Interaction of radiation with matter: Integrable problems

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We construct an extension of the spectral transform theory that allows us to build nonlinear systems of coupled waves that are integrable for arbitrary boundary values. The related time evolution of the spectral transform is in general nonlinear. This result has many important applications in physics, and we apply the procedure to plasma waves (laser-plasma interaction), to quantum electronics [self-induced transparency (SIT) and laser-pulse amplification], and to nonlinear optics (stimulated Raman scattering). In the case of laser-plasma interaction, we obtain an exact model for the description of the total reflexivity due to stimulated Brillouin scattering. For SIT, we show that the presence in the initial state of the medium of *some* atoms in an excited state drastically modifies the related time evolution of the spectral data, possibly making the problem unsolvable. In the case of laser-pulse amplification, we prove that the presence of background noise in the firing laser pulse drastically modifies the long-distance behavior of the solitons. Finally, for the general process of stimulated Raman scattering, we give the correct evolution of the spectral transform and show that the Stokes wave becomes rapidly totally localized; in other words, the system naturally evolves into a pure soliton state, whatever may be the initial profile of the acoustic wave. In the same context, another quite interesting example is studied: it is the first instance of an integrable system which develops a singularity in a *finite time* (for which the solution blows up). The physical application is under study.

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I. INTRODUCTION

The spectral transform method [1] has won physicists' consideration because the nonlinear evolutions that can be handled are of universal application. This is the case, for instance, with the nonlinear Schrödinger and Korteveg–de Vries equations. For such systems, the spectral transform actually gives the solution of the Cauchy problem, which is the *initial-value problem*, and this is relevant when the physical problem is that of the time evolution of an initial profile or disturbance. For instance, it has been demonstrated that a localized initial profile eventually becomes a set of well-separated *solitons* traveling on a vanishing background, and this is indeed a fundamental result.

But when the physical problem is that of the interaction of radiation with matter, it is clear that some boundary values will come into play (at least the input radiation value). Then, in that context the problem to consider is a nonlinear *boundary-value problem* for coupled waves and, up to now, the spectral transform method was not applicable.

The main mathematical property of systems of coupled waves is to have a *singular dispersion relation* [$\omega(k)$ is a nonanalytic function of the complex variable k], and Lamb, Jr. discovered that some of such systems are integrable by an extension of the spectral transform theory [2]. His work on the equations of self-induced transparency has then been generalized in [3] and set up on a general basis in [4]. We have developed a systematic approach of integrable systems of coupled waves [5] by use of the $\bar{\partial}$ formulation of the spectral transform and have applied it to different situations [6–8] for which the relevant problem was an initial-value problem.

We shall prove here that the method can be extended to a more general evolution of the spectral transform, which will allow us to solve boundary-value problems for those integrable systems having a singular dispersion relation. Depending on the problem, the time evolution of the spectral transform can well be nonlinear, but still explicitly solvable in the physically interesting cases.

Our result can be summarized as follows: the system of coupled equations for the fields $q(x,t), a_j(k,x,t)$

$$q_t=\int_{-\infty}^{+\infty} g\, dk\, a_1\bar{a}_2 \ , \tag{1.1a}$$

$$a_{1,x}=qa_2, \quad a_{2,x}-2ika_2=\sigma\bar{q}a_1 \tag{1.1b}$$

[with $\sigma=\pm, x\in\mathbb{R}, t>0$, and $g=g(k,t)$ an arbitrary function in L^2], is integrable for arbitrary boundary values, say

$$a_1 \xrightarrow[x\to+\infty]{} I_1(k,t) \ , \tag{1.2a}$$

$$a_2 \underset{x\to+\infty}{\sim} I_2(k,t)e^{2ikx} \ , \tag{1.2b}$$

or any other choice (at $-\infty$ or mixed $+\infty-\infty$). Starting with $q(x,0)=0$, we refer to this problem as a boundary-value problem and with $q(x,0)$ in $L^1(\mathbb{R})$ to an initial–boundary-value problem. Both cases are physically interesting and we will discuss important consequences.

The above system has to be understood as a paradigm model describing the interaction of two high-frequency waves of envelopes a_1 and a_2, with a low-frequency wave of envelope q. The parameter k measures a frequency mismatch due to the presence of a broad-line resonance between these waves, represented by the *inhomogeneous broadening* factor $g(k)$.

The method can be straightforwardly extended to the system made of (1.1b) with

$$q_t = i\alpha(\tfrac{1}{2}q_{xx} - \sigma|q|^2 q) + \int_{-\infty}^{+\infty} g\, dk\, a_1 a_2 \ , \tag{1.3}$$

but, to avoid formal complications we shall restrict ourselves here to the case (1.1). The system (1.3) has important physical applications [9,7] for it describes the wave coupling including nonlinearity and dispersion of the low-frequency wave q.

The method obviously applies to other spectral problems (we use here the Zakharov-Shabat problem). For instance, starting with the Schrödinger equation, one would solve the Karpman-Kaup equation [10] for which it has been recently shown that, for a physical initial boundary-value problem, the asymptotic state consists of a number of static solitons and a localized totally reflective radiative part [11].

In Sec. II we construct the general scheme of the method and we will recall the minimal basic tools such as to make the method understandable with no particular knowledge of the general theory of the spectral transform. This scheme is then applied to solve a generic boundary-value problem and we obtain in Sec. III the related general time evolution of the spectral transform. We will discuss the boundary-value case with, in particular, the description of the soliton driving by the effect of the boundaries. In Sec. IV we describe four different physical situations where Eq. (1.1) applies but with different boundary values and hence with different spectral transform solutions.

The reader interested only in the application of the method to physical problems can jump directly to Sec. IV. In that section we consider first (Sec. IV A) the case of the interaction of a laser with a two-component plasma (in the fluid approximation) [8]. We obtain an exact model to explain the total reflexivity due to the stimulated Brillouin scattering of the electromagnetic wave with the acoustic wave.

Then (Sec. IV B) we recover the results of self-induced transparency [12,2,3], when the physical situation is that of the interaction of an electromagnetic radiation (laser pulse) with a two-level system of atoms or molecules initially at rest. We will find that the system is highly sensitive to the boundary values: if the initial state contains some atoms in the excited state, then the related evolution of the spectral transform in dramatically different, possibly becoming unsolvable.

In the case when the atoms are initially in the excited state (laser-pulse amplification) [13,14], we prove in Sec. IV C that the existence of a radiative part in the spectrum (as small as we want) dramatically modifies the asymptotic state of the system (far in the medium). Since physically the input pulse can never have a strictly vanishing radiative part, we obtain a generic behavior for the penetration of a laser pulse in an excited medium. In particular, we will discover that the positive velocity of the solitons is compatible only with a nonzero initial radiative part.

In Sec. IV D the general equations of stimulated Raman scattering are considered [15,16] and we obtain the correct time evolution of the spectral data. As a result, we prove that the Stokes wave becomes rapidly *localized* in space and hence that any initial acoustic profile asymptotically evolves into a pure soliton state.

Finally we consider in Sec. IV E a striking example: we discover the first instance of a system which is integrable only for finite time. For such a system, we show that it exists a time t_s for which the energy of the low-frequency wave becomes infinite corresponding to the occurrence of a singularity (in time) of the spectral transform. For the moment, this is only a mathematical curiosity but the physical applications are now under study. Of course, the problem of the nature of the system for $t > t_s$ is a quite interesting open question.

Let us first define some convenient notations. For a complex-valued function $f(k) = a(k) + ib(k)$ of the complex variable $k = \xi + i\eta$, we note

$$\begin{aligned} \bar{f}(k) &= a(\xi + i\eta) - ib(\xi + i\eta) \ , \\ f^*(k) &= a(\xi - i\eta) - ib(\xi - i\eta) = \bar{f}(\bar{k}) \ . \end{aligned} \tag{1.4}$$

We will use also the distributions $\delta^{\pm}(\eta)$ given by

$$\int_{-\infty}^{+\infty} d\eta f(\xi + i\eta)\delta^{\pm}(\eta) = f(\xi \pm i0) \tag{1.5a}$$

and having the property

$$\delta^{+}(\eta) = \bar{\delta}^{-}(\eta) \ . \tag{1.5b}$$

Finally we shall deal with functions in $\mathbf{C}$ which go to a polynomial of k as $k \to \infty$. We denote by $\mathcal{P}_k$ this polynomial, more precisely defined by

$$f(k) - \mathcal{P}_k(f) \underset{k \to \infty}{\longrightarrow} 0 \ . \tag{1.6}$$

II. EXTENSION OF THE SPECTRAL TRANSFORM METHOD

A. Spectral problem and basic solutions

We briefly recall here the basic results on the Zakharov-Shabat spectral problem which we write for the 2×2 matrix $\mu(k,x,t)$

$$\mu_x + ik[\sigma_3, \mu] = Q\mu, \quad Q = \begin{bmatrix} 0 & q(x,t) \\ r(x,t) & 0 \end{bmatrix} . \tag{2.1}$$

The two fundamental solutions of (2.1), say $\mu^{\pm}$, are determined by (a t dependence is understood everywhere)

$$\begin{bmatrix} \mu_{11}^{+}(k,x) \\ \mu_{21}^{+}(k,x) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\int_x^{\infty} d\xi\, q(\xi)\mu_{21}^{+}(k,\xi) \\ \int_{-\infty}^{x} d\xi\, r(\xi)\mu_{11}^{+}(k,\xi)e^{2ik(x-\xi)} \end{bmatrix} , \tag{2.2a}$$

$$\begin{bmatrix} \mu_{12}^{+}(k,x) \\ \mu_{22}^{+}(k,x) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \int_x^{\infty} d\xi\, q(\xi)\mu_{22}^{+}(k,\xi)e^{-2ik(x-\xi)} \\ \int_x^{\infty} d\xi\, r(\xi)\mu_{12}^{+}(k,\xi) \end{bmatrix} , \tag{2.2b}$$

$$\begin{bmatrix}\mu_{11}^-(k,x)\\ \mu_{21}^-(k,x)\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}-\begin{bmatrix}\int_x^{\infty}d\xi\, q(\xi)\mu_{21}^-(k,\xi)\\ \int_x^{\infty}d\xi\, r(\xi)\mu_{11}^-(k,\xi)e^{2ik(x-\xi)}\end{bmatrix}, \tag{2.3a}$$

$$\begin{bmatrix}\mu_{12}^-(k,x)\\ \mu_{22}^-(k,x)\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}+\begin{bmatrix}\int_{-\infty}^{x}d\xi\, q(\xi)\mu_{22}^-(k,\xi)e^{-2ik(x-\xi)}\\ -\int_x^{\infty}d\xi\, r(\xi)\mu_{12}^-(k,\xi)\end{bmatrix}. \tag{2.3b}$$

The first column vector μ_1^+ of the matrix μ^+ is meromorphic in $\mathrm{Im}(k)>0$ where it has a finite number N^+ of poles k_n^+ (assumed to be simple). The second vector μ_2^+ is holomorphic in $\mathrm{Im}(k)>0$. The vector μ_1^- is holomorphic in $\mathrm{Im}(k)<0$, while the second one μ_2^- is meromorphic in $\mathrm{Im}(k)<0$ where it has a finite number N^- of pole k_n^- (simple). One can check directly on (2.2) and (2.3) that we have the relations

$$\operatorname*{Res}_{k=k_n^+}\mu_1^+(k)=iC_n^+\mu_2^+(k_n^+)\ , \tag{2.4a}$$

$$\operatorname*{Res}_{k=k_n^-}\mu_2^-(k)=-iC_n^-\mu_1^-(k_n^-)\ , \tag{2.4b}$$

which define the *normalization coefficients* $C_n^{\pm}$.

The function $\mu(k)$ defined as μ^+ in the upper half plane and μ^- in the lower is then discontinuous on the real k axis. Its discontinuity can be expressed simply in terms of μ itself as

$$\mu_1^+-\mu_1^-=e^{2ikx}\alpha^+\mu_2^+\ , \tag{2.5a}$$

$$\mu_2^+-\mu_2^-=-e^{-2ikx}\alpha^-\mu_1^-\ , \tag{2.5b}$$

which define the *reflection coefficients* $\alpha^{\pm}(k)$,

$$\alpha^+(k)=\int_{-\infty}^{+\infty}d\xi r(\xi)\mu_{11}^+(k,\xi)e^{-2ik\xi}\ , \tag{2.6a}$$

$$\alpha^-(k)=\int_{-\infty}^{+\infty}d\xi q(\xi)\mu_{22}^-(k,\xi)e^{2ik\xi}\ . \tag{2.6b}$$

For future use we define also the transmission coefficients $\beta^{\pm}$

$$\beta^+(k)=1-\int_{-\infty}^{+\infty}d\xi q(\xi)\mu_{21}^+(k,\xi)\ , \tag{2.7a}$$

$$\beta^-(k)=1-\int_{-\infty}^{+\infty}d\xi r(\xi)\mu_{12}^-(k,\xi)\ . \tag{2.7b}$$

The integral equations (2.2) and (2.3) give the following behaviors at large x:

$$\mu^+\underset{x\to-\infty}{\sim}\begin{bmatrix}\beta^+ & -e^{-2ikx}\alpha^-/\beta^-\\ 0 & 1/\beta^+\end{bmatrix}, \quad \mu^+\underset{x\to+\infty}{\sim}\begin{bmatrix}1 & 0\\ e^{2ikx}\alpha^+ & 1\end{bmatrix}, \tag{2.8a}$$

$$\mu^-\underset{x\to-\infty}{\sim}\begin{bmatrix}1/\beta^- & 0\\ -e^{2ikx}\alpha^+/\beta^+ & \beta^-\end{bmatrix}, \quad \mu^-\underset{x\to+\infty}{\sim}\begin{bmatrix}1 & e^{-2ikx}\alpha^-\\ 0 & 1\end{bmatrix}, \tag{2.8b}$$

where we have used that from (2.1), $\det(\mu)$ is x independent. Consequently we have also the unitarity relation

$$\alpha^+\alpha^-+\beta^+\beta^-=1\ . \tag{2.9}$$

To obtain the asymptotic behaviors $-e^{-2ikx}\alpha^-/\beta^-$ of μ_{12}^+ and $e^{-2ikx}\alpha^+/\beta^+$ of μ_{21}^- as x goes to $-\infty$, one must use the relations (2.5) repeatedly in (2.2b) and (2.3a), respectively.

Solving the direct-scattering problem consists of solving for given $Q(x)$ the integral equations (2.2) and (2.3), and then calculating the *spectral data* $\mathcal{S}$,

$$\mathcal{S}=\{\alpha^{\pm}(k),k\in\mathbb{R};k_n^{\pm},C_n^{\pm},n=1,\ldots,N^{\pm}\}\ . \tag{2.10}$$

B. $\bar\partial$ problem and spectral transform

Since the work of Beals and Coiffman [17], we know that the solution of the inverse problem, i.e., the reconstruction of Q from $\mathcal{S}$, is given by a Cauchy-Green integral equation which solves a $\bar\partial$ problem for the function $\mu(k)$ previously defined. Actually the $\bar\partial$ problem is simply the formula which summarizes the analytical properties of $\mu(k)$; it reads

$$\frac{\partial}{\partial\bar k}\mu(k)=\mu(k)R(k)\ , \tag{2.11}$$

where the distribution $R(k)$ is the *spectral transform* and is given from the spectral data $\mathcal{S}$ by

$$R(k)=\frac{i}{2}\begin{bmatrix}0 & -\alpha^-(k)\delta^-(k_I)\\ \alpha^+(k)\delta^+(k_I) & 0\end{bmatrix}e^{2ik\sigma_3x}+2\pi\sum_{n=1}^{N^{\pm}}\begin{bmatrix}0 & C_n^-\delta(k-k_n^-)\\ C_n^+\delta(k-k_n^+) & 0\end{bmatrix}e^{2ik\sigma_3x}\ . \tag{2.12}$$

This formula can be demonstrated simply by noting that

$$\frac{\partial}{\partial\bar k}\mu(k)=\frac{i}{2}\mu(k)[\delta^+(k_I)-\delta^-(k_I)],\quad k\in\mathbb{R}$$

$$\frac{\partial}{\partial\bar k}\mu(k)=-2i\pi\sum_{n=1}^{N^+}\delta(k-k_n^+)\begin{bmatrix}\operatorname*{Res}_{k_n^+}\mu_{11}^+(k) & 0\\ \operatorname*{Res}_{k_n^+}\mu_{21}^+(k) & 0\end{bmatrix},\ \mathrm{Im}(k)>0 \tag{2.13}$$

$$\frac{\partial}{\partial\bar k}\mu(k)=2i\pi\sum_{n=1}^{N^-}\delta(k-k_n^-)\begin{bmatrix}0 & \operatorname*{Res}_{k_n^-}\mu_{12}^-(k)\\ 0 & \operatorname*{Res}_{k_n^-}\mu_{22}^-(k)\end{bmatrix},\quad \mathrm{Im}k<0\ .$$

The above $\bar\partial$ problem is completed by the asymptotic behavior of μ at large k, which is obtained from (2.2) and (2.3) by integration by parts and reads

$$k \to \infty \Longrightarrow \mu(k) = \mathbb{1} + O\left(\frac{1}{k}\right) . \tag{2.14}$$

Finally the solution of Eq. (2.1) obeying the above behavior is obtained by solving the following Cauchy-Green integral equation:

$$\mu(k) = \mathbb{1} + \frac{1}{2i\pi} \int\int \frac{d\lambda \times d\bar{\lambda}}{\lambda - k} \mu(\lambda) R(\lambda) . \tag{2.15}$$

By comparison of the different powers of $1/k$ in Eq. (2.1), we readily obtain

$$Q = i[\sigma_3, \mu^{(1)}] , \tag{2.16}$$

where we have defined $\mu^{(1)}$ through

$$\mu(k) = \mathbb{1} + \sum_{n=1}^{\infty} k^{-n} \mu^{(n)} . \tag{2.17}$$

Therefore the inverse problem is solved by the integral equation (2.15).

C. General evolution of the spectral transform

On a very general level, we want to link a given parametric dependence of the spectral transform $R(k)$ on the real variables (x,t) to the corresponding dependence of the field $Q(x,t)$. Clearly, this will be done by means of the inverse problem, hence by using mainly the results of Sec. II B.

The x dependence of $R(k)$ has already been chosen, let us write it

$$R_x = [R, \Lambda], \quad \Lambda = ik\sigma_3 \tag{2.18}$$

and mention that it can be generalized to nonanalytic functions $\Lambda(k)$ [5]. The above x evolution implies that μ, solution of (2.11), does obey the differential equation (2.1), which is proved as follows. First we check that

$$\frac{\partial}{\partial \bar{k}}[(\mu_x - \mu\Lambda)\mu^{-1}] = 0 , \tag{2.19}$$

and hence the function $(\mu_x - \mu\Lambda)\mu^{-1}$ is analytic in k, let us call it U, that is

$$\mu_x - \mu\Lambda = U\mu . \tag{2.20}$$

The function $U(k)$ can then be calculated from the Liouville theorem and we get [note that $\mu_x\mu^{-1} = O(1/k)$]

$$U = -\mathcal{P}_k(\mu\Lambda\mu^{-1}) . \tag{2.21}$$

It is then trivial to check (2.1) which is

$$\Lambda = ik\sigma_3 \Longrightarrow U = -ik\sigma_3 + Q , \tag{2.22}$$

where Q is given by (2.16).

For the t dependence, we proceed in the same way but seek a more general structure, namely

$$R_t = [R, \Omega] + M . \tag{2.23}$$

First of all, this equation has to be compatible with (2.18), which is guaranteed by

$$[R, \Lambda_t - \Omega_x + [\Omega, \Lambda]] = M_x - [M, \Lambda] . \tag{2.24}$$

To solve this it is convenient to expand the 2×2 matrices on their diagonal (D) and antidiagonal (A) parts. Then we remark in (2.38) that only Ω^D and M^A contribute to the t dependence of R. Moreover, the diagonal part of (2.23), namely

$$[R, \Omega^A] + M^D = 0 , \tag{2.25}$$

inserted in the diagonal part of (2.24) gives for the Λ diagonal

$$[R, [\Omega^A, \Lambda]] = 0 . \tag{2.26}$$

Hence we take

$$\Omega^A = 0 \Longrightarrow M^D = 0 , \tag{2.27}$$

and we will prove that this choice is not restrictive for the resulting integrable system. It follows that the constraint (2.24) will be realized as soon as

$$M_x - [M, \Lambda] = [\Omega_x, R] \tag{2.28}$$

and we see that $\Omega_x \neq 0$ would lead to possible further generalizations. We leave this case to future investigations and take here

$$\begin{aligned} &\Omega_x = 0 \Longrightarrow \Omega = \Omega(k,t) , \\ &M_x = [M, \Lambda] \Longrightarrow M = M_0(k,t) e^{2ik\sigma_3 x} . \end{aligned} \tag{2.29}$$

In the previous works [5] was only considered the case $M = 0$, which actually corresponds to an implicit *a priori* choice of the boundary values. Indeed we shall see here that the freedom in the boundary values requires that M be nonzero.

Apart from the restrictions (2.29) and (2.27), Ω and M_0 are quite general functions of k and t, which can be nonanalytic in the variable k and will appear to be functionals of $R(k,t)$ and, of course, of the boundary values $I_1(k,t)$ and $I_2(k,t)$.

Let us make now a further restriction by taking Ω with no polynomial part, that is

$$\Omega(k,t) = O\left(\frac{1}{k}\right) . \tag{2.30}$$

Adding a polynomial part is trivial but makes the formalism heavier. For instance, the model (1.3) is obtained for $\mathcal{P}_k\Omega = 2i\alpha k^2 \sigma_3$ [5].

The problem is now to compute μ_t in terms of μ once given (2.23). As for the x dependence, we first check that

$$\frac{\partial}{\partial \bar{k}}[(\mu_t - \mu\Omega)\mu^{-1}] = \mu\left[M - \frac{\partial\Omega}{\partial \bar{k}}\right]\mu^{-1} . \tag{2.31}$$

The general solution of the above equation reads

$$\mu_t - \mu\Omega = V\mu , \tag{2.32}$$

$$V = \frac{1}{2i\pi} \int\int \frac{d\lambda \times d\bar{\lambda}}{\lambda - k} \mu(\lambda)[M(\lambda) - \bar{\partial}\Omega]\mu^{-1}(\lambda) . \tag{2.33}$$

(Here $\bar{\partial}\Omega = \partial\Omega/\partial\bar{\lambda}$.) The polynomial part of $V(k)$ vanishes due to (2.30).

Here ends the first step: to obtain the Lax pair (2.20) and (2.32) associated with the given evolution (2.23) of the spectral transform. The following step is now to derive the resulting evolution equation for the field $Q(x,t)$.

D. General integrable system

The compatibility ($\mu_{xt}=\mu_{tx}$) between the two members of the Lax pair furnishes the usual (U,V) system

$$U_t - V_x + [U,V] = 0 \tag{2.34}$$

because we have chosen Λ and Ω verifying

$$\Lambda_t - \Omega_x + [\Omega,\Lambda] = 0 \ . \tag{2.35}$$

Equation (2.34) furnishes an evolution equation for $Q(x,t)$ if and only if the k dependence in (2.34) cancels identically. This is indeed the case since we have, in the computation of V_x,

$$\frac{\partial}{\partial x}\{\mu[M-\bar{\partial}\Omega]\mu^{-1}\} = [U(\lambda),\mu\{M-\bar{\partial}\Omega\}\mu^{-1}] \ . \tag{2.36}$$

Then, noting that $U(k)-U(\lambda)=i\sigma_3(\lambda-k)$, (2.34) becomes

$$Q_t = \frac{1}{2\pi}\int\int d\lambda\times d\bar{\lambda}[\sigma_3,\mu\{M-\bar{\partial}\Omega\}\mu^{-1}] \ . \tag{2.37}$$

This is the general integrable evolution related to the evolution (2.23) of the spectral transform with the restrictions [(2.29), (2.27), and (2.30)] on the arbitrary matrices Ω and M.

With the notations

$$\Omega = \omega(k,t)\sigma_3 \ , \qquad M = \begin{bmatrix} 0 & m^-(k,t) \\ m^+(k,t) & 0 \end{bmatrix} e^{2ik\sigma_3 x} \ , \tag{2.38}$$

the evolution (2.37) becomes

$$q_t = -\frac{1}{\pi}\int\int d\lambda\times d\bar{\lambda}(2\mu_{11}\mu_{12}\bar{\partial}\omega + m^-\mu_{11}^2 e^{-2i\lambda x} - m^+\mu_{12}^2 e^{2i\lambda x}) \ ,$$
$$r_t = -\frac{1}{\pi}\int\int d\lambda\times d\bar{\lambda}(2\mu_{21}\mu_{22}\bar{\partial}\omega + m^-\mu_{21}^2 e^{-2i\lambda x} - m^+\mu_{22}^2 e^{2i\lambda x}) \ . \tag{2.39}$$

This integrable system has to be understood coupled to the differential equations (2.1) for μ_{ij}. Its novelty lies in the presence of the distributions $m^{\pm}$, and we note that even for a *regular* dispersion relation (i.e., $\bar{\partial}\omega=0$), we get a system of coupled waves.

E. Structural constraints and reductions

It is essential for the physical applications to consider reductions from the two-field Q to a one-field equation. We are interested in the case

$$r(x,t) = \sigma\bar{q}(x,t), \sigma=\pm \tag{2.40}$$

for which it is easy to prove [directly on (2.1) and (2.11)] that we have [remember the definition (1.4)]

$$\sigma=+ \Longrightarrow \mu^* = \sigma_1\mu\sigma_1 \Longrightarrow R^* = \sigma_1 R\sigma_1 \ , \tag{2.41}$$

$$\sigma=- \Longrightarrow \mu^* = \sigma_2\mu\sigma_2 \Longrightarrow R^* = \sigma_2 R\sigma_2 \ , \tag{2.42}$$

and consequently from the definitions (2.6) and (2.7)

$$\alpha^+ = \sigma\bar{\alpha}^- , \quad k\in\mathbb{R} \ , \tag{2.43}$$

$$k_n^+ = \bar{k}_n^- , \quad C_n^+ = \sigma\bar{C}_n^- \ . \tag{2.44}$$

It follows that the three scalar functions ω and $m^{\pm}$ must obey some *reduction* constraints coming from the above relations together with some *structural* constraints coming from the nature of the support of $R(k,t)$ given in (2.12) and from the structure of the evolution (2.39).

The first structural constraint can be read on (2.23); it is

$$m^{\pm}(k,t) = m_0^{\pm}(k,t)\delta^{\pm}(k_I) + \sum_n m_n^{\pm}(t)\delta(k-k_n^{\pm}) \ . \tag{2.45}$$

The reduction constraints read (for $k\in\mathbb{R}$)

$$\omega(k,t)\in i\mathbb{R}, \quad \omega(k_n^+,t) = -\bar{\omega}(k_n^-,t) \ , \tag{2.46}$$

$$m_0^+(k,t) = \sigma\bar{m}_0^-(k,t), \quad m_n^+(t) = \sigma\bar{m}_n^-(t) \ . \tag{2.47}$$

The resulting evolution (2.16) of the spectral transform then reads

$$\alpha_t^+ = 2\omega\alpha^+ - 2im_0^+ \tag{2.48}$$

for the continuum ($k\in\mathbb{R}$), and

$$k_{n,t}^+ = 0, \quad C_{n,t}^+ = 2\omega(k_n^+)C_n^+ + \frac{1}{2\pi}m_n^+ \tag{2.49}$$

for the discrete spectrum.

There is now a supplementary structural constraint on ω if we are interested on evolutions (2.39) where the integral on the right-hand side runs on the real axis only. In that case we obviously have to set

$$m_n^{\pm}(t) = 0 \ . \tag{2.50}$$

Then we will need that $\bar{\partial}\omega$ be proportional to the distributions $\delta^{\pm}$. Considering (2.40) in the evolution (2.39) and remembering the property (1.5b), we require

$$\frac{\partial}{\partial\bar{k}}\omega(k,t) = ip(k,t)[\delta^+(k_I)+\delta^-(k_I)], \quad p(k,t)\in\mathbb{R} \ . \tag{2.51}$$

Consequently, in the reduction (2.40), the evolution which we are interested in reads

$$q_t = \frac{2i}{\pi}\int_{-\infty}^{+\infty} d\lambda[2ip(\mu_{11}^+\mu_{12}^+ + \mu_{11}^-\mu_{12}^-) + m_0^-(\mu_{11}^-)^2 e^{-2i\lambda x} - m_0^+(\mu_{12}^+)^2 e^{2i\lambda x}] \ . \tag{2.52a}$$

This equation is coupled to (2.1) which is

$$\mu_{11,x} = q\mu_{21}, \quad \mu_{21,x} - 2ik\mu_{21} = \sigma\bar{q}\mu_{11}, \quad \mu_{12}^{\pm} = \sigma\bar{\mu}_{21}^{\mp} \ . \tag{2.52b}$$

Finally, since the relevant datum for the evolution is the function $p(k,t)$, we need to compute $\omega(k,t)$ out of (2.51), we get

$$\omega(k,t)=-\frac{2i}{\pi}\mathrm{P}\int_{-\infty}^{+\infty}\frac{d\lambda}{\lambda-k}p(\lambda,t)\ . \tag{2.53}$$

Here above, P denotes the Cauchy principal value of the integral.

III. GENERIC INTEGRABLE BOUNDARY-VALUE PROBLEM

A. Method and formulas

We prove here that the equation

$$q_t=\int_{-\infty}^{+\infty}gdka_1\bar{a}_2\ , \tag{3.1a}$$

$$a_{1,x}=qa_2,\quad a_{2,x}-2ika_2=\sigma\bar{q}a_1\ , \tag{3.1b}$$

with the boundary values

$$a_1\underset{x\to+\infty}{\to}I_1(k,t)\ , \tag{3.2a}$$

$$a_2\underset{x\to+\infty}{\sim}I_2(k,t)e^{2ikx}\ , \tag{3.2b}$$

and given $g(k,t)$, can be obtained from the integrable system (2.52) for a convenient choice of Ω and M, which is of ω and $m_0^{\pm}$.

Although any other boundary-value problems can be obtained from (3.2) by a convenient choice of I_j, it will be useful to consider also the following two cases:

$$a_1\underset{x\to-\infty}{\longrightarrow}K_1(k,t)\ , \tag{3.3a}$$

$$a_2\underset{x\to-\infty}{\sim}K_2(k,t)e^{2ikx}\ , \tag{3.3b}$$

and

$$a_1\underset{x\to+\infty}{\longrightarrow}J_1(k,t)\ , \tag{3.4a}$$

$$a_2\underset{x\to-\infty}{\sim}J_2(k,t)e^{2ikx}\ . \tag{3.4b}$$

The proof that these problems are integrable is performed essentially by means of the asymptotic behaviors (2.8) of $\mu^{\pm}$, and by the relation (2.5) between μ^+ and μ^- on the real axis (the Riemann-Hilbert problem).

Comparing (3.2) with (2.8), we deduce that

$$\begin{bmatrix}a_1\\a_2\end{bmatrix}=I_1\mu_1^-+I_2\mu_2^+e^{2ikx}\ . \tag{3.5}$$

In the same way the other boundary values give

$$\begin{bmatrix}a_1\\a_2\end{bmatrix}=\frac{K_1}{\beta^+}\mu_1^++\frac{K_2}{\beta^-}\mu_2^-e^{2ikx}\ , \tag{3.6}$$

$$\begin{bmatrix}a_1\\a_2\end{bmatrix}=J_1\mu_1^++\beta^+J_2\mu_2^+e^{2ikx}\ . \tag{3.7}$$

Now the game consists in expressing $a_1\bar{a}_2$ in terms of the only three quantities appearing in (2.52): $(\mu_{11}^+\mu_{12}^++\mu_{11}^-\mu_{12}^-)$, $(\mu_{11}^-)^2e^{-2ikx}$, and $(\mu_{12}^+)^2e^{2ikx}$. This is indeed possible by repeatedly using the Riemann-Hilbert problem (2.5), which proves the integrability of the system (3.1) with arbitrary boundary values. We give hereafter the formulas for each possible combination of $\mu_{ij}^{\pm}$,

$$\begin{aligned}\mu_{11}^+\mu_{12}^-=&\tfrac{1}{2}(1+\alpha^+\alpha^-)(\mu_{11}^+\mu_{12}^++\mu_{11}^-\mu_{12}^-)\\&+\tfrac{1}{2}\alpha^+(1-\alpha^+\alpha^-)(\mu_{12}^+)^2e^{2ikx}\\&+\tfrac{1}{2}\alpha^-(1-\alpha^+\alpha^-)(\mu_{11}^-)^2e^{-2ikx}\ ,\end{aligned} \tag{3.8}$$

$$\begin{aligned}\mu_{11}^-\mu_{12}^+=&\tfrac{1}{2}(\mu_{11}^+\mu_{12}^++\mu_{11}^-\mu_{12}^-)\\&-\tfrac{1}{2}\alpha^+(\mu_{12}^+)^2e^{2ikx}\\&-\tfrac{1}{2}\alpha^-(\mu_{11}^-)^2e^{-2ikx}\ ,\end{aligned} \tag{3.9}$$

$$\begin{aligned}\mu_{11}^+\mu_{11}^-e^{-2ikx}=&\tfrac{1}{2}\alpha^+(\mu_{11}^+\mu_{12}^++\mu_{11}^-\mu_{12}^-)\\&-\tfrac{1}{2}(\alpha^+)^2(\mu_{12}^+)^2e^{2ikx}\\&+(1-\tfrac{1}{2}\alpha^+\alpha^-)(\mu_{11}^-)^2e^{-2ikx}\ ,\end{aligned} \tag{3.10}$$

$$\begin{aligned}\mu_{12}^+\mu_{12}^-e^{2ikx}=&\tfrac{1}{2}\alpha^-(\mu_{11}^+\mu_{12}^++\mu_{11}^-\mu_{12}^-)\\&+(1-\tfrac{1}{2}\alpha^+\alpha^-)(\mu_{12}^+)^2e^{2ikx}\\&-\tfrac{1}{2}(\alpha^-)^2(\mu_{11}^-)^2e^{-2ikx}\ ,\end{aligned} \tag{3.11}$$

$$\begin{aligned}(\mu_{11}^+)e^{-2ikx}=&\alpha^+(\mu_{11}^+\mu_{12}^++\mu_{11}^-\mu_{12}^-)\\&+(1-\alpha^+\alpha^-)(\mu_{11}^-)^2e^{-2ikx}\ ,\end{aligned} \tag{3.12}$$

$$\begin{aligned}(\mu_{12}^-)^2e^{2ikx}=&\alpha^-(\mu_{11}^+\mu_{12}^++\mu_{11}^-\mu_{12}^-)\\&+(1-\alpha^+\alpha^-)(\mu_{12}^+)^2e^{2ikx}\ .\end{aligned} \tag{3.13}$$

B. Evolution of the spectral transform

The values of $\omega(k,t)$ and $m_0^{\pm}(k,t)$ corresponding to each case of boundary values are obtained simply by inspection. Starting with the development of $a_1\bar{a}_2$ from (3.5)–(3.7), written uniquely in terms of the three quantities $(\mu_{11}^+\mu_{12}^++\mu_{11}^-\mu_{12}^-)$, $(\mu_{11}^-)^2e^{-2ikx}$, and $(\mu_{12}^+)^2e^{2ikx}$, we compare it to the rhs of the integrable equation (2.52). We obtain the following values.

(i) In the case (3.2)

$$\begin{aligned}&\omega(k,t)=\frac{i}{4}\mathrm{P}\int\frac{d\lambda}{\lambda-k}g(\sigma|I_1|^2+|I_2|^2)\ ,\\&m_0^+(k,t)=\frac{i\pi}{2}g[\sigma\bar{I}_1I_2-\tfrac{1}{2}\alpha^+(\sigma|I_1|^2+|I_2|^2)]\ ,\\&m_0^-(k,t)=-\frac{i\pi}{2}g[I_1\bar{I}_2-\tfrac{1}{2}\alpha^-(\sigma|I_1|^2+|I_2|^2)]\ .\end{aligned} \tag{3.14}$$

(ii) In the case (3.3)

$$\begin{aligned}&\omega(k,t)=\frac{i}{4}\mathrm{P}\int\frac{d\lambda}{\lambda-k}g\left[(\sigma|K_1|^2+|K_2|^2)\frac{1+\alpha^+\alpha^-}{1-\alpha^+\alpha^-}+K_1\bar{K}_2\frac{\alpha^+}{(\beta^+)^2}+\bar{K}_1K_2\frac{\sigma\alpha^-}{(\beta^-)^2}\right]\ ,\\&m_0^+(k,t)=\frac{i\pi}{2}g\left[\sigma\bar{K}_1K_2\frac{\beta^+}{\beta^-}+\tfrac{1}{2}\alpha^+(\sigma|K_1|^2+|K_2|^2)\right]\ ,\\&m_0^-(k,t)=-\frac{i\pi}{2}g\left[K_1\bar{K}_2\frac{\beta^-}{\beta^+}+\tfrac{1}{2}\alpha^-(\sigma|K_1|^2+|K_2|^2)\right]\ .\end{aligned} \tag{3.15}$$

(iii) In the case (3.4)

$$\omega(k,t)=\frac{i}{4}\mathrm{P}\int\frac{d\lambda}{\lambda-k}g[\sigma|J_1|^2(1+\alpha^+\alpha^-)+|J_2|^2\beta^+\beta^- +J_1\bar{J}_2\alpha^+\beta^-+\bar{J}_1J_2\sigma\alpha^-\beta^+] ,$$

$$m_0^+(k,t)=\frac{i\pi}{2}g[\sigma\bar{J}_1J_2\beta^+(1-\tfrac{1}{2}\alpha^+\alpha^-)-\tfrac{1}{2}\beta^-(\alpha^+)^2J_1\bar{J}_2 +\tfrac{1}{2}\alpha^+\beta^+\beta^-(\sigma|J_1|^2-|J_2|^2)] , \tag{3.16}$$

$$m_0^-(k,t)=-\frac{i\pi}{2}g\left[J_1\bar{J}_2\beta^-(1-\tfrac{1}{2}\alpha^+\alpha^-) -\frac{\sigma}{2}\beta^+(\alpha^-)^2\bar{J}_1J_2 +\tfrac{1}{2}\alpha^-\beta^+\beta^-(\sigma|J_1|^2-|J_2|^2)\right] .$$

The corresponding evolutions of the spectral transform are given by Eqs. (2.48) and (2.49), namely

$$\alpha_t^+=2\omega\alpha^+-2im_0^+ ,\quad \alpha_t^-=-2\omega\alpha^-+2im_0^- , \tag{3.17a}$$

$$C_{n,t}^+=2\omega(k_n^+)C_n^+ ,\quad C_{n,t}^-=-2\omega(k_n^-)C_n^- , \tag{3.17b}$$

together with $k_{n,t}=0$. This gives the complete solution of the problem. The input data are the boundary values (in either case) and the initial spectral data $\{\alpha^\pm(k,0),C_n^\pm(0)\}$, which can be taken to be zero for the *boundary-value* case or nonzero for the *initial boundary-value* case.

In the physically interesting cases, the boundary values are such that the functions $\beta^\pm$ can be eliminated. However, in general, to close the above systems of equations, we need to express $\beta^\pm$ in terms of $\alpha^\pm$. This can be done by solving (2.9) as a Riemann-Hilbert problem for $\beta(k)$ as follows (see, e.g., [1]). Let us define the functions

$$b^\pm(k)=\beta^\pm(k)\prod_n\frac{k-k_n^\pm}{k+k_n^\pm} , \tag{3.18}$$

which are holomorphic in $\pm\mathrm{Im}(k)>0$ and go to 1 as k goes to ∞. Then we introduce the function

$$f(k)=\ln b^+(k),\quad \mathrm{Im}(k)>0 ,$$

$$f(k)=-\ln b^-(k),\quad \mathrm{Im}(k)<0$$

which is analytic everywhere except on the real axis where it stands a discontinuity. By computing f^+-f^- we obtain a Riemann-Hilbert problem for f which solution is

$$f(k)=\frac{1}{2i\pi}\int_{-\infty}^{+\infty}\frac{d\lambda}{\lambda-k}\left[\ln(1-\alpha^+\alpha^-)+\sum_n\ln\frac{\lambda-k_n^+}{\lambda+k_n^+} +\sum_n\ln\frac{\lambda+k_n^-}{\lambda-k_n^-}\right] , \tag{3.19}$$

and the solution to our problem is

$$\beta^+(k)=\exp[f(k+i0)]\prod_n\frac{k+k_n^+}{k-k_n^+} , \tag{3.20a}$$

$$\beta^-(k)=\exp[-f(k-i0)]\prod_n\frac{k+k_n^-}{k-k_n^-} . \tag{3.20b}$$

It is clear on the above set of equations that the evolution of the spectral data is quite complicated and essentially nonlinear. However, we shall see in the applications that, in physical situations, these evolutions are explicitly solvable. Note that the simplest case is (3.2) and it is explicitly integrable for external I_1 and I_2. A particular case will be considered in Sec. IV A.

Finally, it will be useful for the following to have the evolution of the reflected energy:

$$E(k,t)=|\alpha^+(k,t)|^2=\sigma\alpha^+(k,t)\alpha^-(k,t) , \tag{3.21}$$

which from (3.17a)

$$E_t=2i\sigma(m_0^-\alpha^+-m_0^+\alpha^-) . \tag{3.22}$$

C. Boundary values and soliton driving

We briefly consider here the cases when the boundary values are used to *pump* the solution $q(x,t)$ out of the vacuum $q(x,0)=0$ and to *drive* a soliton. Hence we assume here that there is no continuous part in the spectrum [no background radiation in $q(x,0)$], namely

$$\alpha^+(k,0)=0 , \tag{3.23}$$

and we check that, at $t=0$, any set of Eqs. (3.14), (3.15) or (3.16) is equivalent and reads

$$\omega(k,0)=\frac{i}{4}\mathrm{P}\int\frac{d\lambda}{\lambda-k}g(\sigma|I_1|^2+|I_2|^2) ,$$

$$m_0^+(k,0)=\frac{i\pi}{2}g\sigma\bar{I}_1I_2 , \tag{3.24}$$

$$m_0^-(k,0)=-\frac{i\pi}{2}gI_1\bar{I}_2 .$$

The evolution (3.17a) then shows that the boundary values $I_1(k,t)$ and $I_2(k,t)$ pump the continuum: $\alpha^+(k,t)$ is no longer zero for $t\neq0$, due to the presence of $m_0^\pm$.

Indeed, we have from (3.17)

$$\left.\frac{\partial\alpha^+(h,t)}{\partial t}\right|_{t=0}=\pi g\sigma\bar{I}_1(k,0)I_2(h,0) , \tag{3.25}$$

$$\left.\frac{\partial\alpha^-(h,t)}{\partial t}\right|_{t=0}=\pi gI_1(k,0)\bar{I}_2(k,0) . \tag{3.26}$$

Hence the system builds up radiation from the vacuum and it does it in a *regular* way as from (3.22) we have

$$\left.\frac{\partial E(k,t)}{\partial t}\right|_{t=0}=0 . \tag{3.27}$$

Suppose now that the initial profile contains a pure N-soliton solution and also that the boundary values are given by (3.2). Then we get from (3.17b)

$$C_n^+(t)=C_n^+(0)\exp\left[2\int_0^t dt'\omega(k_n^+,t')\right] , \tag{3.28}$$

where $\omega(k_n^+,t)$ is given in (3.14). The arbitrariness of the

I_j is a tool to build arbitrary C_n, which produces solitons with arbitrarily varying velocities [see, for instance, the one-soliton formula (4.30)]. There is, however, a price to pay: the value of $\alpha^{\pm}(k,t)$ is nonzero for any $t>0$. Hence the soliton driving always goes with a pumping of the background radiation, that is with an energy loss. As a consequence, although we have an explicit expression of the reflection coefficient at all time, we cannot in general write down an explicit solution, except maybe for some particular choices of the set $\{g,I_1,I_2\}$.

D. Comments

(1) In our approach we take the evolution of the spectral transform as a starting point and then we construct the related Lax pair. One may ask if the standard approach would allow for a similar extension of the spectral transform method.

To follow a standard approach would mean to start with the Lax pair (2.20) and (2.32) and try to compute the time evolution of $R(k)$. The method consists here in expressing the auxiliary spectral problem (2.32) as x goes to $\pm\infty$. The problem is that the function $V(k)$ defined by (2.33) has ill-defined limits as $x\to\pm\infty$ if we insert the structures (2.45) and (2.51) of the distributions $\bar{\partial}\omega$ and $m^{\pm}$.

(2) There exists a simple interesting case that we do not study here and which consists in taking

$$\omega=0,\quad m_0^{\pm}=0,\quad m_n^{\pm}\neq 0\ .$$

The integrable Eq. (2.39) then becomes

$$q_t=-\frac{1}{\pi}\sum_{n=1}^{N} m_n^{-}(\mu_{n1}^{-})^2\exp[-2ik_n^{-}x] - m_n^{+}(\mu_{n2}^{+})^2\exp[2ik_n^{+}x]\ , \tag{3.29}$$

where the functions $\mu_{nj}^{\pm}$ are the bound-states eigenfunctions of (2.1). In that case the boundary values of a_1 and a_2 are simply zero. The interesting feature of this system is to possess a pure N-soliton arbitrarily driven. Indeed the evolution (2.49) reads here

$$C_{n,t}^{\pm}=\frac{1}{2\pi}m_n^{\pm}(t)\ , \tag{3.30}$$

where the $m_n^{\pm}(t)$ are arbitrary. In particular, each time one of the m_n vanishes, the corresponding soliton disappears.

IV. APPLICATIONS

A. Interaction of electromagnetic waves with the ion-acoustic wave in plasmas

In a long, two-component, fluid-type plasma irradiated by laser light, the electromagnetic wave (EMW) interacting with the ion-acoustic wave (IAW) induces a reflected EMW which drastically reduces the penetration of the laser. This process of stimulated emission of radiation is the Brillouin (back) scattering (SBS) and results from the low-frequency effect of the high-frequency EMW by means of the ponderomotive force on the electrons which acts as a source for the IAW [18].

Starting with the complex electric field (z and T are the laboratory coordinates)

$$\mathcal{E}(z,T)=\epsilon a_1(x,t)\exp[i(\omega_1T+k_1z)] +\epsilon a_2(x,t)\exp[i(\omega_2T+k_2z)]+O(\epsilon^2) \tag{4.1}$$

and the electronic density $n(z,T)$ defining the IAW q through

$$\frac{n(z,T)}{n_0}-1=\epsilon\frac{2k_1c^2}{i\omega_1^2}q(x,t)\exp[i(\Omega T+Kz)]+O(\epsilon^2), \tag{4.2}$$

in the slow variables $x=\epsilon(z+c_sT)$ and $t=\epsilon^2T$, we have proved in [8] that the hydrodynamic and Maxwell equations for the plasma become

$$q_t=\gamma a_1\bar{a}_2\ ,\qquad a_{1,x}=qa_2,\quad a_{2,x}=\bar{q}a_1 \tag{4.3}$$

under the selection rules

$$\omega_1=\omega_2+\Omega,\quad k_1=k_2+K \tag{4.4}$$

(the values of the various constants will be found in [8]). The physical asymptotic values for this model are

$$a_1\underset{x\to+\infty}{\longrightarrow}1,\quad a_2\underset{x\to-\infty}{\longrightarrow}0\ , \tag{4.5}$$

and, for q vanishing at both ends, Eq. (4.3) implies that we have also $a_2\to 0$ at $+\infty$. Hence an initial datum $q(x,0)$, to be consistent, has to have a vanishing reflection coefficient at $k=0$. The resulting behavior of the solution is somehow singular and has been described in [8].

If, however, we consider the fact that there is a *broadline* response of the IAW to the input EMW, then (4.3) is the sharp-line limit [i.e., as $\gamma(k)$ goes to $\delta(k)$] of the system

$$q_t=\int dk\ \gamma(k)a_1\bar{a}_2\ ,\qquad a_{1,x}=qa_2,\quad a_{2,x}-2ika_2=\bar{q}a_1\ , \tag{4.6}$$

where k measures the frequency mismatch due to the spreading of the resonance. This system is consistent with the boundary values (4.5) with no more constraint, and it is integrable with the dispersion relation (3.16) where we set

$$J_1=1,\quad J_2=0,\quad \sigma=+,\quad g(k)=\gamma(k)\ . \tag{4.7}$$

The resulting evolution (3.22) of $E=|\alpha^+|^2$ is easily solved and we have

$$E(k,t)=\frac{E(k,0)}{E(k,0)+[1-E(k,0)]\exp[-\pi\gamma t]}\ . \tag{4.8}$$

Remember that E measures the reflected energy; indeed, we have from (3.7) and (2.8)

$$|a_2|^2\underset{x\to+\infty}{\longrightarrow}E(k,t)\ . \tag{4.9}$$

The function $\gamma(k)$ being positive for all k (it is typically a Gaussian centered in $k=0$), we have the limit

$$E(k,t)\xrightarrow[t\to+\infty]{}1\ , \tag{4.10}$$

which shows that indeed SBS causes total reflexivity of the EMW. As far as we know, this model provides the first analytical proof of SBS reflexivity in a purely nonlinear context.

The formula (4.8) shows, moreover, that the rapidity with which the wave becomes totally reflected depends on the value of the mismatch wave number k through the distribution $\gamma(k)$. As this distribution is centered in $k=0$ (resonance), and rapidly decreasing around this value, it is clear that the first wave to be reflected is the one which resonantly interacts with the sound wave, as expected. Moreover, far from the resonance (i.e., k far from zero), $\gamma(k)$ vanishes and no reflection occurs. This behavior is qualitatively the one observed in experiments of laser-plasma interaction [18], but, however, a quantitative comparison with experiments is more questionable due to the fact that our model runs on the infinite line.

B. Self-induced transparency

The general context is that of the propagation of a laser pulse in a dielectric medium. When the input electromagnetic wave has a frequency close to one transition frequency, then the dielectric is well modeled as a two-level medium, and the incident light should be strongly absorbed. However, McCall and Hahn [12] discovered that, above some intensity threshold, a laser pulse of duration much shorter than the relaxation time of the two-level medium can propagate with a surprisingly low-energy loss, the medium becoming (self) transparent. This is typically a nonlinear process due to wave coupling.

The theoretical approach is based on the following slowly varying envelope-approximation limit of the general Maxwell-Bloch system (in dimensionless form):

$$\begin{aligned} &\mathcal{E}_\xi=\int dkg\lambda\ ,\\ &\lambda_\tau+2ik\lambda=\mathcal{E}N\ ,\\ &N_\tau=-\tfrac{1}{2}(\bar{\mathcal{E}}\lambda+\mathcal{E}\bar{\lambda})\ . \end{aligned} \tag{4.11}$$

The notations here are those of Lamb [2] who proved that this system is integrable. In short, in the rotating frame, $\mathcal{E}$ is the complex electric-field envelope, λ is the polarization [$\mathrm{Re}(\lambda)$ is the in-phase component, $\mathrm{Im}(\lambda)$ is the in-quadrature component], and N is the population inversion. The parameter k measures the frequency difference between the applied field $\mathcal{E}$ and the resonant frequency gap.

The boundary values associated with (4.11) are

$$N(k,\xi,\tau)\xrightarrow[\tau\to-\infty]{}-1,\quad \lambda(k,\xi,\tau)\xrightarrow[\tau\to-\infty]{}0\ , \tag{4.12}$$

and, launching a short duration laser pulse in $\xi=0$ means that $\mathcal{E}(0,\tau)$ is an exponentially localized function of τ of short width. The asymptotic value of N means that, long before the launching of the laser pulse, all atoms are in the lower level.

The relation between this system of equations and our system (1.1) is given by [2,3]

$$\begin{aligned} &x=\tau,\quad t=\xi,\quad \sigma=-,\quad q=\tfrac{1}{2}\mathcal{E}\ ,\\ &N=|a_2|^2-|a_1|^2,\quad \lambda=2a_1\bar{a}_2\ . \end{aligned} \tag{4.13}$$

Hence the boundary values (4.12) lead us to choose the case (3.3) (boundary values given at $-\infty$) with, from the above definitions,

$$K_1=1,\quad K_2=0\ . \tag{4.14}$$

For these particular values, the evolution (3.17) of the spectral transform is easily solved and we get

$$E(k,t)=E(k,0)\exp[-\pi g(k)t]\ , \tag{4.15}$$

$$\alpha^+(k,t)=\alpha^+(k,0)\exp\left[-i\theta(k,t)-\frac{\pi}{2}g(k)t\right]\ , \tag{4.16}$$

$$\theta(k,t)=\mathrm{P}\int\frac{d\lambda}{\lambda-k}\left[\tfrac{1}{2}g(\lambda)t+\frac{1}{\pi}\ln\frac{1+E(k,t)}{1+E(k,0)}\right]\ , \tag{4.17}$$

$$C_n^+(t)=C_n^+(0)\exp[-i\theta(k_n^+,t)]\ . \tag{4.18}$$

These formulas only apparently differ from those of [3]. Indeed, the reflection coefficient considered there is actually the quantity $-\alpha^-\beta^+/\beta^-$ in which the presence of β^+/β^- cancels the ln term in the pase of α^- [see (3.19) with $E=-\alpha^+\alpha^-$).

The main feature of SIT revealed by (4.15) is that, as t grows, that is as the pulse penetrates in the medium ($\xi\to\infty$), the background radiation exponentially vanishes, the initial firing pulse becoming a set of solitons.

Our method allows us to go beyond these known results. Indeed we can treat, for instance, the interesting case where, long before the pulse launching, some of the atoms are in an excited state. In other words,

$$N(k,\xi,\tau)\xrightarrow[\tau\to-\infty]{}-1+2\Delta(k,\xi)\ , \tag{4.19}$$

$$\lambda(k,\xi,\tau)\xrightarrow[\tau\to-\infty]{}\lambda_0(k,\xi)e^{2ikx}\ . \tag{4.20}$$

To be consistent with (4.13), the functions Δ and λ have to verify

$$2\Delta=\tfrac{1}{4}|\lambda_0|^2\ . \tag{4.21}$$

The initial average population inversion $N(k,\xi,-\infty)$ is given by the real arbitrary function $\Delta(k,\xi)$ which then takes values between 1 (excited state) and 0 (fundamental).

The solution of such a problem is given by the evolutions (3.17) in the case (3.3) with

$$K_1(k,t)=1,\quad K_2(k,t)=\tfrac{1}{2}\lambda_0(k,\xi)|_{\xi=t}\ . \tag{4.22}$$

By inspection of (3.15) we see that this case is much more complicated than the standard one and requires in particular the computation of $\beta^\pm$ out of (3.20). Still, we can obtain interesting qualitative information, starting for instance with an initial profile consisting of a pure soliton, i.e.,

$$\alpha^\pm(k,0)=0,\quad \beta^\pm(k,0)=1\ , \tag{4.23}$$

then we can readily discover on (3.15) the presence of an impurity at the laser input in the initial state of the system induces the *creation* of a background radiation (the time derivative of $\alpha^{\pm}$ is nonzero at $t=0$).

C. Laser-pulse amplification

The physical context is the same as that in the preceding text, except that the pulse is fired in a medium where all atoms are in the upper level. Hence, the boundary values for the Maxwell-Bloch system (4.11) are to be replaced with the following ones:

$$N(k,\xi,\tau)\xrightarrow[\tau\to-\infty]{}1,\quad \lambda(k,\xi,\tau)\xrightarrow[\tau\to-\infty]{}0\ . \tag{4.24}$$

Consequently, by looking at (4.13), we deduce that we still have to select the case (3.3) (boundary values at $-\infty$) with the following values:

$$K_1=0,\quad K_2=1\ . \tag{4.25}$$

The corresponding evolution (3.17) is solved and we obtain

$$E(k,t)=E(k,0)\exp[\pi g(k)t]\ , \tag{4.26}$$

$$\alpha^+(k,t)=\alpha^+(k,0)\exp[i\varphi(k,t)+\frac{\pi}{2}g(k)t]\ , \tag{4.27}$$

$$\varphi(k,t)=\mathrm{P}\int\frac{d\lambda}{\lambda-k}\left[\tfrac{1}{2}g(\lambda)t-\frac{1}{\pi}\ln\frac{1+E(k,t)}{1+E(k,0)}\right]\ , \tag{4.28}$$

$$C_n^+(t)=C_n^+(0)\exp[i\varphi(k_n^+,t)]\ . \tag{4.29}$$

Of course, in this case too, the inhomogeneous factor $g(k)$ is the same as before, that is strictly positive for all k. Therefore the time evolution (the long-distance ξ behavior) of the solution is quite different from the preceding one: $E(k,t)$ grows exponentially. The asymptotic behavior of the solution, in the absence of discrete spectrum (no solitons in the initial pulse), has been given in [14].

We only wish to point out here that the presence of a *bit* of continuous spectrum (or background radiation) is not only always physically realized, but also necessary to be able to propagate solitons *in the right direction.*

To do that we first need to write down the one-soliton solution which is obtained by solving the integral equation (2.15) in the reduction (2.41) with $\sigma=-$, where $R(k)$ is given by (2.12) with $\alpha^{\pm}=0$ and $C_j(0)\neq 0$ for $j=n$. The resulting $q(x,t)$ is then obtained from (2.16) and reads

$$q(x,t)=\frac{-8(\mathrm{Im}k_n^+)^2\bar{C}_n^+(0)e^{\phi}}{4(\mathrm{Im}k_n^+)^2+|C_n^+(0)|^2e^{\phi+\bar{\phi}}}\ , \tag{4.30}$$

$$\phi=2ik_n^+x+\arg C_n^+(t)\ . \tag{4.31}$$

Then the behavior of this soliton is essentially determined by $\arg C_n^+(t)=i\varphi(k_n^+,t)$ in the amplifier case and $\arg C_n^+(t)=-i\theta(k_n^+,t)$ in the SIT case.

Now, the initial datum $q(x,0)$ completely determines the values of the spectral parameters $C_n^+(0)$ and k_n^+ and hence, the same initial datum in both cases produces different soliton dynamics. If we take $\alpha^{\pm}(k,0)$ to be *strictly* vanishing, we have from (4.17)

$$\arg C_n^+(t)=-\frac{i}{2}\int_{-\infty}^{+\infty}\frac{d\lambda}{\lambda-k_n^+}g(\lambda)t \tag{4.32}$$

in the SIT case, and from (4.28)

$$\arg C_n^+(t)=\frac{i}{2}\int_{-\infty}^{+\infty}\frac{d\lambda}{\lambda-k_n^+}g(\lambda)t \tag{4.33}$$

in the amplifier case. Hence the corresponding solitons would propagate at *opposite velocities.*

Such is not the case if we consider also the presence in the spectrum of the radiative part (continuum). In that case the corresponding asymptotic behaviors of the above $\arg C_n^+(t)$ give, respectively,

$$\arg C_n^+(t)\xrightarrow[t\to+\infty]{}-i\int_{-\infty}^{+\infty}\frac{d\lambda}{\lambda-k_n^+}\left[\tfrac{1}{2}g(\lambda)t-\frac{1}{\pi}\ln[1+E(k,0)]\right]\ , \tag{4.34}$$

$$\arg C_n^+(t)\xrightarrow[t\to+\infty]{}-i\int_{-\infty}^{+\infty}\frac{d\lambda}{\lambda-k_n^+}\left[\tfrac{1}{2}g(\lambda)t+\frac{1}{\pi}\ln[1+E(k,0)]\right]\ . \tag{4.35}$$

Hence, apart from the fixed phase shift $\pm(1/\pi)\ln[1+E(k,0)]$, the solitons have asymptotically the same velocities. This is an example where the mathematically *ideal* situation of a pure soliton initial condition would lead to wrong conclusions about the pulse dynamics.

D. Stimulated Raman scattering (SRS)

Since the works of Chu and Scott [15], considerable interest has been devoted to the so-called SRS equations [16] which read in the notations of [15]

$$\begin{aligned}&Y_\tau-i\delta Y=-iA_1\bar{A}_2e^{-i\Delta\kappa\xi}\ ,\\&A_{1,\xi}=-iA_2Ye^{i\Delta\kappa\xi}\ ,\\&A_{2,\xi}=-iA_1\bar{Y}e^{-i\Delta\kappa\xi}\ .\end{aligned} \tag{4.36}$$

Here, A_1 is the scaled slowly varying amplitude of the incident (pump) electromagnetic wave, A_2 is the scaled scattered (Stokes) electromagnetic wave, and Y is the scaled amplitude of the scattering (acoustic) wave. The variables denote the rest frame of the electromagnetic waves (which propagate in the same direction) and $\Delta\kappa$ is the mismatched wave number.

These equations model the same type of nonlinear coupling that was described in Sec. IV A, but in a plasma this process in generally masked by SBS which has a much higher probability. In other domains like the light-pulse propagation in a fiber, this process becomes dominant and is at the origin of lossless pulse propagation.

Again here we consider the above equations as being the sharp-line limit $\gamma(\Delta\kappa)\to\delta(\Delta\kappa)$ of the following system:

$$Y_\tau - i\delta Y = -i\int_{-\infty}^{+\infty} d(\Delta\kappa)\gamma(\Delta\kappa)A_1\overline{A}_2 e^{-i\Delta\kappa\xi} ,$$
$$A_{1,\xi} = -iA_2 Y e^{i\Delta\kappa\xi} , \tag{4.37}$$
$$A_{2,\xi} = -iA_1\overline{Y}e^{-i\Delta\kappa\xi} .$$

This system is now mapped in our system (1.1) through the following transformation:

$$\sigma = -, \quad k = -\tfrac{1}{2}\Delta\kappa, \quad g(k) = 2\gamma(\Delta\kappa) ,$$
$$x = -\xi, \quad t = \tau , \tag{4.38}$$
$$q = iYe^{-i\delta\tau}, \quad a_1 = A_1 e^{-i\delta\tau}, \quad a_2 = A_2 e^{i\Delta\kappa\xi} .$$

The natural boundary values to associate with this system consist of a normalized pump wave in the input zone $\xi = -\infty$ and no Stokes wave coming from the output zone $\xi = +\infty$. Hence (remember $x = -\xi$)

$$a_1 \xrightarrow[x\to+\infty]{} e^{-i\delta t}, \quad a_2 \xrightarrow[x\to-\infty]{} 0 . \tag{4.39}$$

The solution of (1.1) with the above boundary values will then be obtained by choosing, for instance, the case (3.4) with

$$J_1 = e^{-i\delta t}, \quad J_2 = 0 . \tag{4.40}$$

The corresponding evolution of the spectral transform can be solved exactly and we have [remember $E = |\alpha^+|^2$, see (3.21)]

$$E(k,t) = \frac{E(k,0)}{[1+E(k,0)]\exp[\pi g t] - E(k,0)} . \tag{4.41}$$

At this point it is essential to note that $g(k)$ is strictly positive [$g(k) = 2\gamma(\Delta\kappa)$]. Hence

$$E(k,t) \xrightarrow[t\to+\infty]{} 0 , \tag{4.42}$$

which means that the *radiative* part of the spectrum asymptotically vanishes in time. In other words, for an arbitrary initial profile $Y(\xi,0)$ of the acoustic wave, the time-asymptotic state consists of a pure soliton state. It means in particular that the Stokes wave becomes *localized.* Indeed, the formulas (4.38), (3.7), and (2.8) imply

$$|A_2(\Delta\kappa,\xi,\tau)|^2 \to |\alpha^+|^2 = E(\Delta\kappa,\tau) \text{ as } \xi\to-\infty , \tag{4.43}$$

and, as $\tau\to\infty$, $|A_2(\Delta\kappa,-\infty,\tau)|^2\to 0$.

We must stress now the following important points.

(1) In the sharp-line limit case (4.36), which is the case originally studied in [15] and later in all the studies of SRS in the spectral transform scheme [16,19], the boundary values (4.39) would automatically imply for constitency of (4.36) as $\xi\to\pm\infty$,

$$A_2 \to 0 \text{ as } \xi\to-\infty . \tag{4.44}$$

In other words, from (4.43) we would have to impose that the initial acoustic-wave profile $Y(\xi,0)$ be such that $E(\Delta\kappa,0)=0$ and hence $E(\Delta\kappa,\tau)=0$ from (4.41). Hence the sharp-line case is only compatible with a pure soliton initial datum. In that case the time evolution of the spectral data given in [15] is correct. Moreover, this case is not physically relevant as the spreading $\gamma(\Delta\kappa)$ of the resonant band frequency can be as sharp as we want but never a true Dirac δ function.

(2) Our initial–boundary-value problem [more specifically $Y(\xi,\tau)$ given at all ξ and $\tau=0$] is not the one relevant in nonlinear optics where the laboratory coordinates Z,T are given by [16] $\xi = Z, \tau = T - Z/c$. Consequently, a datum at $\tau=0$ has no physical meaning. In order to study in our formalism an initial–boundary-value problem with datum at $T=0$ would require a completely renewed approach of the spectral problem as proposed in [20] and bypasses the purpose of this paper. However, as the boundary values have been shown to play the central role, we think that the singular behavior described here is representative of SRS. Moreover, there are a number of physical situations (such as laser-plasma interaction with a laser-frequency tune at the plasma frequency, or as the nonlinear absorption of radiation in diatomic chains of coupled oscillators) where the present results apply and we plan to report such studies in future work.

(3) A very interesting approach of the system (4.36) with damping has been recently proposed in [21]. It is based on the group-theoretical methods for nonlinear evolution equations which allows the authors to derive explicit particular solutions and their time asymptotic behavior.

E. An integrable system which blows up in finite time

If we consider the system

$$Y_\tau - i\delta Y = i\int_{-\infty}^{+\infty} d(\Delta\kappa)\gamma(\Delta\kappa)A_1\overline{A}_2 e^{-i\Delta\kappa\xi} ,$$
$$A_{1,\xi} = -iA_2 Y e^{i\Delta\kappa\xi} , \tag{4.45}$$
$$A_{2,\xi} = -iA_1\overline{Y}e^{-i\Delta\kappa\xi} ,$$

then its solution is given by the evolution (4.41) with

$$g(k) = -2\gamma(\Delta k) . \tag{4.46}$$

It is clear on (4.41) that the above condition implies that the solution is valid up to the time when E becomes singular, that is for $t < t_s$ with

$$t_s = \frac{1}{2\pi\gamma}\ln\left[1 + \frac{1}{E(k,0)}\right] . \tag{4.47}$$

This singularity corresponds to a scattered wave A_2 which blows up (in finite time) due to an acoustic wave Y where energy goes to infinity. Indeed from (4.43) $E(\Delta\kappa,\tau)$ measures the amplitude of the scattered wave *in the input zone* and hence, as t approaches t_s, the amplitude of the envelope of the scattered wave blows up. Correspondingly, the behavior of the acoustic wave can be evaluated by using the first conservation law related to the system (4.45)

$$\frac{\partial}{\partial\tau}|Y(\xi,\tau)|^2 = \frac{\partial}{\partial\xi}\int d(\Delta\kappa)\gamma(\Delta\kappa)|A_2(\Delta\kappa,\xi,\tau)|^2 , \tag{4.48}$$

which gives by means (4.43)

$$\frac{\partial}{\partial\tau}\int_{-\infty}^{+\infty}d\xi|Y(\xi,\tau)|^2=\int d(\Delta\kappa)\gamma(\Delta\kappa)E(\Delta\kappa,\tau)\ . \quad (4.49)$$

Thanks to the explicit expression (4.41) of the time dependence of E, the above quantity can be integrated with respect to the variable τ and we obtain

$$\int_{-\infty}^{+\infty}d\xi\{|Y(\xi,\tau)|^2-|Y(\xi,0)|^2\} = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}d(\Delta\kappa)\ln[1+E(\Delta\kappa,0)(1-e^{2\pi\gamma\tau})]\ . \quad (4.50)$$

For $\tau>0$, this expression is always positive and

$$\tau\rightarrow\tau_s \Longrightarrow \int_{-\infty}^{+\infty}d\xi|Y(\xi,\tau)|^2\rightarrow\infty \quad (4.51)$$

and hence the energy of the acoustic wave blows up as τ reaches the critical value.

An integrable system is shown here to develop a singularity in a finite time. Such a behavior had been discovered in nonintegrable systems like the Zakharov equation [22] describing the interaction of Langmuir waves with acoustic waves in plasmas, and had been interpreted as being at the origin of plasma turbulence. Consequently, we may wonder if the system (4.45) will show a turbulent behavior for $\tau>\tau_s$ where it is *longer integrable.*

However, the first important question is the applicability of such a system in a real physical situation, where *a priori* the medium should behave like an amplificator. This question is now under study.

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