First-order exact solutions of the nonlinear Schrödinger equation in the normal-dispersion regime

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"First-order" exact solutions of the nonlinear Schrödinger equation (NLSE) with positive group-velocity dispersion are obtained. We find a three-parameter family of solutions that are finite everywhere; particular cases include periodic solutions expressed in terms of elliptic Jacobi functions, stationary periodic solutions, and solutions describing the collision or excitation of two dark solitons with equal amplitudes. A classification of solutions using the plane of their parameters, a geometrical description on the complex plane, and physical interpretations of the solutions obtained are given. A simple relation, which permits transformation of the solutions of the NLSE in the anomalous-dispersion regime into solutions of the NLSE in the normal-dispersion regime, is also discussed.

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I. INTRODUCTION

Since the pioneering works by Zakharov and Shabat [1] and Hasegawa and Tappert [2], dark optical solitons have been an active topic of research both theoretically [3,4] and experimentally [5–9]. All of these studies dealt with solitary pulses of the nonlinear Schrödinger equation (NLSE) in the normal-dispersion regime. Periodic solutions have attracted less attention, even though such solutions could also be important in problems of signal propagation in optical fibers. For the NLSE in the anomalous-dispersion regime, the full classification of the solutions of first order, including periodic solutions and solitary ones, has been given in Ref. [10] using a straight-line relation between the real and imaginary parts of the solution found in [11]. An attempt to make the same kind of classification was made in two recent very similar works by Mihalache and Panoiu [12,13]. Unfortunately, the authors of these works were not able to obtain the solutions in an explicit form or to analyze and interpret them in a useful manner.

The method developed in [10] allowed us to introduce a new type of classification of the exact solutions of NLSE (including periodic solutions) which depends on the order of the polynomial relating the real and imaginary parts of the solution. If the real and imaginary parts of the solutions are related through a polynomial of first order, then they can be called solutions of first order. This relation was suggested in paper [11] and then used for obtaining the solutions in a more elaborate way in [10]. We shall call this the AEKK-ansatz method, from the names of the authors of [10,11].

In this work, we present the full classification of finite first-order exact solutions and introduce periodic solutions of the NLSE with normal dispersion:

$$i\Psi_t-\Psi_{xx}+2|\Psi|^2\Psi=0\ . \tag{1}$$

In studies of spatial solutions and other waves in planar structures, t represents time and x is the transverse dimension, whereas in optical fibers t is the distance along the fiber in soliton units, while x is the "retarded time," indicating that a reference frame moving at the group velocity is being considered.

The construction of the paper is as follows. In Sec. II of this paper we shall write the general form of the three-parameter family of solutions, restricting ourselves solely to finite solutions. In Sec. III we suggest a general classification of first-order solutions on the plane formed by two parameters. In Sec. IV, particular cases of the general solution, which have been reduced to one-parameter families, will be presented. In addition, graphical representations and some physical interpretation will be given. In Sec. V we shall compare the solutions of NLSE in normal- and anomalous-dispersion cases. We shall show that, using the simple transform $x\rightarrow ix$, some solutions corresponding to these two cases can be related to each other.

II. SOLUTION IN GENERAL FORM

The AEKK ansatz can be written in the form [10]

$$\Psi(x,t)=[Q(x,t)+i\sqrt{z(t)}]\exp[i\phi(t)]\ , \tag{2}$$

where $z(t)$ is the solution of the equation

$$z_t^2=-64z(z-a_1)(z-a_2)(z-a_3)\ ,\qquad a_1<a_2<a_3\ ,\quad a_3>0\ , \tag{3}$$

with three parameters a_1, a_2, and a_3. In general, the restrictions on these parameters are the same as in [10]. Here, we consider them to be real and positive.

We can think of the solutions of Eq. (3) as oscillations of a particle with zero energy in a potential well described by the polynomial $U(z)=64z(z-a_1)(z-a_2)(z-a_3)$ [see Fig. 1(a)]. One can see easily that the solutions of Eq. (3) are always bounded. The function Q is now the solution of the equation [10]

$$Q_x^2=(Q-Q_1)(Q-Q_2)(Q-Q_3)(Q-Q_4)\ , \tag{4}$$

where the roots of the polynomial on the right-hand side are given by

$$Q_1 = \sqrt{a_1 - z} + \sqrt{a_2 - z} + \sqrt{a_3 - z} \ ,$$
$$Q_2 = -\sqrt{a_1 - z} - \sqrt{a_2 - z} + \sqrt{a_3 - z} \ ,$$
$$Q_3 = -\sqrt{a_1 - z} + \sqrt{a_2 - z} - \sqrt{a_3 - z} \ ,$$
$$Q_4 = \sqrt{a_1 - z} - \sqrt{a_2 - z} - \sqrt{a_3 - z} \ ,$$

where $Q_4 < Q_3 < Q_2 < Q_1$. The solutions of Eq. (4) can be viewed as oscillations of a particle with zero energy in a potential well specified by the polynomial $V(Q) = -(Q - Q_1)(Q - Q_2)(Q - Q_3)(Q - Q_4)$ [see Fig. 1(b)]. This potential decreases to minus infinity at $Q > Q_1$ and $Q < Q_4$, and its only minimum occurs between the roots Q_3 and Q_2. Hence, all finite solutions can be written as

$$Q = \frac{Q_2(Q_1 - Q_3) - Q_1(Q_2 - Q_3)\mathrm{sn}^2(\beta x, k)}{(Q_1 - Q_3) - (Q_2 - Q_3)\mathrm{sn}^2(\beta x, k)} \ , \tag{5}$$

where $\underline{Q_3 < Q < Q_2}$, $\beta = \sqrt{a_3 - a_1}$, and $k = \sqrt{(a_3 - a_2)/(a_3 - a_1)}$, assuming that the roots Q_3 and Q_2 are real. The solution (5) represents all cases of interest (finite solutions). However, it can be written in a different form by shifting the argument or by using a double-argument transform.

The roots Q_i are real only if all three a_i are positive and $0 < z < a_1$. In this case at least Q_1 is positive. The only difference of this case from the case of anomalous dispersion [10] is the sign of the right-hand side of Eq. (4) for Q [i.e., the sign of polynomial $V(Q)$]. Thus, the solutions of this equation are located in intervals which are complementary to those in the case considered in [10]. Now, only the solution located between the roots Q_3 and Q_2 is finite. In contrast to the case considered in [10], Eq. (4) can have not only finite, but also additional singular solutions located outside the roots Q_1 and Q_4. We shall consider in this paper only solutions which are finite everywhere. Singular ones probably have physical interest only in special cases such as layered media, where they can be matched with other types of solutions on boundaries in such a way as to avoid the singularities. In the case of complex parameters a_1 and a_2, the potential $V(Q)$ has only two real roots, and the solutions of Eq. (4) can only be singular. Taking this into account, we restrict ourselves to the cases where all three parameters a_i are real and the roots Q_i are also real.

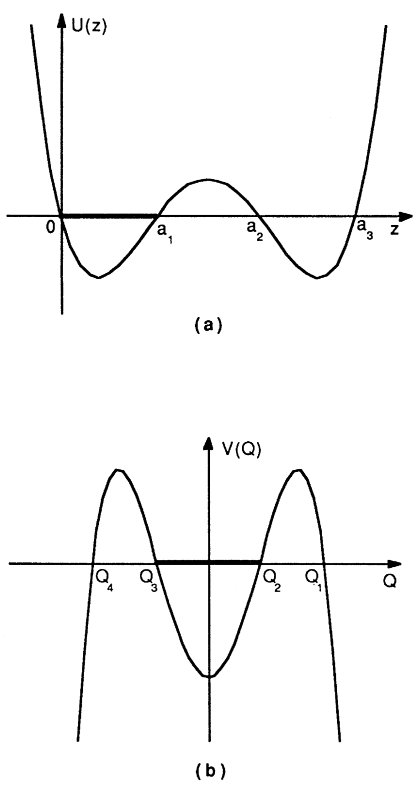

FIG. 1. Potential (a) $U(z)$ and (b) $V(Q)$. Bold lines show the regions where the functions $z(t)$ and $Q(x)$ can provide solutions $\Psi(x,t)$ which are finite everywhere.

This can happen if the solution of Eq. (3) for $z(t)$ is located between the zero root and the root a_1. We obtain

$$z(t) = \frac{a_1 a_3 \mathrm{sn}^2(\mu t, k)}{a_3 - a_1 \mathrm{cn}^2(\mu t, k)} \ , \tag{6}$$

where $\mu = 4\sqrt{a_2(a_3 - a_1)}$ and $k^2 = a_1(a_3 - a_2)/a_2(a_3 - a_1)$. The function $\phi(t)$ is then found to be

$$\phi(t) = \int_0^t (W - 4z)dt$$
$$= 2(a_1 + a_2 - a_3)t + \frac{4a_3}{\mu}\Pi(n; \mu t, k) \ , \tag{7}$$

where $\Pi(n; \mu t, k) = \int_0^{\mu t} d\tau / [1 - n\,\mathrm{sn}^2(\tau, k)]$ is the elliptic integral of the third kind, and $n = a_1/(a_1 - a_3)$.

The formulas (2), (5), (6), and (7) provide the general finite first-order solutions of Eq. (1). This is a three-parameter family of solutions. The squares of the absolute values of the solutions are periodic, with single periods along the x and t axes. These two periods can be changed independently by adjusting parameters and we obtain a general periodic solution with two periods as independent parameters. The third parameter allows us to adjust the amplitude. The period in x can be extracted from Eq. (5):

$$T_x = \frac{2}{\sqrt{a_3 - a_1}} K \left[k = \left[\frac{a_3 - a_2}{a_3 - a_1} \right]^{1/2} \right] ,$$

where K is the complete elliptic function of the first kind, while the period in t is found from Eq. (6):

$$T_t = \frac{1}{2\sqrt{a_2(a_3 - a_1)}} K \left[\left[\frac{a_1(a_3 - a_2)}{a_2(a_3 - a_1)} \right]^{1/2} \right] .$$

Both periods become infinite on the line $a_1 = a_2$. This will lead to soliton solutions, as we will see later. However, they can be treated as limiting cases of periodic solutions. All three parameters can be scaled by the same constant, for example q. Thus, if we set $a_1' = a_1/q$, $a_2' = a_2/q$, and $a_3' = a_3/q$, then the whole solution will be scaled in such a way that $\Psi' = \Psi/q$, $t' = 2q^2 t$, and

$x'=qx$. This transformation allows us to restrict ourselves to a two-dimensional space of parameters when analyzing particular cases, because the scaling does not affect the physical nature of the solution. Scaling transformations of the NLSE have been discussed in various works, e.g., Ref. [14], where scaling of the coefficient of coupling between two optical fibers is also included. The general solution can be simplified and written in terms of elementary functions in some special cases. We shall classify all of them and reduce them to simpler forms in the remaining sections.

III. GENERAL CLASSIFICATION OF THE SOLUTIONS

The solution of the NLSE in the form given by Eqs. (2), (5), (6), and (7) is general, and describes a variety of different physical phenomena. Thus, we need to classify these solutions and simplify them, giving forms which are convenient to use. In this paper we will give a few simple forms of the solutions. To do this, we classify the solutions with respect to parameters a_1, a_2, and a_3. We shall reduce the three-parameter family of solutions to several one-parameter-family solutions. Each of these families describes particular physical phenomena.

Let us consider the plane of parameters (a_1,a_2) (Fig. 1). Because of the symmetry of these parameters in Eq. (3), we can see that if they are interchanged, then the solution will retain its form. For reasons of convenience, we have arranged them in such a way that $a_1<a_2<a_3$. This means that we need only consider the solutions inside the triangle OAB in Fig. 2. All other solutions of the three-parameter family can be written down just by using permutations of these three parameters. We have listed all cases which can be simplified in this figure. At the points of intersection, the solutions of different one-parameter families clearly must coincide. These interactions are designated in Fig. 2 by open circles. Now we shall present the solutions.

IV. PARTICULAR CASES

A. Periodic solutions

Let $a_1+a_2=a_3$. In this case the function z can be converted, using the Landen transformation [15], into (see Appendix)

$$z(t)=a_3\frac{\kappa^2\mathrm{sn}^2(2a_3t,\kappa)\mathrm{cn}^2(2a_3t,\kappa)}{1-\kappa^2\mathrm{sn}^4(2a_3t,\kappa)}=\frac{a_3\kappa^2}{2}\frac{\mathrm{sn}(2a_3t,\kappa)\mathrm{cn}(2a_3t,\kappa)\mathrm{sn}(4a_3t,\kappa)}{\mathrm{dn}(2a_3t,\kappa)}\ , \quad (8)$$

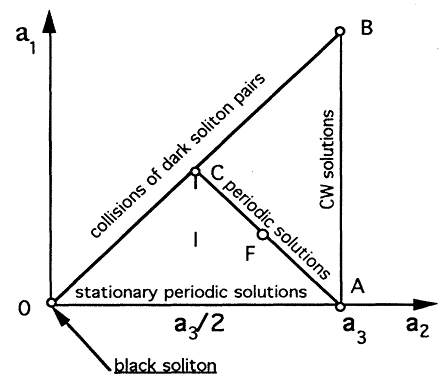

FIG. 2. Illustration of the location of solutions of various types in the plane of parameter space (a_1,a_2).

where the modulus of the elliptic functions is $\kappa=2\sqrt{a_1a_2}/a_3$. The equation for the phase derivative becomes

$$\varphi_t=W-4z(t)=4a_3-4a_3\frac{\kappa^2\mathrm{sn}^2(2a_3t,\kappa)\mathrm{cn}^2(2a_3t,\kappa)}{1-\kappa^2\mathrm{sn}^4(2a_3t,\kappa)}=\frac{4a_3\mathrm{dn}^2(2a_3t,\kappa)}{1-\kappa^2\mathrm{sn}^4(2a_3t,\kappa)}=2a_3[1+\mathrm{dn}(4a_3t,\kappa)]\ , \quad (9)$$

and the phase itself is now

$$\varphi=\int_0^{2a_3t}[1+\mathrm{dn}(2y,\kappa)]dy=2a_3t+\tfrac{1}{2}\int_0^{4a_3t}\mathrm{dn}(u,\kappa)\,du=2a_3t+\tfrac{1}{2}\arcsin[\mathrm{sn}(4a_3t,\kappa)]\ . \quad (10)$$

If we let $\theta=\arcsin[\mathrm{sn}(4a_3t,\kappa)]$, then $e^{i\varphi}=e^{i\theta/2}e^{2ia_3t}$, so $e^{i\varphi}=e^{2ia_3t}[\sqrt{1+\mathrm{cn}(4a_3t,\kappa)}+i\sqrt{1-\mathrm{cn}(4a_3t,\kappa)}]/\sqrt{2}$. We can write

$$\sqrt{a_3-z}=\frac{\sqrt{a_3}\mathrm{dn}(2a_3t,\kappa)}{\sqrt{1-\kappa^2\mathrm{sn}^4(2a_3t,\kappa)}}=\left[\frac{a_3}{2}[1+\mathrm{dn}(4a_3t,\kappa)]\right]^{1/2}\ , \quad (11)$$

and similar expressions for the other terms. We note that $z(t)$ of Eq. (8) can also be simplified:

$$z(t)=\tfrac{1}{2}a_3\kappa^2[1+\mathrm{cn}(4a_3t,\kappa)]\mathrm{sn}^2(2a_3t,\kappa)\ .$$

Thus the periodic solution of the NLSE reduces to the following remarkably simple form:

$$\Psi(x,t)=\kappa\sqrt{a_3}\frac{\mathrm{cn}(2a_3t,\kappa)-i\sqrt{1+\kappa}\,\mathrm{sn}(2a_3t,\kappa)\mathrm{dn}\left[\sqrt{a_3(1+\kappa)}x,\left[\frac{2\kappa}{1+\kappa}\right]^{1/2}\right]}{\sqrt{1+\kappa}\,\mathrm{dn}\left[\sqrt{a_3(1+\kappa)}x,\left[\frac{2\kappa}{1+\kappa}\right]^{1/2}\right]+\mathrm{dn}(2a_3t,\kappa)}\exp[i2a_3t]\ . \quad (12)$$

This is a two-parameter family of solutions, but by using the scaling transformation $\Psi'=\Psi/\sqrt{a_3}$, $t'=2a_3t$, and $x'=\sqrt{a_3}x$, it can be reduced to a one-parameter family of solutions located along the line AC in Fig. 2. Then Eq. (12) can be written down with κ being the only parameter. Thus

$$|\Psi'(x',t')|^2=\kappa^2\frac{\mathrm{cn}^2(t',\kappa)+(1+\kappa)\mathrm{sn}^2(t',\kappa)\mathrm{dn}^2\left[\sqrt{(1+\kappa)}x',\left[\frac{2\kappa}{1+\kappa}\right]^{1/2}\right]}{\left\{\sqrt{1+\kappa}\,\mathrm{dn}\left[\sqrt{1+\kappa)}x',\left[\frac{2\kappa}{1+\kappa}\right]^{1/2}\right]+\mathrm{dn}(t',\kappa)\right\}^2}\ . \tag{12a}$$

We note that the minimum possible value of the numerator is $\kappa^2(1-\kappa)$. This is strictly positive when $\kappa \ll 1$, thus showing that if we exclude the point C itself, then the intensity is never zero along the line AC. The period in t' is then $2K(\kappa)$, while the period in x' is $(2/\sqrt{1+\kappa})K(\sqrt{2\kappa/(1+\kappa)})$. Changing the parameter will change the period of the function along the x and t axes. As we move from A to F to C, we note that κ increases from 0 to 1, and thus, correspondingly, the periods in x and t both increase. At point C these periods become infinite and the periodic nature is lost. In Fig. 3 we present the periodic solution at the point F of Fig. 2. At $t=0$, the field amplitude is found from Eq. (12):

$$\Psi(x,t=0)=\frac{\kappa\sqrt{a_3}}{\sqrt{1+\kappa}\,\mathrm{dn}\left[\sqrt{a_3(1+\kappa)}x,\left[\frac{2\kappa}{1+\kappa}\right]^{1/2}\right]+1}\ . \tag{12b}$$

If κ is small (i.e., near A on the line AF in Fig. 2) then we can expand the dn function and observe that we get a slight modulation superimposed on a fixed amplitude:

$$\Psi(x,t=0)\approx\frac{\kappa\sqrt{a_3}}{2}\left[1-\frac{\kappa}{4}\cos[2\sqrt{a_3(1+\kappa)}x]\right]\ .$$

It is clear that this is a small ripple on a constant value. The field evolves periodically with t and the energy swaps between the periods in x. The dependence of the periods of the solution $|\Psi'|^2$ of Eq. (12a) on κ is shown in Fig. 4.

B. Stationary periodic solution

We now consider the case $a_1=0$. In this case we necessarily have $z=0$. Hence, the Q function does not depend on t. The phase function can be written in the form

$$\varphi=Wt=2(a_2+a_3)t\ . \tag{13}$$

The Q function is

$$Q=\frac{2k\sqrt{a_3}}{1+k}\mathrm{sn}\left[\frac{2\sqrt{a_3}}{1+k}x,k\right], \tag{14}$$

where $k=(\sqrt{a_3}-\sqrt{a_2})/(\sqrt{a_3}+\sqrt{a_2})$. Thus the periodic stationary solution is

$$\Psi(x,t)=\frac{2k\sqrt{a_3}}{1+k}\mathrm{sn}\left[\frac{2\sqrt{a_3}}{1+k}x,k\right]\exp\left[4ia_3\frac{1+k^2}{(1+k)^2}t\right]\ . \tag{15}$$

Apart from the phase factor $\exp\{4ia_3(1+k^2)/[(1+k)^2]t\}$, this solution does not depend on t. Defining

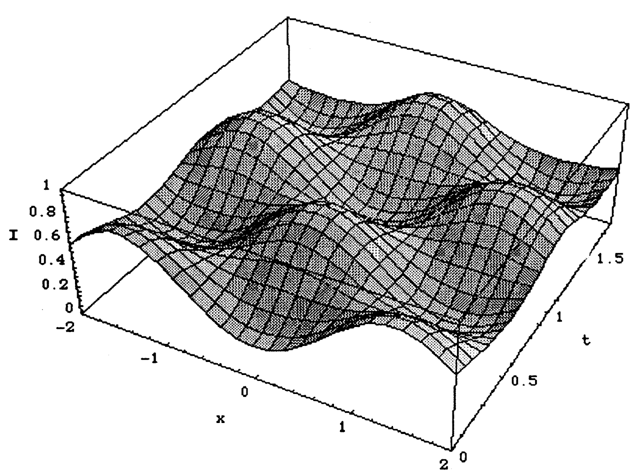

FIG. 3. Example of a periodic solution. This is for the point F in Fig. 2 ($a_1=a_3/4$, $a_2=3a_3/4$, $a_3=2$).

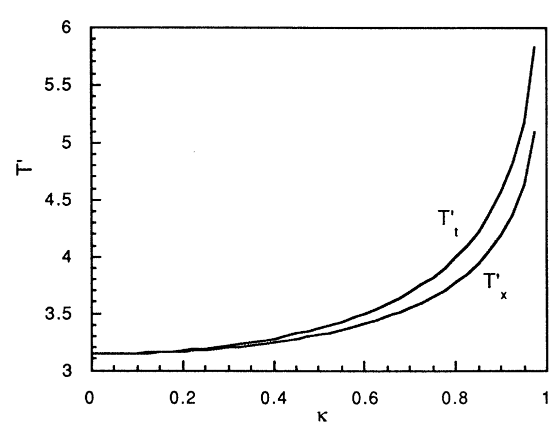

FIG. 4. Periods of $|\Psi'|^2$ given by Eq. (12a) in the x and t directions. Note that κ increase from 0 to 1 as we move from A (where each period is π) to F to C of Fig. 2. At point C ($\kappa=1$) the periods become infinite.

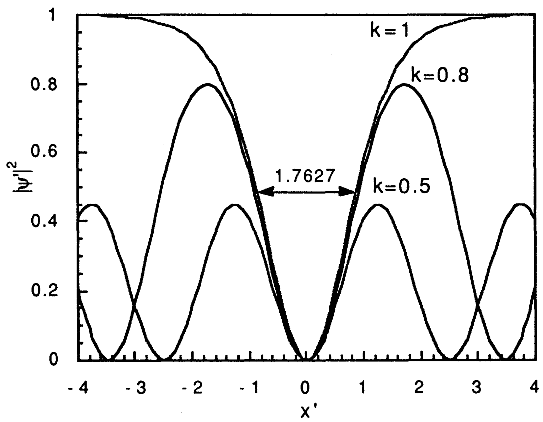

FIG. 5. Stationary periodic solutions. The black soliton ($k=1$) has a single minimum, while the other two values give periodic solutions. The amplitude decreases as k decreases.

$\Psi'=\Psi/\sqrt{a_3}$ and $x'=\sqrt{a_3}x$ shows that $|\Psi'|^2$ depends only on one parameter, namely, k. Thus, the absolute value of the solution is stationary and periodic. This solution corresponds to a cnoidal wave. In an optical fiber, this would represent an unchanging periodic signal propagating along the link. When $k=0$ (point A in Fig. 2), it reduces to the trivial zero solution. Along the line AO in Fig. 2, we see that k increases and reaches 1 at the origin. Then we have the black soliton $\Psi'=\tanh(x')\exp(it')$, where $t'=2a_3t$. We plot $|\Psi'|^2$ for various k in Fig. 5. In close proximity of the point $k=1$, the solution is essentially a train of black solitons with the distance between them increasing as k approaches 1.

For k approaching 1, this limit of the solution looks like a sequence of black solitons with a constant background. In this case, the period of $|\Psi'|^2$, viz., $2K(k)$, increases without limit, while the full width at half height of each single dip is 1.7627, as illustrated in Fig. 5. If we define σ as the ratio of this period to the full width, then for k near 1,

$$\sigma \approx 0.5673 \ln \left[\frac{16}{1-k^2} \right] .$$

Alternatively, the k required to produce a given ratio is found from

$$k^2 \approx 1 - 16\exp(-1.7627\sigma) .$$

As with the case of bright solitons in the anomalous dispersion case [16], we note that if $\sigma > 5$, then the individual pulses will have very little interaction. In the black soliton pulse train, each pulse has a phase difference of π from each of its neighbors.

C. Continuous wave (CW) (solutions independent of x)

We now turn to the case $a_2=a_3$. The function $z(t)$ [see Eq. (6)] now becomes

$$z(t)=\frac{a_1a_3\sin^2(\mu t)}{a_3-a_1\cos^2(\mu t)} , \quad \mu=4\sqrt{a_3(a_3-a_1)} , \tag{16}$$

and the finite solution of Eq. (5) does not depend on x:

$$Q=Q_2=Q_3=-\sqrt{a_1-z} = \left[\frac{a_1(a_3-a_1)}{a_3-a_1\cos^2(\mu t)} \right]^{1/2} \cos(\mu t) . \tag{17}$$

The equation for the phase function is

$$\varphi_t = W - 4z(t) = 2a_1t + \frac{4a_3(a_3-a_1)}{a_3-a_1\cos^2(\mu t)} . \tag{18}$$

It solution is

$$\varphi = 2a_1t + \arctan\left[\left[\frac{a_3}{a_3-a_1} \right]^{1/2} \tan(\mu t) \right] . \tag{19}$$

Noting that

$$e^{i\arctan\xi} = \frac{1+i\xi}{(1+\xi^2)^{1/2}} \tag{20}$$

allows us to use the following representation:

$$e^{i\varphi} = \frac{[p\cos(\mu t)+i\sqrt{a_3}\sin(\mu t)]}{[a_3-a_1\cos^2(\mu t)]^{1/2}} e^{2a_1t} . \tag{21}$$

The solution we get is

$$\Psi=\sqrt{a_1}\exp[i2a_1t] . \tag{22}$$

These solution are CW waves with an amplitude $\sqrt{a_1}$, where a_1 is the parameter for this family. They are located along the vertical line AB in Fig. 2. This solution is obvious and does not need illustration.

D. Collision of two dark solitons

An important special case, showing the approach, collision, and continued propagation of two dark solitons, can be investigated by considering $a_1=a_2$. Then we have

$$Q_x^2=(Q-Q_1)(Q-Q_2)(Q-Q_3)^2 , \tag{23}$$

where $Q_1=2\sqrt{a_1-z}+\sqrt{a_3-z}$, $Q_2=-2\sqrt{a_1-z}+\sqrt{a_3-z}$, and $Q_3=Q_4=-\sqrt{a_3-z}$. Now $z(t)$ reduces to

$$z(t)=\Delta a_1a_3\sinh^2(\mu t) , \tag{24}$$

where $\Delta=1/[a_3\cosh^2(\mu t)-a_1]$ and $\mu=4\sqrt{a_1(a_3-a_1)}$. The solution of Eq. (23) in this case is

$$Q=\frac{Q_2(Q_1-Q_3)-Q_1(Q_2-Q_3)\tanh^2(px)}{(Q_1-Q_3)-(Q_2-Q_3)\tanh^2(px)} ,$$
$$Q_3<Q<Q_2 , \quad p=\sqrt{a_3-a_1} . \tag{25}$$

We can simplify this by noting that

$$Q_{1,2}=\pm 2p\sqrt{a_1\Delta}-Q_3 ,$$

where + is for subscript 1 and − is for subscript 2, and

$$Q_3=-p\sqrt{a_3\Delta}\cosh(\mu t) .$$

This allows us to rewrite Q:

$$Q=p\sqrt{\Delta}\frac{a_3\cosh^2(\mu t)-2a_1-\sqrt{a_1a_3}\cosh(\mu t)\cosh(2px)}{\sqrt{a_3}\cosh(\mu t)+\sqrt{a_1}\cosh(2px)} . \tag{26}$$

The equation for the phase function is

$$\varphi_t=W-4z(t)=4a_1+2a_3-4z(t)=4a_1+2a_3-4\Delta a_1a_3\sinh^2(\mu t)=\frac{\Delta\mu^2}{4}+2a_3 , \tag{27}$$

Its solution is

$$\varphi=2a_3t+\arctan\left[\frac{\sqrt{a_1}}{p}\tanh(\mu t)\right] . \tag{28}$$

Using the identity in Eq. (21) allows us to obtain the following representation:

$$e^{i\varphi}=\sqrt{\Delta}[p\cosh(\mu t)+i\sqrt{a_1}\sinh(\mu t)]e^{2ia_3t} . \tag{29}$$

The field [Eq. (2)] is now found by multiplying the above phase factor (29) with the complex function $Q+i\sqrt{z}$ obtained from Eqs. (24) and (26) above. Thus we again obtain a very simple form:

$$\Psi(x,t)=\frac{(2a_3-4a_1)\cosh(\mu t)-2\sqrt{a_1a_3}\cosh(2px)+i\mu\sinh(\mu t)}{2\sqrt{a_3}\cosh(\mu t)+2\sqrt{a_1}\cosh(2px)}e^{i2a_3t} , \tag{30}$$

where $a_3>a_1$ and μ and p are given above.

The singular solution can be obtained from this one simply by changing the sign in front of the $\cosh(2px)$ terms in the numerator and denominator of Eq. (30). The asymptotic background level of this function, at plus or minus infinity, is $-\sqrt{a_3}$. The solution is symmetric relative to the plane $x=0$, while the mirror image in the plane $t=0$ produces the complex conjugate. A two-dark-soliton solution, in a different form, was obtained earlier by Blow and Doran [3]. After some transformations it can be simplified to the form of Eq. (30).

Let us consider limiting cases. Suppose x and t are large and positive. In this case we can approximate the hyperbolic functions by exponential ones:

$$\Psi(x,t)=\frac{(a_3-2a_1)-\sqrt{a_1a_3}\exp[2p(x-2\sqrt{a_1}t)]+i2\sqrt{a_1(a_3-a_1)}}{\sqrt{a_3}+\sqrt{a_1}\exp[2p(x-2\sqrt{a_1}t)]}e^{i2a_3t} . \tag{31}$$

This is obviously the solution for a single dark soliton [1] moving with velocity $2\sqrt{a_1}$ to the right. Because of symmetry, the mirror-image soliton is moving to the left:

$$\Psi(x,t)=\frac{(a_3-2a_1)-\sqrt{a_1a_3}\exp[2p(-x-2\sqrt{a_1}t)]+i2\sqrt{a_1(a_3-a_1)}}{\sqrt{a_3}+\sqrt{a_1}\exp[2p(-x-2\sqrt{a_1}t)]}e^{i2a_3t} . \tag{32}$$

Due to the initial formulation of this approach, it is convenient to represent the dark soliton on a complex plane of solutions (Fig. 6). In this presentation we ignore the exponential factor $\exp(i2a_3t)$, which produces a fast rotation around the origin. The circle in this figure has radius $\sqrt{a_3}$, meaning that the background intensity is a_3. The backgrounds at infinity and in the region between the solitons can be located at any two points on this circle. In the particular case of Fig. 6, these are the points A and B. All points of the solution at any fixed time t are located on the straight line connecting these two points. The center of the soliton is located at the point C closest to the origin. The distance OC is equal to $\sqrt{a_1}$. We can define the (x,t) values corresponding to this point by specifying

$$\exp[2p(\pm x-2\sqrt{a_1}t)]=\left[\frac{a_3}{a_1}\right]^{1/2} . \tag{33}$$

Differentiation of $|\Psi|^2$ also shows that the intensity has a minimum when this condition holds. This minimum intensity is actually a_1. Equation (33) means that solitons are propagating at infinity in such a way that

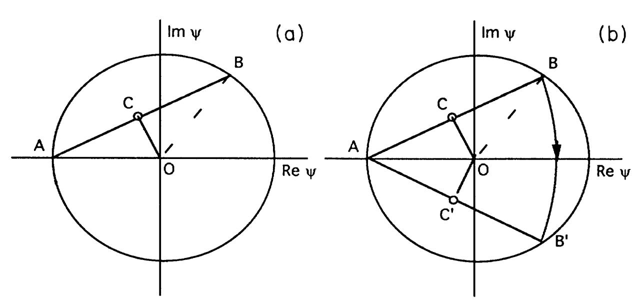

FIG. 6. Geometric representation of (a) dark solitons and (b) collision of dark solitons as a rotation of the line AB in the complex plane. The distance OC is equal to $\sqrt{a_1}$, while the radius OA has a length $\sqrt{a_3}$.

$$x=\pm 2\sqrt{a_1}t\pm\frac{1}{4p}\ln\frac{a_3}{a_1}\ . \tag{34}$$

Hence, the velocity of each soliton is $2\sqrt{a_1}$, and each is shifted, after the collision, by the value

$$\Delta x=\frac{1}{2p}\ln\frac{a_3}{a_1}\ , \tag{35}$$

relative to the place where it would have been if no collision had occurred. Thus, with higher values of a_1, the x values increase more rapidly, and plots require a wider range for x when a_1 is larger. Clearly the shift Δx increases as a_1 decreases. Thus, high-speed dark solitons pass through one another with little interaction, while lower-speed (lower a_1) solitons affect each other more (Fig. 7). From Eq. (30) the field value at $t=0$ never changes sign if $a_3-2a_1<\sqrt{a_1a_3}$, i.e., if $a_1/a_3>0.25$. On the other hand, when $a_1/a_3<0.25$, the field has symmetrically placed zeros and thus a maximum in intensity at $x=0$. As a_1/a_3 decreases, the central maximum gets higher, and approaches the background level a_3 as $a_1\to 0$. Physically, we can regard this as the repulsion of slowly moving dark solitons (at low a_1), rather than an actual collision (as occurs at high a_1). This transition happens gradually as a_1 is changed, and can be seen as an analogy with equally charged particles or balls coming near to each other—high-speed particles continue almost in straight lines, while slow ones deviate from straight lines to curved paths to repel each other.

The soliton phases change in the cross sections from point A at minus infinity, to the point B in the region between the solitons, and back again to the point A at plus infinity. Thus, the background phase outside the solitons remains constant. During the collision of the solitons, the straight line moves on the plane in such a way that a point A stays on the circle but the point B moves to the other side of the circle so that the solitons exchange their phase shifts. The length of the straight line is equal to

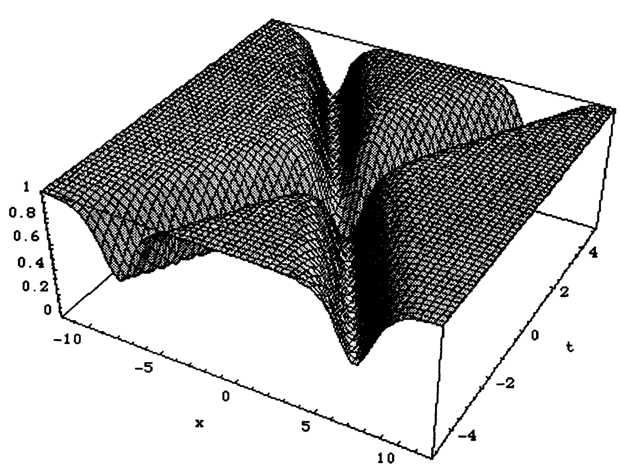

FIG. 7. Collision of two dark solitons ($a_3=1$, $a_1=a_2=0.5$; point C in Fig. 2).

$$\Delta 1=2\sqrt{(a_3-a_1)}\left[\frac{\sqrt{a_3}\cosh(\mu t)-\sqrt{a_1}}{\sqrt{a_3}\cosh(\mu t)+\sqrt{a_1}}\right]^{1/2}.$$

The velocity of each soliton cannot exceed $2\sqrt{a_3}$. In the limit $a_1=a_3$, the solitons disappear. These asymptotic results can be obtained using a standard inverse-scattering approach [1]. In our case, we have the full exact solution which describes the collision in more detail at the point of impact.

This solution simplifies in the case of $a_1=a_3/2$:

$$\Psi(x,t)=\frac{q}{\sqrt{2}}\frac{-\cosh(qx)+i\sqrt{2}\sinh(q^2t)}{\sqrt{2}\cosh(q^2t)+\cosh(qx)}e^{iq^2t}\ , \tag{36}$$

where $q=\sqrt{2a_3}$. This also agrees with minus 1 times the limit $a_1=a_2=a_3/2$ of Eq. (12) in Sec. I. The point B in Fig. 6 is 90° out of phase with point A in this case.

1. Applications: Excitation of pairs of dark solitons using symmetric initial conditions

The form of Eq. (30) at $t=0$ is

$$\Psi(x,0)=\frac{(a_3-2a_1)-\sqrt{a_1a_3}\cosh(2px)}{\sqrt{a_3}+\sqrt{a_1}\cosh(2px)}\ . \tag{37}$$

Equation (37) represents a symmetric function which gives an excitation of paris of dark solitons in the Cauchy problem without any additional radiation. It is known that any other initial condition produces some radiation along with the solitons [4]. Hence, the initial condition (37) can be considered as optimal for exciting a pair of dark solitons. The critical value of a_1 is $a_1=a_3/4$. Then

$$\Psi(x,t=0)=\sqrt{a_3}\frac{[1-\cosh(\sqrt{3a_3}x)]}{[2+\cosh(\sqrt{3a_3}x)]}$$

$$=-2\sqrt{a_3}\frac{\sinh^2\left[\frac{\sqrt{3a_3}}{2}x\right]}{\left[1+2\cosh^2\left[\frac{\sqrt{3a_3}}{2}x\right]\right]}\ . \tag{38}$$

In this case the function is equal to zero at the center, $x=0$, and this is the only minimum. Below this critical value, the initial condition (37) has two zeros. Above this value the function is always positive, but still has a single minimum at $x=0$.

E. Black soliton

In the $a_1\to 0$ limit of the solution given by Eq. (30) (i.e., approaching the origin of Fig. 2 along the line CO), we have two black solitons, with the distance between them approaching infinity. Hence, the solution in the form of one black soliton can be obtained only if we ensure that one of these solitons is at $x=0$. The same applies for Eq. (5). To have a solution for Q corresponding to one black soliton, the solution (5) must be shifted along the x axis by a quarter of the period of the elliptic functions. Then, after setting the limit $a_1=a_2=0$, we get

$$Q=\sqrt{a_3}\tanh(\sqrt{a_3}x) .$$

Obviously the solution of the NLSE corresponding to this $Q(x)$ is the black soliton

$$\Psi=\sqrt{a_3}\tanh(\sqrt{a_3}x)\exp(i2a_3t) . \quad (39)$$

This agrees with the $k=1$ solution found in Sec. IV B, which was the limiting case of the stationary periodic solution (i.e., approaching the origin of Fig. 2 along the line AO), when the period has become infinite.

V. CORRESPONDENCE BETWEEN THE SOLUTIONS OF NLSE IN NORMAL- AND ANOMALOUS-DISPERSION REGIMES

Obviously, if we make the substitution $x\rightarrow ix$ in Eq. (1), we obtain the NLSE in the anomalous-dispersion regime:

$$i\Psi_t+\Psi_{xx}+2|\Psi|^2\Psi=0 . \quad (1')$$

We can expect, then, that the solutions of Eq. (1′) with the same transformation $x\rightarrow ix$ will give the solutions of Eq. (1). For example, the bright soliton solution of Eq. (1′), namely, $\Psi(x,t)=q\,\mathrm{sech}(qx)\exp(iq^2t)$, on transformation gives the solution $\Psi(x,t)=q\sec(qx)\exp(iq^2t)$, which is indeed a solution of Eq. (1). However, it is clearly a singular solution which up to now has not found wide physical application. Considering this simple example, we can conclude that any solution of first order of Eq. (1a) found in [10] can be transformed into a solution of Eq. (1). However, to find finite solutions, we have to be careful, and in using the transformation $x\rightarrow ix$, we have to select the proper signs for square roots of the parameters a_i. For example, this is relevant for the solution given by Eq. (38) of [10], describing the process of modulation instability. Thus, in order to obtain the finite solution given by Eq. (30) of this work, we have to choose the opposite sign in front of the $\sqrt{a_1}$ terms relative to those chosen in Eq. (38) of [10]. If this is not done, the solution obtained again will be singular. In principle, all the solutions of Eq. (1′) can be obtained from the solutions of Eq. (1) using the simple transformation above, but care must be taken with signs in order to do this correctly. Also, we should remember that there are no finite solutions corresponding to the solutions of Eq. (1′) for some values of the parameters a_i. These cases are easily seen by considering the potential $V(Q)$.

Comparing the solutions of Eqs. (1) and (1′), we can see that there is no overall physical equivalence of the solutions corresponding to the same values of parameters a_i. For example, the black soliton solution of Eq. (1) and the bright soliton solution of Eq. (1′) are located at different points of the plane (a_1,a_2). The solution of Eq. (1) describing the collision between two dark solitons corresponds to the solution of Eq. (1′) describing the modulation instability of the CW solution. Hence, the simple transformation $x\rightarrow ix$ converts the solutions in such a way that they describe totally different physical processes.

VI. CONCLUSIONS

We have shown that the straight-line relation between real and imaginary parts is useful in obtaining, as well as in analyzing, the solutions for the NLSE in the normal-dispersion regime. It allows for a clear geometric interpretation of the solutions on the plane of parameters as well as in the complex plane of solutions.

Of all periodic (and soliton) solutions of NLSE in the normal-dispersion regime with single periods along each axis, these are the simplest exact solutions. They can be utilized as a basis for the construction of solutions of higher order describing more complicated multiperiod physical phenomena. Certain nonlinear transformations, for example the Darboux transformation [17], can be used for this purpose.

In conclusion, a classification of "first-order" exact solutions of nonlinear Schrödinger equation with positive group-velocity dispersion has been presented. We find a three-parameter family of solutions which are finite everywhere; particular cases include periodic solutions expressed in terms of elliptic Jacobi functions, stationary periodic solutions, and solutions describing the collision of two dark solitons with equal amplitudes. Classification of singular solutions can be done using the techniques developed here for finite solutions.

Note added in proof. Recently, the particular solution of the NLSE describing the collision of two dark solitons [our Eq. (30)] was obtained by Gagnon [18]. We are grateful to Dr. Gagnon for sending us the manuscript of this work prior to publication.

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APPENDIX

We have, from Eq. (5),

$$z(t)=\frac{a_1a_3\mathrm{sn}^2(\mu t,k)}{a_3-a_1\mathrm{cn}^2(\mu t,k)} , \quad (A1)$$

where $\mu=4\sqrt{a_2(a_3-a_1)}$ and $k^2=a_1(a_3-a_2)/[a_2(a_3-a_1)]$. In the case $a_2+a_1=a_3$, Eq. (A1) can be written as

$$z(t)=\frac{a_1a_3\mathrm{sn}^2(\mu t,k)}{a_2+a_1\mathrm{sn}^2(\mu t,k)} , \quad (A2)$$

where $\mu=4a_2$ and $k=a_1/a_2$. The Landen (Gauss) transform provides the following transformation:

$$\mathrm{sn}(\mu t,k)=\frac{2a_2\mathrm{sn}(2a_3t,\kappa)\mathrm{cn}(2a_3t,\kappa)}{a_3\mathrm{dn}(2a_3t,\kappa)},\quad \kappa=\frac{2\sqrt{a_1a_2}}{a_3} . \quad (A3)$$

Hence,

$$z(t)=a_3\frac{\kappa^2\mathrm{sn}^2(2a_3t,\kappa)\mathrm{cn}^2(2a_3t,\kappa)}{1-\kappa^2\mathrm{sn}^4(2a_3t,\kappa)} . \quad (A4)$$

If required, this is easily converted to the double-argument form given in the text.

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