Two-parameter family of exact solutions of the nonlinear Schrödinger equation describing optical-soliton propagation

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By using a direct method for obtaining exact solutions of the nonlinear Schrödinger equation that describes the evolution of spatial or temporal optical solitons, a two-parameter family of solutions is given. These exact solutions describe the periodic wave patterns that are generated by the spatial or temporal modulational instability, the periodic evolution of the bright solitons superimposed onto a continuous-wave background, and the breakup of a single pulse into two dark waves which move apart with equal and opposite transverse components of the velocities.

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Temporal or spatial optical solitons have been the subject of much interest in the past few years because of both their scientific and their practical importance [1–17] (for a recent review, see [18]). Potential applications in the field of optical switching devices and high-rate fiber-optic communication links can be easily anticipated. Temporal solitons in optical fibers are pulses which propagate without changing their form (or a change which is at most periodic) as a result of a balance between nonlinearly induced self-phase-modulation and group-velocity dispersion [1–10]. It is well known that three-dimensional propagation of intense laser beams leads to catastrophic breakdown owing to the self-focusing instability, i.e., at high powers, self-focusing overcomes diffraction and the beam collapses [19]. Nevertheless, it is possible to observe a stable equilibrium between diffraction and the nonlinear Kerr effect when the light is propagating in the form of spatial-optical-soliton beams [11–14].

The propagation of both temporal and spatial optical solitons can be described by the nonlinear Schrödinger equation (NLSE)

$$i\frac{\partial\psi}{\partial\xi}+\varepsilon\frac{\partial^2\psi}{\partial\tau^2}+2|\psi|^2\psi=0 , \tag{1}$$

which is written in dimensionless form.

Here, ψ represent a normalized complex amplitude of the optical field, ξ is a normalized longitudinal coordinate, and τ is a normalized retarded time measured in a frame of reference moving along the fiber at the group velocity in the case of temporal optical solitons and a normalized transverse coordinate in the case of spatial optical solitons.

For temporal solitons, $\varepsilon=1$ corresponds to the anomalous dispersion regime where bright solitons can exist [3] and $\varepsilon=-1$ corresponds to the normal dispersion regime where dark solitons occur [9–10]. In the spatial domain, bright or dark solitons occur for self-focusing ($\varepsilon=1$) or self-defocusing ($\varepsilon=-1$) nonlinear media, respectively [11–14].

The NLSE (1) is one of the completely integrable nonlinear partial-differential equations and its solutions may be obtained by different methods, e.g., by using the inverse-scattering technique [2,20–23], the Lie-group theory [24], or the Darboux-transformation method [25]. We mention also the work on the inverse-scattering-transform perturbation theory for soliton propagation and on the extended first- and second-order perturbation expansion for temporal-soliton propagation in optical fibers [26].

Recently, a direct method for obtaining exact solutions of the NLSE for both $\varepsilon=1$ and -1 was given [27,28]. The method is based on the construction of a certain completely integrable finite-dimensional dynamical system whose solutions determine the exact solutions of the NLSE (1). This method comes from the observation that the one-soliton solutions and the periodic solutions which describe the development of the self-phase-modulation instability [29,30] belong to a large class of complex solutions

$$\psi(\xi,\tau)=u(\xi,\tau)+iv(\xi,\tau) ,$$

the so-called first-order solutions of the NLSE (1) for which a linear relationship

$$u(\xi,\tau)-a_0(\xi)v(\xi,\tau)-b_0(\xi)=0 \tag{2}$$

holds between the real part $u(\xi,\tau)$ and the imaginary part $v(\xi,\tau)$ of the complex function $\psi(\xi,\tau)$, where the coefficients a_0 and b_0 depend only on the spatial variable ξ. We note that the two-soliton or, more generally, the n-soliton solution ($n\geq 2$) and the periodic solutions with more than one period in the τ variable do not belong to this set of solutions of the NLSE (1) for which the linear relationship (2) holds.

In the general case, the exact solutions obtained in [27] for $\varepsilon=1$ and in [28] for $\varepsilon=-1$ form a three-parameter family of solutions of the NLSE (1) which are expressed in terms of Jacobi elliptic functions and the incomplete elliptic integral of the third kind. Let $\psi(\xi,\tau)$ be the solution of the NLSE (1) corresponding to the real parameters α_i ($i=1,2,3$). It was shown in [27,28] that at least one of the parameters α_i is positive and in the following we suppose that $\alpha_3>0$. An important scaling holds for the nonlinear partial-differential equation (1). If $\psi'(\xi,\tau)$ is a solution of this equation, then

$$\psi(\xi,\tau)=A\psi'(A^2\xi,A\tau) \tag{3}$$

is also a solution of this equation, where A is an arbitrary scaling factor. We can choose $A=2\alpha_3>0$ and therefore the solution $\psi(\xi,\tau)$ corresponds to the parameters α_i ($i=1,2,3$) and, respectively the solution $\psi'(\xi,\tau)$ corresponds to the parameters $a_1=\alpha_1/2\alpha_3$, $a_2=\alpha_2/2\alpha_3$, and $a_3=\frac{1}{2}$.

By using the method developed in [27,28], we obtain in this paper several particular exact solutions of the NLSE (1) which are very important from physical point of view. These exact solutions form a two-parameter family of solutions of the NLSE (1) and describe the evolution of the modulational instability, the bright solitons on a continuous-wave background (the periodic evolution of the bright-soliton amplitude), and the formation of a diverging pair of dark waves. For simplicity we will write down only the explicit form of $\psi'(\xi,\tau)$ and then, by using the scaling relation (3), it is easy to obtain the general solution $\psi(\xi,\tau)$.

Consider first the case $\varepsilon=1$. For the particular choice $a_1=a_2=a$ and $0\leq a\leq a_3=\frac{1}{2}$ (i.e., two parameters are equal to a positive number a which may take any value between 0 and $\frac{1}{2}$), we obtain the following one-parameter family of solutions $\psi'(\xi,\tau)$ which describe the periodic wave patterns that are generated by the self-phase-modulation instability:

$$\psi'(\xi,\tau)=-\frac{[(1-4a)\cosh(\mu\xi)\mp(2a)^{1/2}\cos(\beta\tau)-i\mu\sinh(\mu\xi)]}{\sqrt{2}[\cosh(\mu\xi)\mp(2a)^{1/2}\cos(\beta\tau)]}e^{i\xi}\ , \tag{4}$$

where

$$\mu=[8a(1-2a)]^{1/2}\ , \tag{5a}$$

$$\beta=[2(1-2a)]^{1/2}\ . \tag{5b}$$

The solutions with different signs in (4) correspond to a shift in the variable τ equal to the semiperiod of the modulation: $\tau\to\tau+\pi/\beta$.

The temporal modulational instability in the nonlinear fiber-optic context [29,30] occurs through the interplay between self-phase-modulation and anomalous group-velocity dispersion and manifests itself as the breakup of a continuous-wave radiation into a train of ultrashort optical pulses. Its spatial analog corresponds to the development of a ring pattern on the transverse intensity profile of a continuous beam in a nonlinear self-focusing medium. In the case of spatial modulational instability, the diffraction takes the role played by the anomalous group-velocity dispersion in the case of temporal modulational instability.

In the particular case $a=\frac{1}{4}$, the solution (4) becomes [27]

$$\psi'(\xi,\tau)=-\frac{(\cos\tau\pm i\sqrt{2}\sinh\xi)}{\sqrt{2}(\sqrt{2}\cosh\xi\mp\cos\tau)}e^{i\xi}\ . \tag{6}$$

If we write $\psi'(\xi,\tau)=w(\xi,\tau)e^{i\xi}$, then from (6) we obtain

$$w(\xi,\tau)\to-(1/\sqrt{2})e^{\pm i\pi/2}$$

as $\xi\to\pm\infty$. Thus in the process of evolution from $\xi=-\infty$ to $\xi=+\infty$, we observe the phenomenon of the return to the initial amplitude $1/\sqrt{2}$ but the phase is reversed ($\Delta\varphi=\pi$).

In Figs. 1–3 we show the periodic wave patterns that are generated by the self-phase-modulational instability for $a=0.125$, 0.25, and 0.375.

For the particular choice $a=a_1\leq a_2=a_3=\frac{1}{2}$, we find the following one-parameter family of solutions with finite boundary conditions at $\tau\to\pm\infty$:

$$\psi'(\xi,\tau)=\frac{[2(1-a)\cos(\rho\xi)\pm(2a)^{1/2}\cosh(\beta\tau)+i\rho\sin(\rho\xi)]}{[-2a^{1/2}\cos(\rho\xi)\mp2^{1/2}\cosh(\beta\tau)]}e^{2ia\xi}\ , \tag{7}$$

where

$$\rho=2(1-2a)^{1/2} \tag{8}$$

and β is given by Eq. (5b).

We see from (7) that

$$\psi'(\xi,\tau)\to-a^{1/2}e^{2ia\xi}$$

as $|\tau|\to\infty$, i.e., for $|\tau|\gg1$, this wave form approaches a continuous wave with amplitude $a^{1/2}$. The solutions with different signs in (7) correspond to a shift in the variable ξ equal to the semiperiod of the modulation: $\xi\to\xi+\pi/\rho$. We note that the soliton solution (7) was first obtained we believe in [31] by using the inverse-scattering technique for finite boundary conditions at $\tau=\pm\infty$ and then in [32]

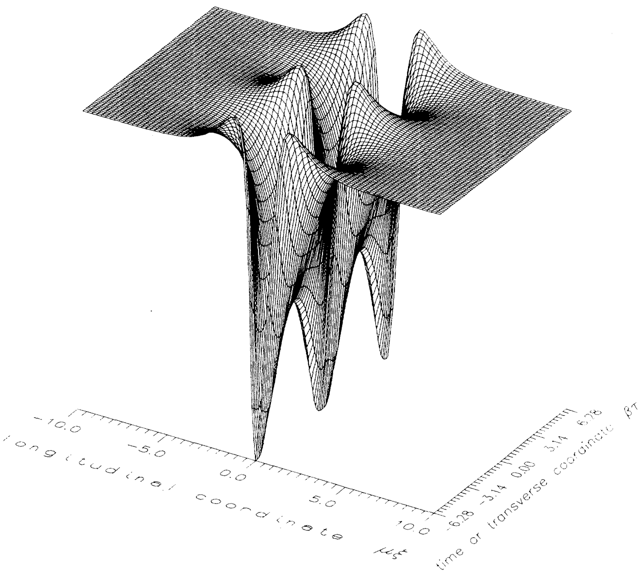

FIG. 1. Intensity profile $|\psi'|^2$ of the modulational instability patterns vs normalized longitudinal coordinate $\mu\xi$ and normalized time or transverse coordinate $\beta\tau$ for $a=0.125$.

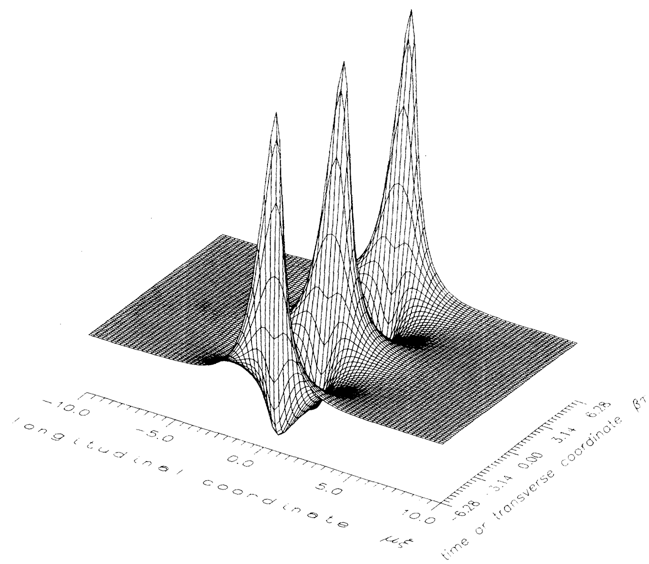

FIG. 2. Same as Fig. 1, but for $a=0.25$.

by means of direct methods. For $0<a<\frac{1}{2}$, the solution (7) describes the bright solitons superimposed onto a continuous-wave background. The soliton amplitude evolves periodically along the longitudinal direction with a period $(1-2a)^{-1/2}$ [33]. In Figs. 4–6 we show the evolution of a bright soliton on a continuous-wave background for $a=0.125$, 0.25, and 0.375.

In the particular case $a_1=a_2=a_3=a=\frac{1}{2}$, we obtain from (7) the rational (algebraic) bright-soliton solution (see Fig. 7)

$$\psi'(\xi,\tau)=\sqrt{2}\frac{[(\frac{3}{2}-2\xi^2-\tau^2)+4i\xi]}{(1+4\xi^2+2\tau^2)}e^{i\xi} . \tag{9}$$

The canonical bright one-soliton solution of the NLSE (1) is obtained from (7) in the limit $a\to 0$:

$$\psi'(\xi,\tau)=\mp\sqrt{2}\,\mathrm{sech}(\sqrt{2}\tau)e^{2i\xi} . \tag{10}$$

Next we consider the case $\varepsilon=-1$. For the particular choice $a_1=a_2=a$ and $0\le a\le a_3=\frac{1}{2}$, we find the following one-parameter family of dark waves with finite boundary conditions at $\tau\to\pm\infty$:

$$\psi'(\xi,\tau)=-\frac{[(1-4a)\cosh(\mu\xi)\mp(2a)^{1/2}\cosh(\beta\tau)-i\mu\sinh(\mu\xi)]}{\sqrt{2}[\cosh(\mu\xi)\mp(2a)^{1/2}\cosh(\beta\tau)]}e^{i\xi} , \tag{11}$$

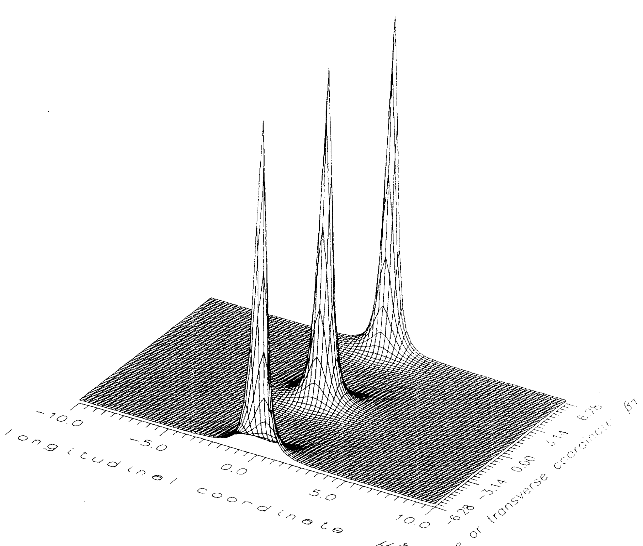

FIG. 3. Same as Fig. 1, but for $a=0.375$.

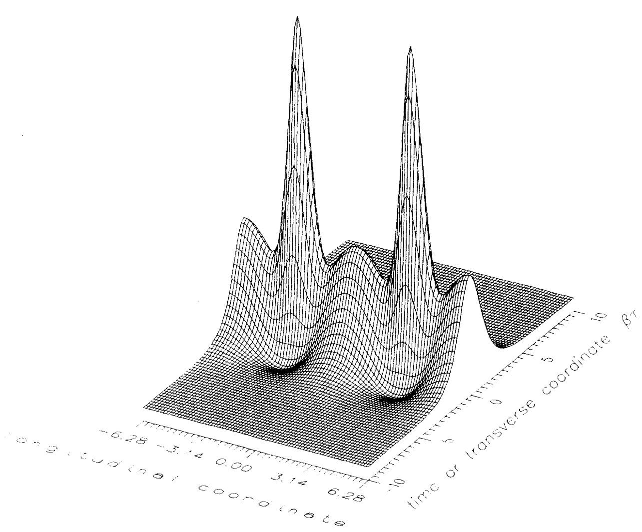

FIG. 4. Intensity profile $|\psi'|^2$ of the soliton on a continuous-wave background vs normalized longitudinal coordinate $\rho\xi$ and normalized time or transverse coordinate $\beta\tau$ for $a=0.125$.

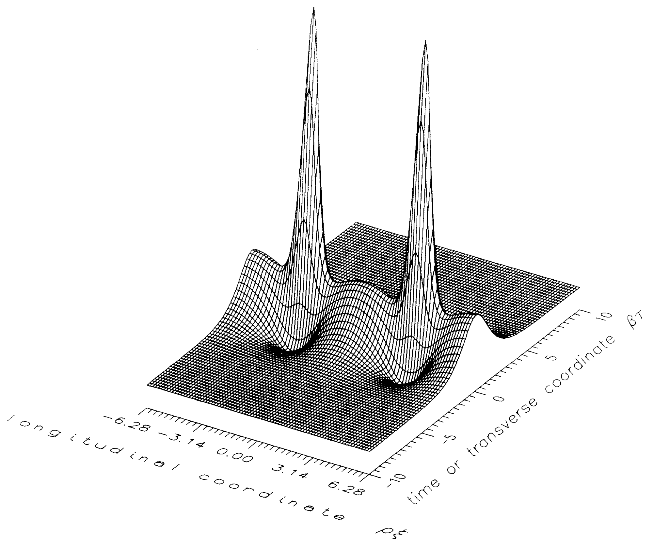

FIG. 5. Same as Fig. 4, but for $a = 0.25$.

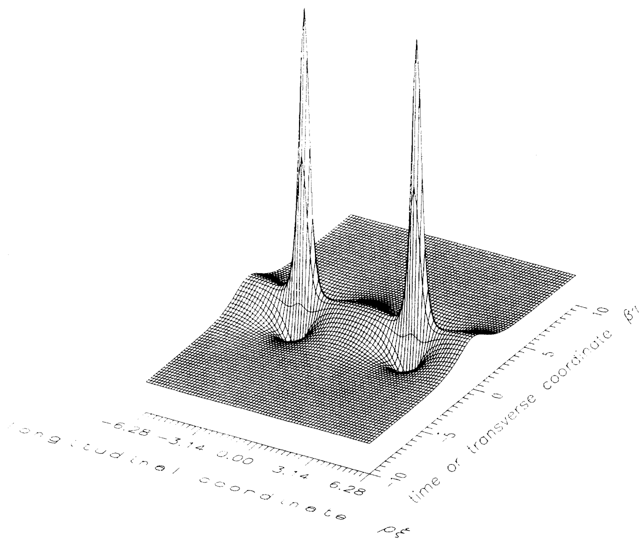

FIG. 6. Same as Fig. 4, but for $a = 0.375$.

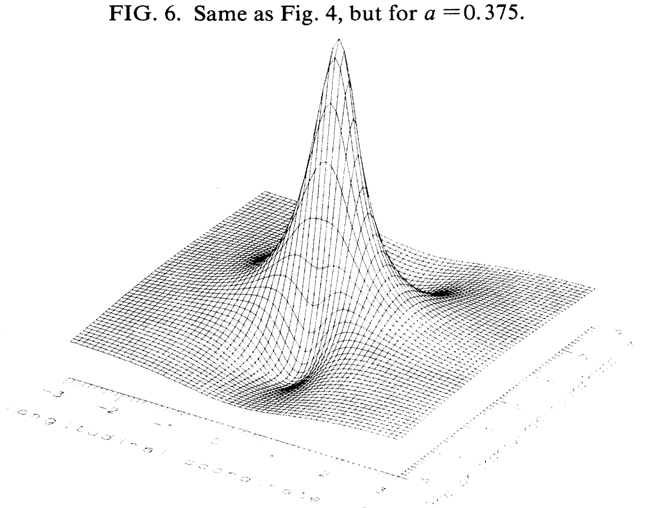

FIG. 7. Intensity profile $|\psi'|^2$ of the algebraic soliton vs normalized longitudinal coordinate ξ and normalized time or transverse coordinate τ.

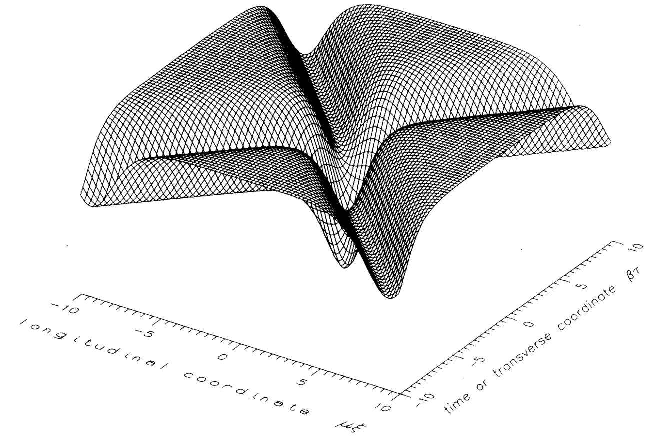

FIG. 8. Intensity profile $|\psi'|^2$ of the dark wave vs normalized longitudinal coordinate $\mu\xi$ and normalized time or transverse coordinate $\beta\tau$ for $a = 0.125$

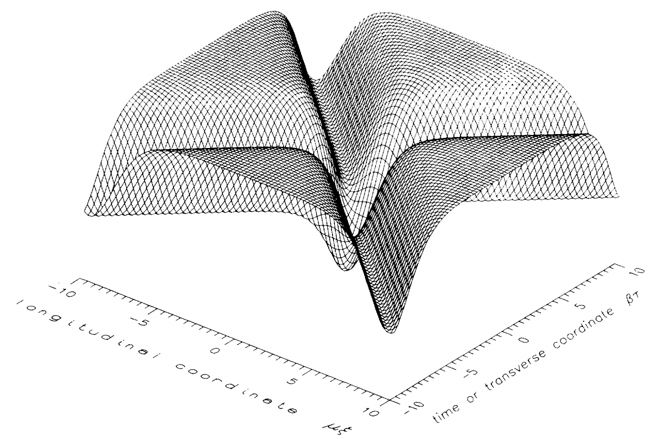

FIG. 9. Same as Fig. 8, but for $a = 0.25$.

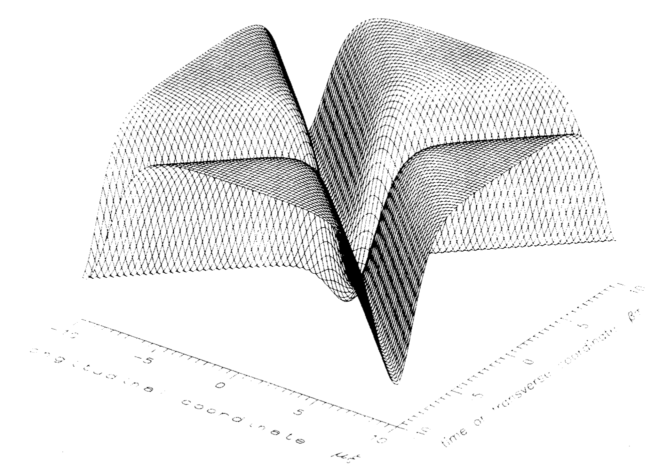

FIG. 10. Same as Fig. 8, but for $a = 0.375$.

where μ and β are given by Eqs. (5a) and (5b), respectively. Note that the solution corresponding to the upper signs in (11) is singular and the solution corresponding to the lower signs in (11) is regular. The dark optical solitons can be created without a threshold, i.e., by an infinitely small driving pulse, as opposed to the process of generation of bright optical solitons which can be created from a localized pulse if the area under its envelope is more than a certain threshold value. The inverse-scattering transform predicts that the dark solitons can be created in pairs [2,5,34]. We note that in [5] the initial-value problem for Eq. (1) was solved numerically with $\varepsilon=-1$. For initially symmetric data of the form

$$\psi(0,\tau)=[1+A\,\mathrm{sech}(\beta\tau)]^{-1}\ ,$$

it was shown numerically that this single pulse breaks up into two dark solitons which move apart with equal and opposite transverse components of the velocities, as predicted by the inverse-scattering theory. Recent experiments clearly demonstrate the formation of two counterpropagating nonfundamental dark spatial solitons (gray solitons) in bulk self-defocusing media such as ZnSe [13] or thermally defocusing liquids [17].

In the particular case $a=\frac{1}{4}$, the regular solution (11) becomes

$$\psi'(\xi,\tau)=-\frac{(\cosh\tau-i\sqrt{2}\sinh\xi)}{(\sqrt{2}\cosh\xi+\cosh\tau)}e^{i\xi}\ . \tag{12}$$

If we write $\psi'(\xi,\tau)=w(\xi,\tau)e^{i\xi}$, we thus have

$$w(\xi,\tau)\to-(1/\sqrt{2})e^{\mp i\pi/2}$$

as $\xi\to\pm\infty$; therefore, in the process of evolution from $\xi=-\infty$ to $\xi=+\infty$, the amplitude $1/\sqrt{2}$ is recovered but the phase is reversed ($\Delta\varphi=\pi$).

Figures 8–10 show the evolution of the intensity profile for the solutions (11) and (12) for $a=0.125$, 0.25, and 0.375. In addition, we illustrate in Figs. 11–13 the splitting of an input ($\xi=0$) single dark pulse into two dark waves of equal amplitudes and opposite transverse velocities for the same values of the parameter a.

We note that the parameter a is related to the contrast of the solitons, which is defined in photometry as

$$C=(I_{\max}-I_{\min})/(I_{\max}+I_{\min})$$

and gives the visibility of the solitons. In the origin ($\xi=0$), the contrast is given by the expression

$$C=\frac{4a[1+(2a)^{1/2}-2a]}{(2a)^{1/2}+4a+(1-4a)[1+(2a)^{1/2}-2a]} \tag{13}$$

and for the separated pulses by the simple expression

$$C=\frac{a}{1-a}\ . \tag{14}$$

For the particular choice $a=a_1\le a_2=a_3=\frac{1}{2}$, we find the following singular solution (double periodic in the variable ξ and periodic in the variable τ):

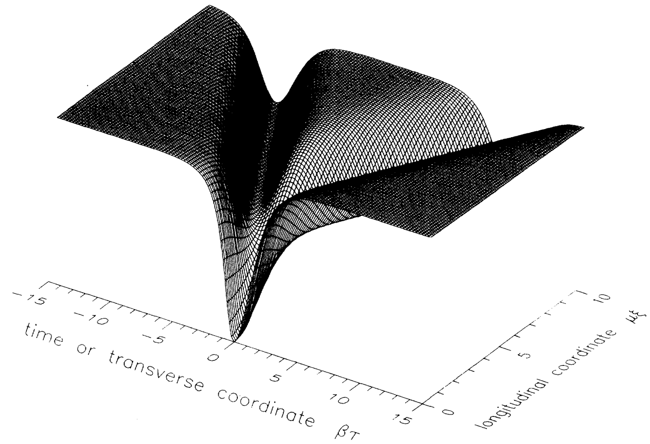

FIG. 11. The splitting of a single dark pulse input in two dark waves for $a=0.125$.

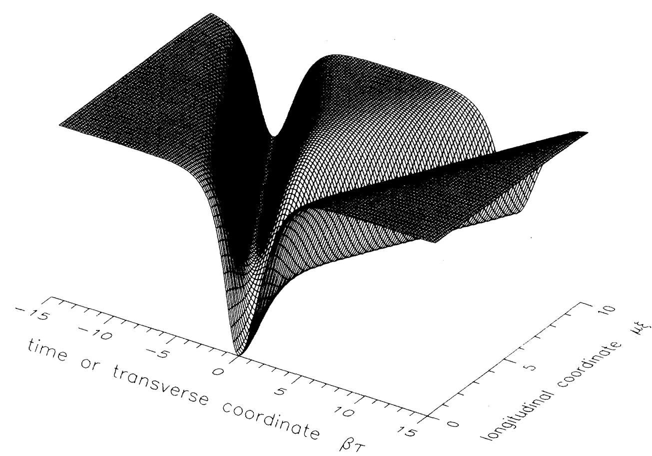

FIG. 12. Same as Fig. 11, but for $a=0.25$.

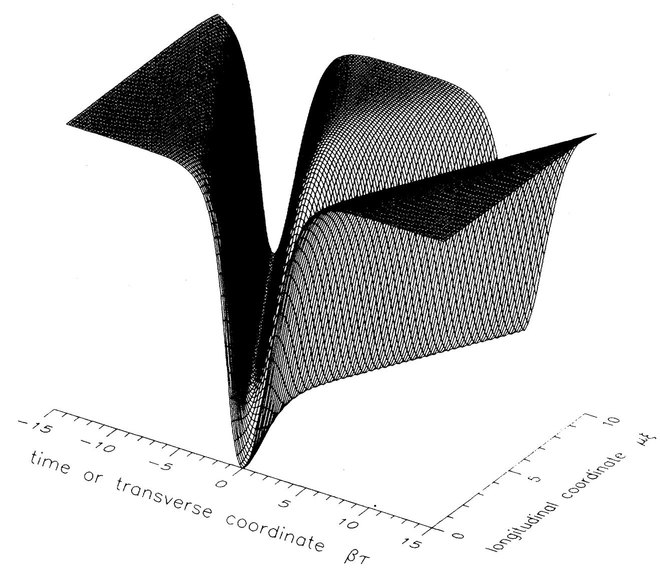

FIG. 13. Same as Fig. 11, but for $a=0.375$.

$$\psi'(\xi,\tau)=\frac{[2(1-a)\cos(\rho\xi)\pm(2a)^{1/2}\cos(\beta\tau)+i\rho\sin(\rho\xi)]}{[-2a^{1/2}\cos(\rho\xi)\mp 2^{1/2}\cos(\beta\tau)]}e^{2ia\xi}, \tag{15}$$

where β and ρ are given by Eqs. (5b) and (8), respectively.

In the particular case $a=a_1=a_2=a_3=\frac{1}{2}$, we obtain from (15) the singular rational (algebraic) dark-soliton solution:

$$\psi'(\xi,\tau)=\sqrt{2}\frac{[(\frac{3}{2}-2\xi^2+\tau^2)+4i\xi]}{(1+4\xi^2-2\tau^2)}e^{i\xi}. \tag{16}$$

In conclusion, the direct method developed recently in [27,28] allows us to obtain a two-parameter family of exact solutions of the NLSE describing the propagation of temporal or spatial optical solitons. This class of exact solutions contains the rational (algebraic) bright soliton, the solution which describes the superposition of a bright soliton on a continuous-wave background, the solution corresponding to the periodic wave patterns generated by the modulation instability, and the solution which shows the splitting of an even dark pulse into a pair of shallow grey pulses.

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