Measurement of time-dependent quantum phases

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We evaluate the exact (Pancharatnam) phase differences between the final state $|\psi(t)\rangle$ and *various initial states* for a spin-$\frac{1}{2}$ particle in a rotating magnetic field $\mathbf{B}(t)$. For initial states $|n;\mathbf{B}_{\rm ef}(0)\rangle$, which are eigenstates of the spin component along the direction of the initial effective field $\mathbf{B}_{\rm ef}(0)$, the exact phase has an energy-dependent part and an energy-independent part. It is shown that these states $|n;\mathbf{B}_{\rm ef}(0)\rangle$ are cyclic and their corresponding Aharonov-Anandan phases are evaluated. In the adiabatic limit we discuss different choices of time-dependent bases and the relationship between the exact phase, the Born-Fock-Schiff phase, and Berry's phase. We propose neutron interference experiments to test separately the exact and the adiabatic evolution laws, as well as to measure exact and adiabatic time-dependent phases.

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I. INTRODUCTION

Despite the fact that the rise of quantum mechanics has been associated with the discovery of the wave properties of particles and that in numerous different experiments particle interference [1–3] has been demonstrated, the theoretical status and the physical interpretation of all phases in quantum mechanics have not been fully established. We find the explanation of this state of affairs in the standard statistical interpretation of quantum mechanics [4,5]. According to this interpretation only the modulus square of the wave function is interpretable and measurable. The following statement is a major part of this interpretation: If, in a Hilbert space $\mathcal{H}$, $|\alpha\rangle$ represents a physical state of the system, all vectors of the form $e^{i\varphi}|\alpha\rangle$, φ real, represent the *same* physical state and are said to form a *ray*.

Berry's paper [6] on "quantal phase factors accompanying adiabatic changes" stimulated experimentalists to measure various phases: Berry's phase [7,8], Pancharatnam's phase [9,10], the noncyclic Berry phase [11], the nonclassical Berry phase [12], etc.

Immediately after the appearance of Berry's paper it seemed that these phases were not the usual phases of quantum mechanics so that Berry's phase obtained [13] the attributes *nonintegrable, geometrical,* topological, additional phase of geometrical origin, new phase, canonical phase, quantum adiabatic phase, mysterious phase, etc. But it has been more and more recognized [14–19] that these are actually nothing but different names and attributes given to various parts of the total phase in the exact solution of the Schrödinger equation with a time-dependent Hamiltonian. Ramaseshan and Nityananda pointed out [20] that the classical phase discovered by Pancharatnam [21] in the study of the interference of polarized light was an anticipation of Berry's phase. So, they have reminded us that relative phases of states have always been measured in the interference experiments. That is why in the recent literature [11] the phase difference $\chi\equiv\mathcal{P}\langle\Phi|\Psi\rangle$ between two *arbitrary* nonorthogonal states $|\Psi\rangle$ and $|\Phi\rangle$, defined by

$$\langle\Phi|\Psi\rangle=|\langle\Phi|\Psi\rangle|\exp(i\chi)$$

is called Pancharatnam's phase.

Due to the fact that Pancharantnam's phase was derived [21] in the classical theory of light, the first optical experiments [9,10,22] with Berry's phase, which in fact measured the phase shift of classical electromagnetic waves, led to a controversy [23,24] as to whether one should view the optical Berry's phase as originating at the quantum or at the classical level. In order to settle the controversy Kwiat and Chiao did an experiment [12] with photons in essentially $n=1$ Fock states.

Now, the important question is the following: Is there any basic principle which allows us to qualify certain (class of) phase differences between states belonging to the same ray as interpretable and measurable while others should be qualified as uninterpretable and unmeasurable?

To our knowledge, a basic principle which would justify this distinction has never been proposed. In our opinion, all phase differences between states belonging to the same ray (time dependent or time independent, adiabatic or exact, dynamical or geometrical) are measurable if the criterion of measurability is the existence of an appropriate interference pattern. Pancharatnam's general definition of the relative phase of two arbitrary nonorthogonal states is inspired by the dependence of the intensity of their interference pattern on the real part of their scalar product. In this sense the relative phase of two states belonging to different rays is also measurable. However, along with the interference pattern this case clearly requires a measurement of the absolute value of the scalar product.

The present paper aims at elaborating the latter statements by an analysis of already performed experiments and proposes experiments with spin in constant and rotating magnetic fields.

In Sec. II we solve the Schrödinger equation for spin $\frac{1}{2}$ in a rotating magnetic field by using (1) the basis $|n\rangle$, $n=\pm\sigma_z$, (2) the Born-Fock-Schiff (BFS) basis, and (3) the time-revolution operator. Then starting from the exact solutions we derive their form in the adiabatic approximation (Sec. III).

In Sec. IV we compare the Born-Fock-Schiff, Pancharatnam, Berry, and Aharonov-Anandan phases.

In Sec. V we discuss several combinations of static and rotating magnetic fields along two paths in the neutron interferometer that should allow one to check the exact time evolution of spin state and to measure various phases (Berry, Born-Fock-Schiff, Pancharatnam) associated with the adiabatic approximation of the exact solutions.

II. EXACT TIME EVOLUTION OF SPIN-$\frac{1}{2}$ STATES IN A ROTATING MAGNETIC FIELD

The evolution of the spin state in a magnetic field, which is the sum of a static magnetic field $\mathbf{B}_0=B_0\mathbf{e}_z$ and of a rotating magnetic field $\mathbf{B}_1(t)=B_1[\cos\omega t\,\mathbf{e}_x+\sin\omega t\,e_y]$ [Fig. 1(a)], is governed by the Hamiltonian [4]

$$H(t)=-\mu\vec{\sigma}[\mathbf{B}_0+\mathbf{B}_1(t)]\ , \tag{1}$$

where $\vec{\sigma}$ is a vector of Pauli matrices and $\mu=\gamma\hbar/2$ (γ is the gyromagnetic ratio). With the frequencies

$$\omega_0=-\gamma B_0,\quad \omega_1=-\gamma B_1 \tag{2}$$

the Hamiltonian reads as

$$H(t)=(\hbar/2)\{\omega_0\sigma_z+\omega_1[\cos\omega t\,\sigma_x+\sin\omega t\,\sigma_y]\}\ . \tag{3}$$

The corresponding Schrödinger equation was exactly solved by Rabi [25] and Dänzer [26]. For the purpose of our analysis it is convenient to write this solution in a form which uses the so-called effective field [26,4] defined by

$$\mathbf{B}_{\text{ef}}=\frac{1}{\gamma}(\Delta\omega\mathbf{e}_Z-\omega_1\mathbf{e}_X),\quad \Delta\omega=\omega-\omega_0\ , \tag{4}$$

where $(\mathbf{e}_X,\mathbf{e}_Y,\mathbf{e}_Z)$ are unit vectors in the coordinate system $OXYZ$ which rotates with the magnetic field. This field makes an angle ϑ_{ef} with the Z axis of the rotating frame as well as with the z axis of the laboratory frame [Fig. 1(b)], such that

$$\tan\vartheta_{\text{ef}}=-\omega_1/\Delta\omega\ . \tag{5}$$

A. Time evolution in the $|\pm\rangle$ basis of σ_z

The time-independent Hamiltonian

$$\tilde{H}=\frac{\hbar}{2}\begin{bmatrix}-\Delta\omega & \omega_1\\ \omega_1 & \Delta\omega\end{bmatrix} \tag{6}$$

has the eigenvalues

$$\varepsilon_n=-n\varepsilon,\quad \varepsilon=\frac{\hbar}{2}(\Delta\omega^2+\omega_1^2)^{1/2} \tag{7}$$

and the eigenstates

$$\begin{aligned}|\varphi^+\rangle&=\cos\frac{\vartheta_{\text{ef}}}{2}|+\rangle+\sin\frac{\vartheta_{\text{ef}}}{2}|-\rangle\ ,\\ |\varphi^-\rangle&=-\sin\frac{\vartheta_{\text{ef}}}{2}|+\rangle+\cos\frac{\vartheta_{\text{ef}}}{2}|-\rangle\end{aligned} \tag{8}$$

in the basis $|n\rangle, n=\pm$ of σ_z.

The evolution of an arbitrary initial state

$$|\psi(0)\rangle=\alpha|\varphi^+\rangle+\beta|\varphi^-\rangle \tag{9}$$

is given by [4,16]

$$\begin{aligned}|\psi(t)\rangle=e^{-i\omega t/2}&\left[\alpha e^{i\varepsilon t/\hbar}\cos\frac{\vartheta_{\text{ef}}}{2}-\beta e^{-i\varepsilon t/\hbar}\sin\frac{\vartheta_{\text{ef}}}{2}\right]|+\rangle\\ &+e^{i\omega t/2}\left[\alpha e^{i\varepsilon t/\hbar}\sin\frac{\vartheta_{\text{ef}}}{2}+\beta e^{-i\varepsilon t/\hbar}\cos\frac{\vartheta_{\text{ef}}}{2}\right]|-\rangle\ .\end{aligned} \tag{10}$$

Equation (10) further determines the time dependence of various quantities of experimental interest such as, for instance, the spin polarization vector defined by

$$\mathbf{P}(t)=\langle\psi(t)|\vec{\sigma}|\psi(t)\rangle \tag{11}$$

also shown in Fig. 1(b).

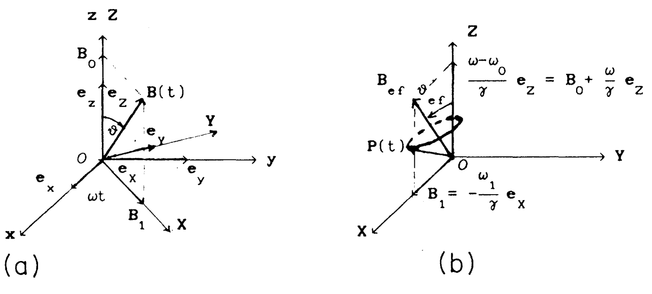

FIG. 1. (a) The vectors $\mathbf{B}_0$ and $\mathbf{B}_1(t)$ in the laboratory frame $0xyz$ and in the rotating frame OXYZ, whose axes OX and OY rotate around the Oz axis with angular velocity ω. (b) The effective field $\mathbf{B}_{\text{ef}}$ and the polarization vector $\mathbf{P}(t)$ in the rotation frame $OXYZ$.

B. Exact time evolution in the Born-Fock-Schiff basis

For the sake of making the adiabatic approximation of the exact solution it is convenient [16] to write the exact solution in terms of the Born-Fock-Shiff basis $|n;\mathbf{B}(t)\rangle_*$ which (for an arbitrary Hamiltonian) is defined by the eigenvalue problem

$$H(t)|n;\mathbf{B}(t)\rangle_* = E_n(t)|n;\mathbf{B}(t)\rangle_*, \quad n=\pm \; , \tag{12}$$

and by the phase fixing condition

$$\left\langle n;\mathbf{B}(t)\left|\frac{\partial}{\partial t}\right|n;\mathbf{B}(t)\right\rangle_* = 0 \; . \tag{13}$$

Schiff has pointed out [27] that in general, the states which satisfy the "eigenvalue problem" (12) and the phase fixing condition (13) may be determined in two steps. In the first step one solves "the eigenvalue problem"

$$H(t)|n;\mathbf{B}(t)\rangle = E_n(t)|n;\mathbf{B}(t)\rangle \; . \tag{12'}$$

Then, by applying the unitary transformation

$$|n;\mathbf{B}(t)\rangle_* = e^{i\gamma_n(t)}|n;\mathbf{B}(t)\rangle \; , \tag{14}$$

where

$$\gamma_n(t) = i\int_0^t d\tau \left\langle n;\mathbf{B}(\tau)\left|\frac{\partial}{\partial\tau}\right|n;\mathbf{B}(\tau)\right\rangle \tag{15}$$

one determines the state $|n;\mathbf{B}(t)\rangle_*$ which satisfies also the condition (13). In the subsequent discussion we shall refer to $\gamma_n(t)$ as the Born-Fock-Schiff phase. Condition (13) was first introduced by Born and Fock [28], in order to simplify the system of equations obtained when solving the Schrödinger equation in the adiabatic approximation. In recent literature [13] this condition is called the parallel transport law, or the connection.

The states which satisfy the eigenvalue equation (12′), in the case of spin-$\frac{1}{2}$ particle, are the eigenstates of the component of the spin operator along $\mathbf{B}(t)-\vec{\sigma}_{\mathbf{B}(t)}$. The most often used eigenstates of $\vec{\sigma}_{\mathbf{B}(t)}$ are

$$\begin{aligned} |+;\mathbf{B}(t)\rangle_C &= e^{-i\omega t/2}\cos\frac{\vartheta}{2}|+\rangle + e^{i\omega t/2}\sin\frac{\vartheta}{2}|-\rangle \; , \\ |-;\mathbf{B}(t)\rangle_C &= -e^{-i\omega t/2}\sin\frac{\vartheta}{2}|+\rangle + e^{i\omega t/2}\cos\frac{\vartheta}{2}|-\rangle \; , \end{aligned} \tag{16}$$

where $\tan\vartheta = \omega_1/\omega_0$. Their eigenvalues are

$$E_n = -nE; \quad E = \frac{\hbar}{2}(\omega_0^2+\omega_1^2)^{1/2} \; . \tag{17}$$

The BFS phase associated with the latter states is

$$\gamma_n^c(t) = n(\omega t/2)\cos\vartheta \; . \tag{18}$$

By substituting (16) and (18) into (14) we find [16] the BFS basis for spin $\frac{1}{2}$ in a rotating magnetic field:

$$\begin{aligned} |+;\mathbf{B}(t)\rangle_* &= e^{i(\omega t/2)\cos\vartheta}|+;\mathbf{B}(t)\rangle_C \; , \\ |-;\mathbf{B}(t)\rangle_* &= e^{-i(\omega t/2)\cos\vartheta}|-;\mathbf{B}(t)\rangle_C \; . \end{aligned} \tag{19}$$

The exact solution in the BFS basis (19) reads [16]:

$$\begin{aligned} |\psi(t)\rangle = e^{-i(\omega t/2)\cos\vartheta} & \left[\left[A\cos^2\frac{\vartheta-\vartheta_{\mathrm{ef}}}{2} - \tfrac{1}{2}C\sin(\vartheta-\vartheta_{\mathrm{ef}}) \right] e^{i\varepsilon t/\hbar} \right. \\ & \left. + \left[A\sin^2\frac{\vartheta-\vartheta_{\mathrm{ef}}}{2} + \tfrac{1}{2}C\sin(\vartheta-\vartheta_{\mathrm{ef}}) \right] e^{-i\varepsilon t/\hbar} \right] |+;\mathbf{B}(t)\rangle_* \\ + e^{i(\omega t/2)\cos\vartheta} & \left[\left[-\tfrac{1}{2}A\sin(\vartheta-\vartheta_{\mathrm{ef}}) + C\sin^2\frac{\vartheta-\vartheta_{\mathrm{ef}}}{2} \right] e^{-i\varepsilon t/\hbar} \right. \\ & \left. + \left[\tfrac{1}{2}A\sin(\vartheta-\vartheta_{\mathrm{ef}}) + C\cos^2\frac{\vartheta-\vartheta_{\mathrm{ef}}}{2} \right] e^{-i\varepsilon t/\hbar} \right] |-;\mathbf{B}(t)\rangle_* \; , \end{aligned} \tag{20}$$

where A and C are coefficients in the initial state written in the BFS basis

$$|\psi(0)\rangle = A|+;\mathbf{B}(0)\rangle_* + C|-;\mathbf{B}(0)|\rangle_* \; . \tag{21}$$

C. The evolution operator

Instead of focusing on state vectors a useful alternative approach is to introduce the time-evolution operator $U(t)$ defined by $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. This operator satisfies the equation

$$i\hbar\frac{\partial}{\partial t}U(t) = H(t)U(t), \quad U(0) = I \; , \tag{22}$$

which can be solved in closed form for a spin-$\frac{1}{2}$ system in a rotating magnetic field. The solution reads [29]

$$U(t) = e^{-i(\omega t/2)\sigma_z} e^{i(\varepsilon t/\hbar)(\sigma_x\sin\vartheta_{\mathrm{ef}}+\sigma_z\cos\vartheta_{\mathrm{ef}})} \; . \tag{23}$$

After straightforward algebra the above product may be expressed as a single exponential [30]

$$U(t) = e^{i[\phi(t)/2]\vec{\sigma}\cdot\mathbf{u}(t)} \; , \tag{24}$$

where

$$\cos\frac{\phi(t)}{2} = \cos\frac{\omega t}{2}\cos\frac{\varepsilon t}{\hbar} + \sin\frac{\omega t}{2}\sin\frac{\varepsilon t}{\hbar}\cos\vartheta_{\mathrm{ef}} \tag{25}$$

and the coordinates of $\mathbf{u}$ are given by

$$u_x(t)\sin\frac{\phi(t)}{2}=\sin\frac{\varepsilon t}{\hbar}\sin\vartheta_{\rm ef}\cos\frac{\omega t}{2}\ ,$$
$$u_y(t)\sin\frac{\phi(t)}{2}=\sin\frac{\varepsilon t}{\hbar}\sin\vartheta_{\rm ef}\sin\frac{\omega t}{2}\ , \tag{26}$$
$$u_z(t)\sin\frac{\phi(t)}{2}=\sin\frac{\varepsilon t}{\hbar}\cos\vartheta_{\rm ef}\cos\frac{\omega t}{2}-\sin\frac{\omega t}{2}\cos\frac{\varepsilon t}{\hbar}\ .$$

In Sec. III B we shall determine $U(t)$ in the adiabatic approximation.

III. THE ADIABATIC APPROXIMATION

The general condition for the applicability of the adiabatic approximation is

$$\left|\frac{\hbar_*\langle k|\frac{\partial H}{\partial t}|n\rangle_*}{E_n-E_k}\right| \ll |E_n-E_k|\ . \tag{27}$$

In the case of spin $\frac{1}{2}$ in a rotating magnetic field it reduces to a condition on the ratio of frequencies

$$|-\omega\omega_1/2(\omega_0^2+\omega_1^2)| \ll 1 \tag{28}$$

and is satisfied if

$$\eta\equiv\omega/(\omega_0^2+\omega_1^2)^{1/2}\ll 1\ . \tag{29}$$

Then, the following approximate relations are valid

$$\cos(\vartheta_{\rm ef}-\vartheta)\simeq 1-[\omega\omega_0/(\omega_0^2+\omega_1^2)]^2/2\simeq 1\ ,$$
$$\sin(\vartheta_{\rm ef}-\vartheta)\simeq\omega\omega_1/(\omega_0^2+\omega_1^2)\ , \tag{30}$$
$$\cos^2\frac{\vartheta_{\rm ef}-\vartheta}{2}\simeq 1-[\omega\omega_0/(\omega_0^2+\omega_1^2)]^2/2\simeq 1\ ,$$
$$\sin^2\frac{\vartheta_{\rm ef}-\vartheta}{2}\simeq[\omega\omega_0/(\omega_0^2+\omega_1^2)]^2/2\simeq 0\ .$$

These relations reflect the fact that in the adiabatic case the real field $\mathbf{B}(t)$ and the effective field $\mathbf{B}_{\rm ef}(t)$ are approximately equal. Therefore their respective angles ϑ and $\vartheta_{\rm ef}$ are approximately equal as well. Also, we have the following important approximate relation:

$$\varepsilon\simeq E+\frac{\omega\hbar}{2}\cos\vartheta\ . \tag{31}$$

A. Adiabatic evolution in the Born-Fock-Schiff basis

To first order the exact state expressed in the time-dependent basis (19) is given by

$$|\psi(t)\rangle = Ae^{iEt/\hbar}|+;\mathbf{B}(t)\rangle_* + Ce^{-iEt/\hbar}|-;\mathbf{B}(t)\rangle_*\ . \tag{32}$$

Particularly simple is the adiabatic evolution of two special initial states, $|+;\mathbf{B}(0)\rangle_*$ ($A=1$, $C=0$) and $|-;\mathbf{B}(0)\rangle_*$ ($A=0$, $C=1$):

$$|\psi_n(t)|\rangle = e^{inEt/\hbar}|n;\mathbf{B}(t)\rangle_*\ . \tag{33}$$

B. Adiabatic evolution in Berry's formulation

Nowadays, using (14) instead of the BFS old-fashioned form (33), the adiabatic solution is written as

$$|\psi_n(t)\rangle = e^{-iE_nt/\hbar}e^{i\gamma_n(t)}|n;\mathbf{B}(t)\rangle\ , \tag{34}$$

where the states $|n;\mathbf{B}(t)\rangle$ are assumed to be single-valued functions in the parameter space. In particular, if $\mathbf{B}(T)=\mathbf{B}(0)$ this implies that the functions $|n;\mathbf{B}(t)\rangle$ have the property

$$|n;\mathbf{B}(T)\rangle = |n;\mathbf{B}(0)\rangle\ . \tag{35}$$

The states

$$|+;\mathbf{B}(t)\rangle_C^l = e^{il\omega t/2}|+;\mathbf{B}(t)\rangle_C\ ,$$
$$|-;\mathbf{B}(t)\rangle_C^j = e^{ij\omega t/2}|-;\mathbf{B}(t)\rangle_C\ ,\quad l,j=1,-1$$

satisfy the eigenvalue equation (12′) and have the property (35). The states $|+;\mathbf{B}(t)\rangle_W\equiv e^{i\omega t/2}|+;\mathbf{B}(t)\rangle_C$, $|-;\mathbf{B}(t)\rangle_W\equiv e^{-i\omega t/2}|-;\mathbf{B}(t)\rangle_C$ in addition have an appealing property (see Sec. IV A):

$$\lim_{t\to T}\mathcal{P}_W\langle n;\mathbf{B}(0)|n;\mathbf{B}(t)\rangle_W=\mathcal{P}_W\langle n;\mathbf{B}(0)|n;\mathbf{B}(T)\rangle_W=0\ . \tag{36}$$

This set is essentially the one recently used by Weinfurter [31]. The associated phase is given by

$$\gamma_n^W(t)=-n\omega t(1-\cos\vartheta)/2\ . \tag{37}$$

For $t=T$ one has

$$\gamma_n^W(T)=-n\pi(1-\cos\vartheta)\ , \tag{38}$$

which is precisely the value obtained by Berry [6] without specifying the time-dependent basis explicitly.

With the single-valued basis $|+;\mathbf{B}(t)\rangle_M\equiv e^{i\omega t/2}|+;\mathbf{B}(t)\rangle_C$, $|-\mathbf{B}(t)\rangle_M\equiv e^{i\omega t/2}|-;\mathbf{B}(t)\rangle_C$ used in Ref. [16] one finds

$$\gamma_n^M(t)=-\omega t(1-n\cos\vartheta)/2\ . \tag{39}$$

Kobe *et al.* [15] define the bases $|+;\mathbf{B}(t)\rangle_K\equiv e^{-i\omega t/2}|+;\mathbf{B}(t)\rangle_C$, $|-;\mathbf{B}(t)\rangle_K\equiv e^{-i\omega t/2}|-;\mathbf{B}(t)\rangle_C$ and obtain

$$\gamma_n^K(t)=\omega(1+n\cos\vartheta)/2 \tag{40}$$

which they call the Yang phase.

Notice that for $t=0$ all the above three sets of states reduce to $|n;\mathbf{B}(0)\rangle_*$:

$$|n;\mathbf{B}(0)\rangle_M=|n;\mathbf{B}(0)\rangle_K=|n;\mathbf{B}(0)\rangle_W=|n;\mathbf{B}(0)\rangle_*\ . \tag{41}$$

One can easily check that they all lead, according to Eq. (14), to the same time-dependent states $|\pm;\mathbf{B}(t)\rangle_*$, given in (19). This means that there is complete agreement between the two forms (33) and (34) of the adiabatic solution. This is no surprise since for the given initial state $|n;\mathbf{B}(0)\rangle_*$ both the exact and the adiabatic solutions of the Schrödinger equation are of course uniquely determined.

C. The evolution operator in the adiabatic approximation

In this section we show that in the adiabatic approximation the evolution operator reduces to the product of two unitary operators $N(t)$ and $M(t)$ defined by

$$|n;\mathbf{B}(t)\rangle_* = N(t)|n;\mathbf{B}(0)\rangle_* , \tag{42}$$

$$M(t)|n;\mathbf{B}(t)\rangle = e^{inEt/\hbar}|n;\mathbf{B}(t)\rangle, \quad n=\pm . \tag{43}$$

Straightforward evaluation leads to

$$N(t) = e^{-i(\omega t/2)\sigma_z} e^{i(\omega t/2)\cos\vartheta(\sigma_x \sin\vartheta + \sigma_z \cos\vartheta)} , \tag{44}$$

$$M(t) = e^{i(Et/\hbar)(\sigma_x \sin\vartheta + \sigma_z \cos\vartheta)} . \tag{45}$$

Now, by starting with the form (23) of the evolution operator, and using the approximate relations (31) and $\vartheta_{\text{ef}} \simeq \vartheta$ one easily sees that, in the *adiabatic approximation, $U(t)$* is a product of $N(t)$ and $M(t)$:

$$U(t) \simeq N(t)M(t) . \tag{46}$$

The same result was obtained by Anandan and Stodolsky [32] and Weinfurter and Badurek [11] with the aid of the adiabatic theorem written in the operator form.

IV. PHASES

In the past a number of phases associated with spin $\frac{1}{2}$ in a magnetic field have been defined and studied. Here we shall review some of them and establish their mutual relations.

A. Pancharatnam's phase

As we have mentioned in the Introduction, the phase difference $\chi = \mathcal{P}\langle\Phi|\Psi\rangle$ between two arbitary quantum states $|\Psi\rangle$ and $|\Phi\rangle$, is defined by

$$\langle\Phi|\Psi\rangle = e^{i\chi}|\langle\Phi|\Psi\rangle| . \tag{47}$$

It is called, in the recent literature [33,11], Pancharatnam's phase. This definition is in fact the generalization to an arbitrary pair of states of the notion of the relative phase which has been used previously for two states belonging to the same ray.

1. Evaluation of Pancharatnam's phases $\mathcal{P}_*\langle n;\mathbf{B}(0)|\psi(t)\rangle$

Let us apply the definition (47) to evaluate the *exact phase difference* $\chi_n(t)$ between $|\psi(t)\rangle$ and the initial state $|n;\mathbf{B}(0)\rangle_*$. The easiest method is to use the exact evolution operator (24). After some algebra we find

$$\begin{aligned} {}_*\langle n;\mathbf{B}(0)|\psi(t)\rangle &= \cos\frac{\phi(t)}{2} \\ &\quad + [u_x(t)\sin\vartheta + u_z(t)\cos\vartheta]ni\sin\frac{\phi(t)}{2} , \end{aligned} \tag{48}$$

where $\cos[\phi(t)/2]$ and $\mathbf{u}(t)$ are given in (28) and (26), respectively. This further gives

$$\chi_n(t) = n \arctan \frac{\cos\frac{\omega t}{2}\sin\frac{\varepsilon t}{\hbar}\cos(\vartheta - \vartheta_{\text{ef}}) - \sin\frac{\omega t}{2}\cos\frac{\varepsilon t}{\hbar}\cos\vartheta}{\cos\frac{\omega t}{2}\cos\frac{\varepsilon t}{\hbar} + \sin\frac{\omega t}{2}\sin\frac{\varepsilon t}{\hbar}\cos\vartheta_{\text{ef}}} . \tag{49}$$

In the adiabatic limit $[\vartheta_{\text{ef}} \to \vartheta,\ \varepsilon \to E + (\omega\hbar/2)\cos\vartheta]$ the above expression reduces to

$$\chi_n(t) \simeq n\frac{Et}{\hbar} + g_n(t) , \tag{50}$$

where

$$g_n(t) = -n\left[\text{arctang}\left[\cos\vartheta\tan\frac{\omega t}{2}\right] - \frac{\omega t}{2}\cos\vartheta\right] , \tag{51}$$

$$g_n(T) \equiv \lim_{t\to T} g_n(t) = -n\pi(1-\cos\vartheta) . \tag{52}$$

Had we used instead the adiabatic approximation (46) we would have obtained the same expression for $\chi_n(t)$. Namely,

$$\begin{aligned} {}_*\langle n;\mathbf{B}(0)|\psi(t)\rangle &\simeq {}_*\langle n;\mathbf{B}(0)|N(t)M(t)|n;\mathbf{B}(0)\rangle_* \\ &= e^{inEt/\hbar}{}_*\langle n;\mathbf{B}(0)|n;\mathbf{B}(t)\rangle_* . \end{aligned} \tag{53}$$

But

$$\begin{aligned} {}_*\langle n;\mathbf{B}(0)|n;\mathbf{B}(t)\rangle_* &= \cos\left[\frac{\omega t}{2}\right]\cos\left[\frac{\omega t}{2}\cos\vartheta\right] + \cos\vartheta\sin\left[\frac{\omega t}{2}\right]\sin\left[\frac{\omega t}{2}\cos\vartheta\right] \\ &\quad + in\left[\cos\left[\frac{\omega t}{2}\right]\sin\left[\frac{\omega t}{2}\cos\vartheta\right] - \cos\vartheta\sin\left[\frac{\omega t}{2}\right]\cos\left[\frac{\omega t}{2}\cos\vartheta\right]\right] . \end{aligned} \tag{54}$$

Therefore

$$|{}_*\langle n;\mathbf{B}(0)|\psi(t)\rangle| \simeq |{}_*\langle n;\mathbf{B}(0)|n;\mathbf{B}(t)\rangle_*| = \left[\cos\frac{2\omega t}{2}+\cos^2\vartheta\sin\frac{2\omega t}{2}\right]^{1/2}, \quad (55)$$

$$\chi_n(t)=\mathcal{P}_*\langle n;\mathbf{B}(0)|\psi(t)\rangle \simeq n\frac{Et}{\hbar}+g_n(t)\ , \quad (56)$$

since

$$\mathcal{P}_*\langle n;\mathbf{B}(0)|n;\mathbf{B}(t)\rangle_* = g_n(t)\ . \quad (57)$$

The easiest way to establish the relation (56) is by starting with the adiabatic approximation (33). This also makes apparent that the term $g_n(t)$ is in fact the sum of two different contributions

$$g_n(t)=\gamma_n(t)+\mathcal{P}\langle n;\mathbf{B}(0)|n;\mathbf{B}(t)\rangle\ . \quad (58)$$

Note that both terms in the right-hand side depend on the chosen basis, while their sum is clearly basis independent (see Table I).

2. Evaluation of the Pancharatnam phases ${}_W\langle n;\mathbf{B}_{\rm ef}(0)|\psi(t)\rangle$

It is interesting that there exists another exact phase difference which reduces to (50) in the adiabatic approximation. This is the phase difference between $|\psi(t)\rangle$ and the initial state $|n;\mathbf{B}_{\rm ef}(0)\rangle_W$, which is an eigenstate of the spin component in the direction of the effective field $\mathbf{B}_{\rm ef}$ for $t=0$. Of course this state is not an eigenstate of $H(0)$.

By applying the exact evolution operator in the form (23) to the state $|n;\mathbf{B}_{\rm ef}(0)\rangle_W$ we find

$$U(t)|n;\mathbf{B}_{\rm ef}(0)\rangle_W = e^{-i(\omega t/2)\sigma_z}e^{in\varepsilon t/\hbar}|n;\mathbf{B}_{\rm ef}(0)\rangle_W \quad (59)$$

and therefore

$$\mathcal{P}_W\langle n;\mathbf{B}_{\rm ef}(0)|\psi(t)\rangle = n\left[\frac{\varepsilon t}{\hbar}-\arctan\left[\cos\vartheta_{\rm ef}\tan\frac{\omega t}{2}\right]\right]. \quad (60)$$

Taking the adiabatic limit yields

$$\lim_{\rm ad}\mathcal{P}_W\langle n;\mathbf{B}_{\rm ef}(0)|\psi(t)\rangle = n\left[\frac{Et}{\hbar}+\frac{\omega t}{2}\cos\vartheta-\arctan\left[\cos\vartheta\tan\frac{\omega t}{2}\right]\right], \quad (61)$$

which is precisely the phase $\chi_n(t)$ of Eq. (50). This result is understandable, taking into account the fact that in the adiabatic limit $\mathbf{B}_{\rm ef}(t)$ is approximately equal to the real field $\mathbf{B}(t)$.

B. Aharonov-Anandan phases

The exact solution in the form (10) shows that among all possible initial states only those with ($\alpha=1,\beta=0$) and ($\alpha=0$, $\beta=1$) undergo cyclic evolution. These states evolve according to

$$|\psi_n(t)\rangle = e^{in(\varepsilon t/\hbar-\omega t/2)}|n;\mathbf{B}_{\rm ef}(t)\rangle_W\ , \quad (62)$$

where

$$|+;\mathbf{B}_{\rm ef}(t)\rangle_W=\cos\frac{\vartheta_{\rm ef}}{2}|+\rangle+e^{i\omega t}\sin\frac{\vartheta_{\rm ef}}{2}|-\rangle\ ,$$
$$|-;\mathbf{B}_{\rm ef}(t0\rangle_W=-\sin\frac{\vartheta_{\rm ef}}{2}e^{-i\omega t}|+\rangle+\cos\frac{\vartheta_{\rm ef}}{2}|-\rangle \quad (63)$$

are eigenstates of the spin component along the effective

TABLE I. The functions $\gamma_n(t)$ and $\mathcal{P}\langle n;\mathbf{B}(0)|n;\mathbf{B}(t)\rangle$ for the three bases $|n;\mathbf{B}(t)\rangle_M$, $|n;\mathbf{B}(t)\rangle_W$, and $|n;\mathbf{B}(t)\rangle_K$, which are all single valued in the space of the parameter B, for BFS basis and for the basis $|n;\mathbf{B}(t)\rangle_C$ which changes sign after $t=T$.

Basis $\lvert n;\mathbf{B}(t)\rangle$	BFS phase $\gamma_n(t)$	$\mathcal{P}\langle n;\mathbf{B}(0)\lvert n;\mathbf{B}(t)\rangle$	$\gamma_n(T)$	$g_n(T)$ for $\cos\vartheta\neq 0$
$\lvert n;\mathbf{B}(t)\rangle_M$	$-\frac{\omega t}{2}(1-n\cos\vartheta)$	$\frac{\omega t}{2}-n\arctan\left[\cos\vartheta\tan\frac{\omega t}{2}\right]$	$-\pi(1-n\cos\vartheta)$	$-n\pi(1-\cos\vartheta)$
$\lvert n;\mathbf{B}(t)\rangle_W$	$-n\frac{\omega t}{2}(1-\cos\vartheta)$	$n\left[\frac{\omega t}{2}-\mathrm{arctang}\left[\cos\vartheta\tan\frac{\omega t}{2}\right]\right]$	$-n\pi(1-\cos\vartheta)$	$-n\pi(1-\cos\vartheta)$
$\lvert n;\mathbf{B}(t)\rangle_K$	$\frac{\omega t}{2}(1+n\cos\vartheta)$	$-\frac{\omega t}{2}-n\arctan\left[\cos\vartheta\tan\frac{\omega t}{2}\right]$	$\pi(1+n\cos\vartheta)$	$-n\pi(1-\cos\vartheta)$
$\lvert n;\mathbf{B}(t)\rangle_*$	0	$n\left[\frac{\omega t}{2}\cos\vartheta-\arctan\left[\cos\vartheta\tan\frac{\omega t}{2}\right]\right]$	0	$-n\pi(1-\cos\vartheta)$
$\lvert n;\mathbf{B}(t)\rangle_C$	$n\frac{\omega t}{2}\cos\vartheta$	$-n\frac{\omega t}{2}\arctan\left[\cos\vartheta\tan\frac{\omega t}{2}\right]$	$n\pi\cos\vartheta$	$-n\pi(1-\cos\vartheta)$
				for $\cos\vartheta=0$ the phase $g_n(T)$ vanishes for all bases

field and have the property

$$|n;\mathbf{B}_{\rm ef}(T)\rangle_W=|n;\mathbf{B}_{\rm ef}(0)\rangle_W=|\psi_n(0)\rangle \ . \tag{64}$$

Therefore, after time T the state $|\psi_n(T)\rangle$ is equal to the initial state multiplied by a phase factor. Using the notation of Aharonov-Anandan [34]

$$|\psi_n(T)\rangle=e^{i\phi_n}|\psi_n(0)\rangle \ , \tag{65}$$

where

$$\phi_n=n\left[\frac{\varepsilon}{\hbar}-\frac{\omega}{2}\right]T=n\pi\left[\frac{2\varepsilon}{\hbar\omega}-1\right] . \tag{66}$$

Taking now into account the expression (60) for the phase $\mathcal{P}_W\langle n;\mathbf{B}_{\rm ef}(0)|\psi(t)\rangle$ we see that ϕ_n is just a particular value of this phase for $t=T$.

$$\begin{aligned}\phi_n&=\mathcal{P}_W\langle n;\mathbf{B}_{\rm ef}(0)|\psi(T)\rangle\\&=\lim_{t\to T} n\left[\frac{\varepsilon t}{\hbar}-\arctan\left[\cos\vartheta_{\rm ef}\tan\frac{\omega t}{2}\right]\right] .\end{aligned} \tag{67}$$

Aharonov and Anandan also define another set of states

$$|\tilde{\psi}_n(t)\rangle\equiv e^{-if_n(t)}|\psi_n(t)\rangle \ , \tag{68}$$

where $f_n(t)$ satisfies

$$f_n(T)-f_n(0)=\phi_n \ . \tag{69}$$

In view of (62) one clearly has

$$|\tilde{\psi}_n(t)\rangle=|n;\mathbf{B}_{\rm ef}(t)\rangle_W \tag{70}$$

and

$$f_n(t)=n\left[\frac{\varepsilon t}{\hbar}-\frac{\omega t}{2}\right] . \tag{71}$$

Finally let us consider the two quantities [34]

$$\beta_n=\int_0^T\langle\tilde{\psi}_n|i\frac{\partial}{\partial t}|\tilde{\psi}_n\rangle dt \tag{72}$$

and

$$\begin{aligned}\alpha_n&=-\hbar^{-1}\int_0^T\langle\psi_n(t)|H|\psi_n(t)\rangle dt\\&=-\int_0^T\left\langle\psi_n\left|i\frac{\partial}{\partial t}\right|\psi_n\right\rangle dt \ ,\end{aligned} \tag{73}$$

which in Ref. [34] are called geometrical and dynamical phase, respectively. Explicit evaluation for the cyclic states (62) gives

$$\beta_n=-n\pi(1-\cos\vartheta_{\rm ef}) \ , \tag{74}$$

$$\alpha_n=-n\pi\left[\cos\vartheta_{\rm ef}-\frac{2\varepsilon}{\hbar\omega}\right] . \tag{75}$$

Notice that

$$\phi_n=\beta_n+\alpha_n \ . \tag{76}$$

In the adiabatic limit $[\vartheta_{\rm ef}\to\vartheta$ and $\varepsilon\to E+(\hbar\omega/2)\cos\vartheta]$ the above exact phases approach Berry's values

$$\lim_{\rm ad}\beta_n=-n\pi(1-\cos\vartheta) \ , \tag{77}$$

$$\lim_{\rm ad}\alpha_n=nET=-E_nT \ . \tag{78}$$

This is understandable since in the adiabatic limit $B_{\rm ef}(t)\to\mathbf{B}(t)$ and therefore $|n;\mathbf{B}_{\rm ef}(t)\rangle_W\to|n;\mathbf{B}(t)\rangle_W$. This is in agreement with our previous conclusions.

V. NEUTRON INTERFEROMETRY WITH STATIC AND TIME-DEPENDENT MAGNETIC FIELDS

Neutrons are the most suitable quantum objects for the experimental study of the spin states, both in static as well as in variable magnetic fields.

A. Verification of the law of transformation of spinors under rotation: Measurement of dynamical phase

Neutron interference experiments with static magnetic fields were originally aimed at verifying the transformation properties of spinors under rotation [35]. In these experiments (Fig. 2) a neutron beam is split into two beams, one of which passes through the static magnetic field $\mathbf{B}_s=B_s\mathbf{e}_z$, the other propagates freely. Along both trajectories there is, in addition, a small guiding magnetic field which determines the quantization axis (z axis). The wave function of neutrons entering the interferometer is a product of the space function (plane wave) and spin state $|n\rangle_z$ which is an eigenstate of σ_z,

$$\Psi_{\rm in}(\mathbf{r},t)=C_{\rm in}e^{i\mathbf{k}\cdot\mathbf{r}}e^{-i(\hbar^2k^2/2m)t/\hbar}|n\rangle_z \ . \tag{79}$$

Here, k is the initial neutron wave vector. The wave function in the detector L is a superposition of wave functions $\Psi_{ijk}(\mathbf{L},t)$ and $\Psi_{spk}(\mathbf{L},t)$ associated with waves which propagated along paths ijk and spk, respectively,

$$\Psi(\mathbf{L},t)=\Psi_{ijk}(\mathbf{L},t)+\Psi_{spk}(\mathbf{L},t) \ . \tag{80}$$

If $\mu B\ll\hbar^2k^2/2m$, the reflection on the field boundaries are negligible and the magnetic field affects only the spin part of $\Psi_{spk}(\mathbf{r},t)$. This implies that the time evolution of $\Psi_{spk}(\mathbf{r},t)$ may be approximately described [35,36] by applying upon the initial spin state $|n\rangle_z$, the operator

$$U_s(t)=\exp(i\mu\sigma_zB_st/\hbar) \ . \tag{81}$$

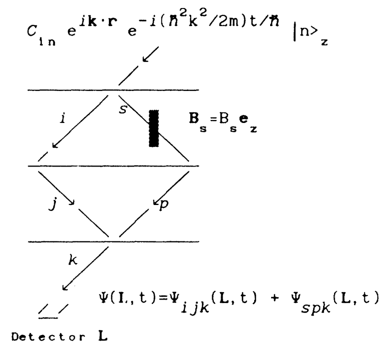

FIG. 2. Scheme of an interference experiment for verifying the law of transformation of spinors under rotation. In modern language this experiment measures the dynamical phase.

This leads to the state

$$\exp(i\mu\sigma_z B_s(t/\hbar))|n\rangle_z = \exp(in\mu B_s t/\hbar)|n\rangle_z$$
$$= \exp(-iE_{\rm ns}t/\hbar)|n\rangle_z\ , \qquad (82)$$

where $E_{\rm ns}=-n\mu B_s$ are the eigenvalues of the neutron spin energy in the constant magnetic field $\mathbf{B}_s$. The neutron velocity v, the length of the field l, and time of passage t_p are simply related by $l=t_p v=t_p\hbar k/m$. Therefore, the effect of the static field directed along the axis of quantization (z axis) consists in inducing the phase difference

$$\alpha_n(t_p)=-E_{ns}t_p/\hbar=n\mu B_s l/v\hbar \qquad (83)$$

between the states $\Psi_{spk}(\mathbf{L},t)$ and $\Psi_{ijk}(\mathbf{L},t)$, so that

$$\Psi_{spk}(\mathbf{L},t)=\Psi_{ijk}(\mathbf{L},t)e^{i\alpha_n(t_p)}\ . \qquad (84)$$

At the exit from the interferometer, the two waves which propagate along paths ijk and spk are superposed and the number of neutrons is counted in the detector. The number of neutrons $I(\mathbf{L})$, counted in the detector $\mathbf{L}$ during finite interval Δt, is proportional to

$$I(\mathbf{L})\propto\int_0^{\Delta t}dt\,|\Psi_{ijk}(\mathbf{L},t)+\Psi_{spk}(\mathbf{L},t)|^2$$
$$=|\Psi_{ijk}(\mathbf{L})|^2 2[1+\cos\alpha_n(t_p)]\ , \qquad (85)$$

where

$$|\Psi_{ijk}(\mathbf{L})|^2=\int_0^{\Delta t}dt\,|\Psi_{ijk}(\mathbf{L},t)|^2$$
$$=\int_0^{\Delta t}dt\,|\Psi_{spk}(\mathbf{L},t)|^2\ . \qquad (86)$$

Since $U_s(t_p)$ has the same form as the operator of rotation around the z axis by the angle $-2\mu B_s l/v\hbar$, the interpretation of the experiment was that it verified the law of transformation of spinors under rotation [35,36].

Currently the phase $\alpha_n(t)=-E_{\rm ns}t/\hbar$ is called, following Berry's nomenclature, the dynamical phase. By adopting this modern nomenclature, one could therefore say that in the interference experiment with static magnetic field (which obeys the transformation law of spinors under rotation) one also measures the dynamical phase.

B. Proposed interference experiment for the verification of the exact and adiabatic evolution law in the rotating magnetic field

The time evolution of the spin state in a rotating magnetic field could be also verified using the interferometer described in Sec. V A. But, instead of a static field $\mathbf{B}_s$ one should apply the rotating magnetic field $\mathbf{B}(t)=\mathbf{B}_0+\mathbf{B}_1(t)$ along one path (for example, along path ijk as shown in Fig. 3). Initially the neutrons should be polarized along the direction of the magnetic field at $t=0$, $\mathbf{B}(0)=B_0\mathbf{e}_z+\mathbf{B}_1(0)$, which means that the initial wave function should be

$$\Psi_{\rm in}(\mathbf{r},t)=C_{\rm in}e^{i\mathbf{k}\cdot\mathbf{r}}e^{-i(\hbar^2k^2/2m)t/\hbar}|n;\mathbf{B}(0)\rangle_*\ . \qquad (87)$$

Wave functions of waves arriving at the detector $\mathbf{L}$ along paths ijk and spk are then

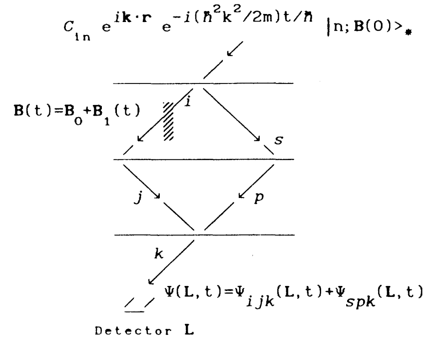

FIG. 3. Scheme of an interference experiment for verifying the exact and the adiabatic evolution law in a rotating magnetic field.

$$\Psi_{spk}(\mathbf{L},t)\sim e^{i\mathbf{k}\cdot\mathbf{L}}e^{-i(\hbar^2k^2/2m)t/\hbar}|n;\mathbf{B}(0)\rangle_*\ ,$$
$$\Psi_{ijk}(\mathbf{L},t)\sim e^{i\mathbf{k}\cdot\mathbf{L}}e^{-i(\hbar^2k^2/2m)t/\hbar}e^{i[\phi(t_p)/2]\vec{\sigma}\cdot\mathbf{u}(t_p)}|n;\mathbf{B}(0)\rangle_*\ . \qquad (88)$$

Taking (48) into account we easily show that the number of neutrons in the detector $\mathbf{L}, I(\mathbf{L})$, measured during time interval Δt is proportional to

$$I(\mathbf{L})\propto\int_0^{\Delta t}dt\,|\Psi_{ijk}(\mathbf{L},t)+\Psi_{spk}(\mathbf{L},t)|^2$$
$$=|\Psi_{ijk}(\mathbf{L})|^2 2\left[1+\cos\frac{\phi(t_p)}{2}\right]\ . \qquad (89)$$

We have obtained again the simple cosine law, except that now $\phi(t_p)/2$ is not the phase, but the angle of the exact evolution operator (24) of spin in the rotating magnetic field. Taking into account that between $\cos[\phi(t_p)/2]$ and $\cos[\chi_n(t_p)]$ there exists the relation

$$\cos\frac{\phi(t_p)}{2}=|{}_*\langle n;\mathbf{B}(0)|U(t_p)|n;\mathbf{B}(0)\rangle_*|\cos\chi_n(t_p)\ , \qquad (90)$$

we conclude that by combining the measurement of $I(L)$ with one of $|{}_*\langle n;\mathbf{B}(0)|U(t_p)|n;\mathbf{B}(0)\rangle_*|$ one could in principle determine the phase difference $\chi_n(t_p)$.

If the evolution is *adiabatic* $[\omega/(\omega_0^2+\omega_1^2)^{1/2}\ll 1]$, it follows from (53)–(57) that

$$\cos\frac{\phi(t_p)}{2}\cong\left[\cos^2\frac{\omega t_p}{2}+\cos^2\vartheta\sin^2\frac{\omega t_p}{2}\right]^{1/2}$$
$$\times\cos\left[n\frac{Et_p}{2}+g_n(t_p)\right]\ . \qquad (91)$$

In particular for $t_p=T$ we find

$$|\langle n;\mathbf{B}(0)|U(T)|n;\mathbf{B}(0)\rangle|\cong 1\ ,$$
$$g_n(T)=-n\pi(1-\cos\vartheta)\ . \qquad (92)$$

Therefore

$$I(\mathbf{L})\propto|\Psi_{ijk}(\mathbf{L})|^2\left[1+\cos\left[n\frac{ET}{\hbar}-n\pi(1-\cos\vartheta)\right]\right]\ . \qquad (93)$$

We see that for $t_p=T$ the expression for $I(\mathbf{L})$ is particularly simple. The argument of cosine is a sum of dynamical, $nET/\hbar$, and geometrical phases $-\gamma_n^W(T)=-n\pi(1-\cos\vartheta)$.

C. Berry's interference experiment

In the experiment proposed by Berry [6] an adiabatically rotating magnetic field $\mathbf{B}(t)=B_0\mathbf{e}_z+\mathbf{B}_1(t)$ should be applied along one path (for example, along ijk) and a static magnetic field of intensity $B=(B_0^2+B_1^2)^{1/2}$ in the direction of $\mathbf{B}(0)$ should be applied along the other path (for example, spk). The length l of both fields has to be such that the time of passage $t_p=l/v$ through the fields has to be equal to the period T of the adiabatically rotating magnetic field $\mathbf{B}_1(t)$. Initial neutrons should be polarized in the direction of $\mathbf{B}(0)$. Then, the initial wave function is

$$\Psi_{\rm in}(\mathbf{r},t)\sim C_{\rm in}e^{i\mathbf{k}\cdot\mathbf{r}}e^{-i(\hbar^2k^2/2m)t/\hbar}|n;\mathbf{B}(0)\rangle_W \ . \tag{94}$$

The waves arriving to the detector $\mathbf{L}$ along paths ijk and spk (Fig. 4) are described by wave functions

$$\begin{aligned}\Psi_{spk}(\mathbf{L},t)&\sim e^{i\mathbf{k}\cdot\mathbf{L}}e^{-i(\hbar^2k^2/2m)t/\hbar}e^{-iE_nT/\hbar}|n;\mathbf{B}(0)\rangle_W \ ,\\ \Psi_{ijk}(\mathbf{L},t)&\sim e^{i\mathbf{k}\cdot\mathbf{L}}e^{-i(\hbar^2k^2/2m)t/\hbar}e^{-iE_nT/\hbar}\\ &\quad\times\exp[i\gamma_n^W(T)]|n;\mathbf{B}(0)\rangle_W \ ,\end{aligned} \tag{95}$$

where E_n is given in (17). Since $\Psi_{ijk}(\mathbf{L},t)$ and $\Psi_{spk}(\mathbf{L},t)$ belong to the same ray the intensity of their superposition $\Psi_{ijk}(\mathbf{L},t)+\Psi_{spk}(\mathbf{L},t)$ is a simple function of Berry's phase $\gamma_n^W(T)$ and therefore

$$I(\mathbf{L})\propto|\Psi_{ijk}(\mathbf{L})|^2[1+\cos\gamma_n^W(T)] \ . \tag{96}$$

By comparing (96) with (93) we see that the difference is in the argument of cosine. With a rotating field (of length $l=vT$) along one path, the argument of the cosine equals the sum of $-E_nT/\hbar$ and $\gamma_n^W(T)$. If in addition a static magnetic field $\mathbf{B}(0)$ (of the same length) is present along another path the argument of the cosine is $\gamma_n^W(T)$. Therefore, the role of the static field is just to eliminate the term $-E_nT/\hbar$ so that $I(\mathbf{L})$ depends only on $\gamma_n^W(T)$.

D. Berry's interference experiment with arbitrary time of passage

If the (common) length l of the static magnetic field $\mathbf{B}_s=\mathbf{B}(0)$ and of the adiabatically rotating magnetic field

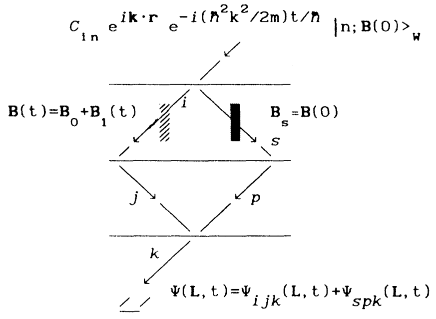

FIG. 4. Scheme of an interference experiment for measuring the Berry phase $\gamma_n^W(T)$ ($l=\hbar kT/m$). If l is arbitrary this experiment verifies Eq. (99).

$\mathbf{B}(t)=B_0\mathbf{e}_z+\mathbf{B}_1(t)$ is arbitrary the states

$$\begin{aligned}\Psi_{spk}(\mathbf{L},t)&\sim e^{i\mathbf{k}\cdot\mathbf{L}}e^{-i(\hbar^2k^2/2m)t/\hbar}e^{-iE_nt_p/\hbar}|n;\mathbf{B}(0)\rangle_W\\ &=e^{i\mathbf{k}\cdot\mathbf{L}}e^{-i(\hbar^2k^2/2m)t/\hbar}e^{-iE_nt_p/\hbar}|n;\mathbf{B}(0)\rangle_* \ ,\\ \Psi_{ijk}(\mathbf{L},t)&\sim{}^{i\mathbf{k}\cdot\mathbf{L}}e^{-i(\hbar^2k^2/2m)t/\hbar}e^{-iE_nt_p/\hbar}\\ &\quad\times\exp[i\gamma_n^W(t_p)]|n;\mathbf{B}(t_p)\rangle_W\\ &=e^{i\mathbf{k}\cdot\mathbf{L}}e^{-i(\hbar^2k^2/2m)t/\hbar}e^{-iE_nt_p/\hbar}|n;\mathbf{B}(t_p)\rangle_*\end{aligned} \tag{97}$$

do not belong to the same ray. Consequently the intensity in the detector depends on the phase difference between the spin states as well as on the absolute value of their scalar product:

$$I(\mathbf{L})\propto|\Psi_{ijk}(\mathbf{L})|^2[1+|{}_*\langle n;\mathbf{B}(0)|n;\mathbf{B}(t_p)\rangle_*|\cos g_n(t_p)] \ . \tag{98}$$

This means that in order to determine the phase $g_n(t_p)$ the measurement of $|{}_*\langle n;\mathbf{B}(0)|n;\mathbf{B}(t_p)\rangle_*|$ is necessary, in addition to the measurement of $I(\mathbf{L})$. Taking (54) into account $I(\mathbf{L})$ in (98) may be also written as

$$I(L)\propto|\Psi_{ijk}(\mathbf{L})|^2[1+\cos\eta(t_p)] \ , \tag{99}$$

where

$$\cos\eta(t_p)=\cos\left[\frac{\omega t_p}{2}\right]\cos\left[\frac{\omega t_p}{2}\cos\vartheta\right]+\cos\vartheta\sin\left[\frac{\omega t_p}{2}\right]\sin\left[\frac{\omega t_p}{2}\cos\vartheta\right] \ . \tag{100}$$

One easily sees that for $t_p=T$ the expression (99) becomes identical to the expression in (96), namely,

$$\begin{aligned}I(\mathbf{L})\propto1-\cos(\pi\cos\vartheta)&=1+\cos[\pi(1-\cos\vartheta)]\\ &=1+\cos\gamma_n^W(T) \ .\end{aligned}$$

Berry associated the attribute "geometrical" with $\gamma_n^W(T)$ because he had noticed that $\gamma_n^W(T)$ is equal to $-n\Omega/2$, where Ω is the spherical angle spanned by $\mathbf{B}(t)$ during the time interval $(0,T)$. However, expression (100) suggests another geometrical interpretation, namely, the angle $\eta(t_p)$ is equal to the length of the third side of the spherical triangle on the sphere of unit radius whose two other sides are $\omega t_p/2$ and $(\omega t_p/2)\cos\vartheta$, respectively, their mutual angle being ϑ (Fig. 5). The analogous interpretation was given by Pancharatnam [21] in the study of the interference of polarized light. This interpretation was recently applied by Berry [33].

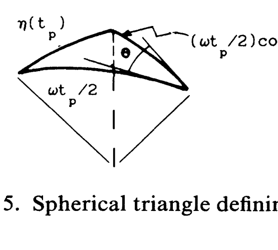

FIG. 5. Spherical triangle defining $\eta(t_p)$.

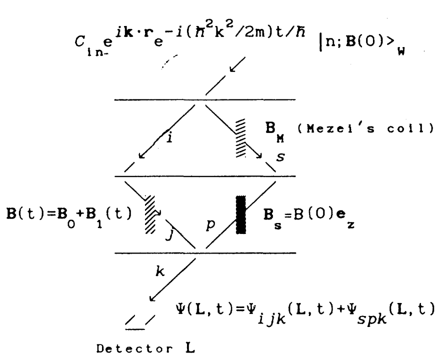

FIG. 6. Proposed experiment for measuring $\gamma_n^W(t_p)$ for arbitrary t_p.

E. A proposed interference experiment for measuring $\gamma_n^W(t_p)$

It seems that by combining two static fields along one path and adiabatically rotating a field along the other path one could measure $\gamma_n^W(t_p)$ for arbitrary t_p (Fig. 6).

The first static field should transform the state $|n;\mathbf{B}(0)\rangle_W$ into an eigenstate of σ_z. This is achievable with the aid of Mezei's coil [37] which creates the field

$$B_M=(\pi h v/2\mu l_M)[\sin(\vartheta/2)e_x+\cos(\vartheta/2)\mathbf{e}_z] \tag{101}$$

on the length l_M. In Mezei's coil [37] the neutrons undergo half-precession during time $t_M=l_M/v$ (Fig. 7). In the second static field, $\mathbf{B}_s=B(0)\mathbf{e}_z$ on the length l, the latter state acquires the dynamical phase $-E_n t_p/\hbar$ $(t_p=l/v)$. The second beam passes through the rotating field $\mathbf{B}(t)$. Thus, the waves hitting the detector L are

$$\Psi_{spk}(\mathbf{L},t)\sim e^{i\mathbf{k}\cdot\mathbf{L}}e^{-i(\hbar^2k^2/2m)t/\hbar}e^{in\pi/2}e^{-iE_nt_p/\hbar}|n\rangle_z\ ,$$
$$\Psi_{ijk}(\mathbf{L},t)\sim e^{i\mathbf{k}\cdot\mathbf{L}}e^{-i(\hbar^2k^2/2m)t/\hbar}e^{-iE_nt_p/\hbar}\times\exp[i\gamma_n^W(t_p)]|n;\mathbf{B}(t_p)\rangle_W\ , \tag{102}$$

and the number of detected neutrons is given by

$$I(\mathbf{L})\propto 1+\cos(\vartheta/2)\cos\left[\gamma_n^W(t_p)-\frac{n\pi}{2}\right]. \tag{103}$$

We see that for given ϑ the intensity $I(\mathbf{L})$ depends directly on $\gamma_n^W(t_p)$, which means that in this experiment the phase $\gamma_n^W(t)$ is measurable for arbitrary values of t.

VI. SUMMARY AND CONCLUSION

The solution of Schrödinger's equation for a spin-$\frac{1}{2}$ particle in a rotating magnetic field is written in various forms, namely, on using the basis $|\pm\rangle$ of eigenstates of σ_z, then the Born-Fock-Schiff basis, and finally the evolution operator. These different forms serve to evaluate the exact (Pancharatnam) phase difference between the final state $|\psi(t)\rangle$ and various initial states.

For the initial state $|n;\mathbf{B}(0)\rangle$, which are eigenstates of the Hamiltonian at $t=0$, the exact phase difference reduces, in the adiabatic approximation, to the sum of $-E_nt/\hbar$ and $g_n(t)=-n\{\arctan[\cos\tan(\omega t/2)]-(\omega t/2)\cos\vartheta\}$. For $t=T$ the phase $g_n(T)$ is equal to Berry's phase: $g_n(T)=\gamma_n(T)=-n\pi(1-\cos\vartheta)$.

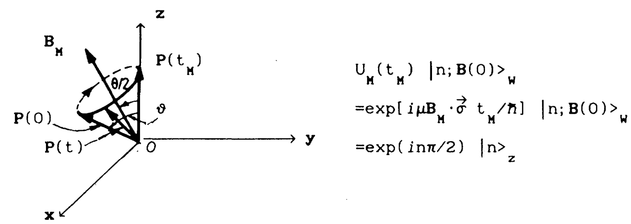

FIG. 7. The principle of Mezei's coil.

For the initial states $|n;B_{\text{ef}}(0)\rangle$, which are eigenstates of the spin component along the direction of the initial effective field $B_{\text{ef}}(0)$, the exact phase has an energy-dependent part and an energy-independent part [see Eq. (60)]. It is shown that the states $|n;\mathbf{B}_{\text{ef}}(0)\rangle_W$ are cyclic and the corresponding Aharonov-Anandan phases (total, dynamical, and geometrical) are evaluated. In the adiabatic limit these exact phases approach Berry's adiabatic values. This is understandable because in the adiabatic limit $\mathbf{B}_{\text{ef}}(t)\to\mathbf{B}(t)$ and $|n;\mathbf{B}_{\text{ef}}(t)\rangle_W\to|n;\mathbf{B}(t)\rangle_W$.

In this work we have proposed neutron interference experiments with pairs of states which do not *necessarily* belong to the same ray.

In such experiments it should be possible to verify the exact and the adiabatic evolution law thanks to the following two facts: (1) the intensity of interference depends linearly on the real part of the scalar product of the superposed states; (2) the real part of the scalar product is a simple function of quantities which characterize the exact and/or the adiabatic evolution law. The exact law is verifiable by substituting in the experiment of Rauch *et al.* [35] the static magnetic field by the rotating magnetic field. The adiabatic law for arbitrary t is verifiable by allowing an arbitrary time of passage $(t_p\neq T)$ through the static and rotating magnetic field in the interference experiment proposed by Berry.

The real part of the mentioned scalar product is also equal to the product of its absolute value by Pancharatnam's phase. From this we conclude that for the determination of Pancharatnam's phase, between states which do not belong to the same ray, the measurement of the absolute value of their scalar product is required in addition of the interference measurement.

Finally, we have proposed an experiment with two static fields and one rotating magnetic field for the measurement of Born-Fock-Schiff phase $\gamma_n^W(t)$ for arbitrary t.

ACKNOWLEDGMENTS

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