.001$, $D/g=1$, $B=C=0$.

VI. OUTLOOK: THE ONE-ATOM LASER

In the one-atom maser the cavity losses are balanced by excited atoms depositing photons while they pass through the cavity. If there is no atomic beam carrying energy into the resonator, but rather a single atom that stays in permanent interaction with the field, a steady state can only be achieved if the losses of the field are balanced by a permanent incoherent excitation of that atom. This is the standard pumping mechanism. In the master equation (1.1) it is taken into account by the term (1.6) with $s>0$.

In this regime (1.1) describes a "one-atom laser," that is, a single permanently pumped atom in interaction with a field mode in a resonator with losses (mainly owing to the outgoing laser light). Its dynamics is given by Eqs. (3.6a)–(3.6d) or (3.7), respectively, where now $s>0$, so that we really have to deal with the three-term recurrence relations (3.8) and (3.13) with matrices $F_n \neq 0$. The same situation arises at finite temperature ($\nu>0$) of the photon reservoir.

These three-term recursions can be solved numerically by using the method of matrix continued fractions [17]. The results of this analysis are reported in [7].

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APPENDIX

Inasmuch as the presence of the factor $\delta_{kk'}$ in the duality relation (2.27) for the $\rho_n^{(k)}$ of (2.23) and the $\check{\rho}_{n'}^{(k')}$ of (2.26) is immediately obvious, it suffices to derive this statement for $k=k'$. In addition, we can choose the common k value non-negative, because the sign of k is changed by taking the adjoint.

For $k \geq 0$, consider the generating functions

$$\rho^{(k)}(x)=\sum_{n=0}^{\infty} x^n \rho_n^{(k)} = \left[\frac{1}{1+\nu}\frac{1}{1+x}\right]^{k+1} a^{\dagger k}{:}\exp\left[-\frac{a^\dagger a}{(1+\nu)(1+x)}\right]{:} \tag{A1}$$

and

$$\check{\rho}^{(k)}(y)=\sum_{n=0}^{\infty}\frac{(n+k)!}{n!} y^n \check{\rho}_n^{(k)} = \left[\frac{1+\nu}{1+\nu(1+y)}\right]^{k+1} {:}\exp\left[\frac{y a^\dagger a}{1+\nu(1+y)}\right]{:} a^k , \tag{A2}$$

where the summations are performed with the aid of the identity

$$\left[\frac{1}{1+z}\right]^{k+1}\exp\left[\frac{xz}{1+z}\right]=\sum_{n=0}^{\infty}(-z)^n L_n^{(k)}(x)\ . \tag{A3}$$

Two other identities, viz.,

$$:\exp(za^\dagger a):=(1+z)^{a^\dagger a} \tag{A4}$$

and

$$a^{\dagger k}z^{a^\dagger a}a^k=\left[\frac{\partial}{\partial z}\right]^k z^{a^\dagger a}\ , \tag{A5}$$

are used in establishing

$$\rho^{(k)}(x)\check{\rho}^{(k)}(y)=\left[\frac{1}{1+x}\,\frac{1}{1+\nu(1+y)}\right]^{k+1}\times\left[\frac{\partial}{\partial\lambda}\right]^k\lambda^{a^\dagger a}\ , \tag{A6}$$

with

$$\lambda=\left[1-\frac{1}{(1+\nu)(1+x)}\right]\left[1+\frac{y}{1+\nu(1+y)}\right]\ . \tag{A7}$$

As a consequence we find

$$\begin{aligned}\mathrm{tr}\{\rho^{(k)}(x)\check{\rho}^{(k)}(y)\}&=\left[\frac{1}{1+x}\,\frac{1}{1+\nu(1+y)}\right]\times\left[\frac{\partial}{\partial\lambda}\right]^k\frac{1}{1-\lambda}\\&=k!\left[\frac{1}{1+x}\,\frac{1}{1+\nu(1+y)}\,\frac{1}{1-\lambda}\right]^{k+1}\\&=k!\left[\frac{1}{1-xy}\right]^{k+1}\ .\end{aligned} \tag{A8}$$

This implies

$$\sum_{n,n'=0}^{\infty}x^n\frac{(n'+k)!}{n'!}y^{n'}\mathrm{tr}\{\rho_n^{(k)}\check{\rho}_{n'}^{(k)}\}=\sum_{n=0}^{\infty}\frac{(n+k)!}{n!}(xy)^n \tag{A9}$$

or, equivalently,

$$\mathrm{tr}\{\rho_n^{(k)}\check{\rho}_{n'}^{(k)}\}=\delta_{nn'} \tag{A10}$$

and thus confirms the duality relation (2.27). In other words, the normalization factors in (2.23) and (2.26) are well chosen, indeed.

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*Also at the Max-Planck-Institut für Quantenoptik, Ludwig-Prandtl-Straße 10, D-8046 Garching, Germany.

[1] E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).

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[3] H. Risken, in *Progress in Optics,* edited by E. Wolf (North-Holland, Amsterdam, London, 1970), Vol. 8.

[4] Ch. Ginzel, H.-J. Briegel, B.-G. Englert, and A. Schenzle (unpublished).

[5] Experiments with temperatures $T\lesssim 0.1$ K corresponding to a mean number $\nu\lesssim 10^{-5}$ of thermal photons in the cavity are in progress [H. Kosciessa, G. Raithel, *et al.* (private communication)].

[6] This dual conjugate $\check{\mathcal{L}}$ need not be identical with the Hermitian conjugate $\mathcal{L}^*$ whose defining property

$$(P_1,\mathcal{L}P_2)=(\mathcal{L}^*P_1,P_2)$$

(to hold for all states P_1 and P_2) is based upon the inner product

$$(P_1,P_2)=\mathrm{Tr}\{P_1^\dagger P_2\}\ .$$

The two conjugates are symmetrically related by

$$\mathcal{L}^*P=(\check{\mathcal{L}}P^\dagger)^\dagger \quad\text{and}\quad \check{\mathcal{L}}P=(\mathcal{L}^*P^\dagger)^\dagger\ .$$

For all semigroup generators

$$\mathcal{L}P=\frac{1}{i\hbar}[H,P]-\sum_j\left(V_j^\dagger V_jP+PV_j^\dagger V_j-2V_jPV_j^\dagger\right)\ ,$$

however, for which (1.8) is an example, the dual conjugate $\check{\mathcal{L}}$ equals the Hermitian conjugate $\mathcal{L}^*$.

[7] Ch. Ginzel, H.-J. Briegel, U. Martini, B.-G. Englert, and A. Schenzle, Phys. Rev. A (to be published).

[8] In the notation of (3.7) it is understood that $\check{\alpha}_{-1,k},\check{\beta}_{-1,k},\check{\gamma}_{-1,k},\check{\eta}_{-1,k}\equiv 0$ for any value of k. In addition, we apply the conventions

$$M_{-1}^{(0)}=0,\quad G_{-1}^{(0)}=0\ ,$$

which incorporates (3.6d) into (3.6b), and

$$M_{-1}^{(k)}=\begin{bmatrix}-A\frac{k}{2}-i\omega k & 0\ 0 & -2g\\ 0\ 0 & 0\ 0 & 0\ 0\\ -gk\left[s-\frac{1}{2}\right] & 0\ 0 & -A\left[\frac{k}{2}-\frac{1}{2}\right]-C-i\omega k+i\Delta\end{bmatrix},$$

$$G_{-1}^{(k)}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & -gk & 0 & 0\end{bmatrix}$$

for $k=1,2,3,\ldots$, which incorporates (3.6c) into (3.6a). For negative k the complex conjugate equations have to be taken.

[9] In the one-atom maser these leakage effects are mainly owing to the holes in the cavity through which the atoms pass.

[10] According to (4.8) the decay rate of the electric field in a cavity is in particular reduced by the coupling of the field to an atom. This effect has also been discussed by G. S. Agarwal and R. R. Puri, Phys. Rev. A **33**, 1757 (1986) as well as by H. J. Carmichael, R. J. Brecha, M. G. Raizen, H. J. Kimble, and P. R. Rice, *ibid.* **40**, 5516 (1989).

[11] In the special case where $k=0$, and without atomic damping, i.e., for $B=C=0$, the eigenvalues (4.9) have independently been found, using "quasiprobability distributions," by Daeubler, Risken, and Schoendorf [12].

[12] B. Daeubler, H. Risken, and L. Schoendorf, Phys. Rev. A

46, 1654 (1992).

[13] See also the references in [10].

[14] The numerical computation is facilitated by the observation that, for $k=0$, the recurrence relation (4.5) can be turned into one involving the $\check{\alpha}$ coefficients only:

$$\check{\alpha}_n=-4g^2 n\frac{[\Gamma+A(n+1)][\Gamma+A(n-1)+B][\Gamma+A(n-\frac{1}{2})+C]}{\det(\Gamma-M_{n-1})(\Gamma+An+B)}\check{\alpha}_{n+1}\ ,$$

with

$$\begin{aligned}\det(\Gamma-M_n)=&[\Gamma+A(n+1)](\Gamma+An+B)\\&\times\{[\Gamma+A(n+\tfrac{1}{2})+C]^2+\Delta^2\}\\&+4g^2(n+1)\left[\Gamma+A(n+\tfrac{1}{2})+\frac{B}{2}\right]\\&\times[\Gamma+A(n+\tfrac{1}{2})+C]\ .\end{aligned}$$

This is supplemented by the relations

$$\check{\beta}_n=-\frac{\Gamma+A(n+1)}{\Gamma+An+B}\check{\alpha}_{n+1}\ ,$$

$$\check{\gamma}_n=\frac{\Gamma+A(n+1)}{4g(n+1)}\left[i+\frac{\Delta}{\Gamma+A(n+\frac{1}{2})+C}\right]\check{\alpha}_{n+1}\ ,$$

$$\check{\eta}_n=\frac{\Gamma+A(n+1)}{4g(n+1)}\left[-i+\frac{\Delta}{\Gamma+A(n+\frac{1}{2})+C}\right]\check{\alpha}_{n+1}$$

for $\Delta\neq 0$ (and slightly modified in the degenerate case $\Delta=0$) which express the other coefficients in terms of the $\check{\alpha}_n$ thus found. So, for any given eigenvalue $\Gamma=\Gamma_{N,0}^{(\pm\pm)}$, starting from the maximal nonvanishing coefficient $\check{\alpha}_{N+1}^{(\Gamma)}$, which is fixed by normalization, one iteratively obtains all coordinates of the eigenstate P_Γ through these relations. This method is also applicable for $\nu\neq 0$ and $s\neq 0$ as well as to the coordinates of the left eigenstates.

[15] D. Meschede, H. Walther, and N. Klein, Phys. Rev. Lett. **54**, 551 (1985).

[16] As every eigenvalue Γ fixes a corresponding diagonal with index k, the summation over the index k' in (4.41) is already settled by the identities

$$\alpha_{n'k'}^{(\Gamma)}=\delta_{k'k}\alpha_{n'k}^{(\Gamma)},\quad \beta_{n'k'}^{(\Gamma)}=\delta_{k'k}\beta_{n'k}^{(\Gamma)}\ ,$$

$$\gamma_{n'k'-1}^{(\Gamma)}=\delta_{k'k+1}\gamma_{n'k}^{(\Gamma)},\quad \eta_{n'k'+1}^{(\Gamma)}=\delta_{k'k-1}\eta_{n'k}^{(\Gamma)}\ .$$

[17] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1984).

[18] J. Eiselt and H. Risken, Phys. Rev. A **43**, 346 (1991).