Generalized covariance condition and quantization in curved configuration space

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We expound the generalized covariance condition in quantization. This condition is combined with the other standard conditions to determine the quantum-mechanical operator of a quadratic Hamiltonian H_2, represented by a quadratic function of momenta with a nonsingular coefficient matrix depending on coordinates, and the operator of an arbitrary quadratic quantity F_2 belonging to a system whose Hamiltonian is of the H_2 type. It is shown that the operator of F_2 depends also on H_2 and that $\hat{H}_2$ or $\hat{F}_2$ cannot contain a curvature term. We also present a modified ordering method with certain conditions under which the operator of a quadratic quantity can be found by replacing some linear functions of momenta with their operators. Finally, a path-integration formula is formed and shown to yield the expression of $\hat{H}_2$ that we derive by the canonical quantization method.

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I. INTRODUCTION

As is well known, there exist ambiguities in passing from a complicated classical Hamiltonian and other quantities to their quantum-mechanical operators. Adding the covariance condition to be expounded in the present paper to the other standard conditions will weaken the ambiguities and enlarge the class of quantities for which the operators can be determined uniquely. We will study a special but important Hamiltonian H_2, a quadratic Hamiltonian represented by a quadratic function of momenta with a nonsingular coefficient matrix g^{kl} depending on coordinates, as well as an arbitrary quadratic quantity F_2 for a system whose Hamiltonian is of the H_2 type. No trouble can be caused by adding to H_2 and F_2 a pure coordinate part and a linear function of momenta. DeWitt [1] derived the operator $\hat{H}_2$ of H_2 by forming a path-integration formula and first found, we believe, a curvature term λR in addition to the part proportional to the Laplace-Beltrami operator, where λ is a numerical constant and R is the scalar curvature of the Riemannian configuration space with the metric g_{kl}^{-1}. The method was revised by Cheng [2] and Sniatycki [3] to give $\lambda=\hbar^2/6$. Other methods were also developed and yield different results (see, for example, Ref. [4–8]). In fact, it is also easy to construct another path-integration formula so that the curvature term does not appear in $\hat{H}_2$ [9]. Obviously, the problem could not be simpler for an arbitrary F_2. The ambiguities in determining $\hat{H}_2$ and $\hat{F}_2$ also add to the difficulties of the quantization of constrained systems [10]. It would be interesting to try and search for the operators of the quadratic quantities H_2 and F_2 for systems mentioned above by using only some of the most obvious arguments including the covariance condition. If a definite answer can be found in such a way and affirmed to be a standard one, then it can, in addition to its original meaning, also be regarded as one of the criteria for testing a quantization method. We will argue that by removing some obstacles and taking into account the standard conditions comprehensively, the canonical quantization method will be capable of determining the operators of H_2 and F_2.

Up to the present, the canonical quantization approach lacks a correct rule for H_2 that has a complicated coefficient matrix ($g^{kk'}$), and the usual procedures often contradict some standard conditions. Consider a particle moving on a unit spherical surface with the Hamiltonian

$$h_2=\frac{1}{2m}\left[p_\theta^2+\frac{1}{\sin^2\theta}p_\varphi^2\right],$$

where p_θ, p_φ stand for the canonical momenta conjugate to the angles θ and φ, respectively. Denoting by $\hat{p}_\theta, \hat{p}_\varphi$ the operators of p_θ and p_φ under the volume element $\sin\theta\, d\theta\, d\varphi$, one has

$$\hat{p}_\theta=(\sin\theta)^{-1/2}\left[-i\hbar\frac{\partial}{\partial\theta}\right](\sin\theta)^{1/2},$$

$$\hat{p}_\varphi=-i\hbar\frac{\partial}{\partial\varphi}.$$

We now ask whether the operators of p_θ^2 and p_φ^2, which will be denoted by $(p_\theta^2)_Q$ and $(p_\varphi^2)_Q$, are equal to the square of the operators of p_θ and p_φ:

$$(p_\theta^2)_Q=(\hat{p}_\theta)^2?,\quad (p_\varphi^2)_Q=(\hat{p}_\varphi)^2\ ?$$

If both of these relations are valid and if the additive principle also holds (by additive principle we mean that the operator of the sum of quantities F and G is equal to the sum of the operators $\hat{F}$ and $\hat{G}$), then we obtain a wrong operator for h_2. The additive principle must hold. Consequently, at least one of the two relations is wrong. Such a conclusion seems to be unimaginable in the view of the usual quantization procedure for some deep reasons. First, it is believed according to the usual procedure, that the operator of classical quantity can always be determined without reference to the Hamiltonian of the system under consideration. Such an idea may also contradict the additive principle. Second, in the usual procedure the generalized covariance condition is often ignored and violated.

Naturally the physical contents of the quantum theory should be independent of the choice of coordinates, and this requirement gives a certain restriction to the quantization method and leads to the generalized covariance condition which will be expounded in Sec. II. When working with a given coordinate (q) and correctly carrying out the quantization, it is not necessary to consider the problem of coordinate transformation. However, when H_2 has a complicated coefficient matrix, one must treat the generalized covariance condition and other standard conditions carefully in order to identify the correct expression of $\hat{H}_2$ and other operators. We will show that by using the covariance condition and the additive principle as well as some general considerations based on the correspondence principle and the Hermiticity of the operator of a physical quantity, one can uniquely determine the operators of H_2 and F_2.

In Sec. III we will construct the quantization method and rule for H_2 and F_2. It will be shown that the operator of H_2 or F_2 cannot contain a curvature term and that the operator of F_2 depends also on H_2. In Sec. IV we will present a special ordering method with which one can find the operator of a quadratic quantity by replacing some linear functions of momenta with their operators. In Sec. V we will form a path-integration formula which yields the operator expression of H_2 obtained by the canonical quantization method. The final section is for concluding remarks.

II. GENERALIZED COVARIANCE CONDITION IN QUANTIZATION

In this section we will first define a few notations and phraseologies and then expound the generalized covariance condition. Consider a classical system which is described by the generalized coordinates $(q_1,q_2,\ldots,q_s)$, canonical momenta $(p_1,p_2,\ldots,p_s)$, and Hamiltonian $H(q,p,t)$. We assume that each coordinate q_k is still a continuous variable in quantum mechanics. Under the coordinate transformation

$$q_k \to q_k'(q_1,q_2,\ldots,q_s,t) , \tag{2.1}$$

the new momenta and Hamiltonian are

$$p_l'=\sum_k \frac{\partial \dot{q}_k}{\partial \dot{q}_l'}p_k=\sum_k \frac{\partial q_k}{\partial q_l'}p_k , \tag{2.2}$$

$$\begin{aligned} H'(q',p',t)&=H(q,p,t)-\sum_k \dot{q}_k p_k+\sum_l \dot{q}_l' p_l' \\ &=H(q,p,t)+\sum_l \frac{\partial q_l'(q,t)}{\partial t}p_l' . \end{aligned} \tag{2.3}$$

One sees from (2.2) that the classical canonical momenta constitute a covariant vector. Next, on expanding $H(q,p,t)$ in powers of the momenta and denoting by H_n the nth-power part, one gets from (2.3),

$$H_n'(q',p',t)=H_n(q,p,t) \quad (n\neq 1) \tag{2.4}$$

$$H_1'(q',p',t)=H_1(q,p,t)+\sum_l \frac{\partial q_l'(q,t)}{\partial t}p_l' , \tag{2.5}$$

which show that each H_n for $n\neq 1$ is a scalar and H_1 is also a scalar with respect to the time-independent coordinate transformation.

In quantum-mechanical theory a weight factor $W(q,t)$ may be introduced rather arbitrarily to express the inner product of wave functions at time t as

$$(\Phi_A,\Phi_B)=\int W(q,t)dq_1\ldots dq_s \Phi_A^*(q,t)\Phi_B(q,t) .$$

The meaning of wave functions, the operator expressions of quantities, and the form of the Schrödinger equation are influenced by $W(q,t)$. The inner product is determined by states and is independent of $W(q,t)$ and of the choice of coordinates. Let $\hat{p}_l,\hat{H}(q,\hat{p},t)$ be the momentum operator and Hamiltonian operator under coordinate (q) and weight factor $W(q,t)$ and let $\Phi_A(q,t),\Phi_B(q,t)$ be the wave functions of states A and B, respectively. For another weight factor which is identically equal to 1, we write the above quantities as $\hat{p}_l^{(1)}$, $\hat{H}^{(1)}(q,\hat{p}^{(1)},t)$, $\Phi_A^{(1)}(q,t)$, and $\Phi_B^{(1)}(q,t)$. The produce $dq_1dq_2\cdots dq_s$ will often be written as (dq) for simplicity. The dependence of the various quantities on the weight factor can be determined with the help of the inner product and matrix element formulas, and one has

$$\Phi(q,t)=W(q,t)^{-1/2}\Phi^{(1)}(q,t) , \tag{2.6}$$

$$\hat{p}_l=W(q,t)^{-1/2}\hat{p}_l^{(1)}W(q,t)^{1/2} , \tag{2.7}$$

$$\hat{H}(q,\hat{p},t)=W(q,t)^{-1/2}\hat{H}^{(1)}(q,\hat{p}^{(1)},t)W(q,t)^{1/2} , \tag{2.8}$$

where

$$\hat{p}_l^{(1)}=-i\hbar\frac{\partial}{\partial q_l} . \tag{2.9}$$

When the weight factor is identically equal to 1, the Schrödinger equation of course takes the form

$$i\hbar\frac{\partial \Phi^{(1)}(q,t)}{\partial t}=\hat{H}^{(1)}(q,\hat{p}^{(1)},t)\Phi^{(1)}(q,t) ,$$

from which, and from (2.6)–(2.8), one gets

$$\frac{i\hbar}{\sqrt{W(q,t)}}\frac{\partial\sqrt{W(q,t)}\Phi(q,t)}{\partial t}=\hat{H}(q,\hat{p},t)\Phi(q,t) . \tag{2.10}$$

This shows how the time evaluation of the wave function is described by the Hamiltonian operator $\hat{H}$ under a time-dependent weight factor.

It is self-evident that when a function $F(q,p)$ is used to represent a physical quantity, then the same physical quantity should be represented in the coordinate (q') by $F'(q',p')$ such that

$$F'(q'(q),p'(q,p))=F(q,p) .$$

That is, a physical quantity is always represented by a scalar function. Take a single momentum, for example. If a physical quantity is represented in the coordinate (q) by p_1, then it is represented by the scalar $\sum_l f_{(1)}'^{l}(q')p_l'$ where $f_{(1)}'^{l}(q')$ constitute a contravariant vector whose components in the coordinate (q) are $(1,0,0,\ldots)$.

If we let $F(q,p,t)$ be a classical scalar with respect to general coordinate transformation and $\hat{F}(q,\hat{p},t)$ the operator of the physical quantity represented by F, then the matrix element of $\hat{F}$ must be independent of the choice of coordinate. Namely, for each pair of states A and B and for an arbitrary two kinds of coordinates, one has

$$\int W(q,t)(dq)\phi^*_A(q,t)\hat{F}(q,\hat{p},t)\Phi_B(q,t) = \int W'(q',t)(dq')\Phi'^*_A(q',t)\hat{F}'(q',\hat{p}',t)\Phi'_B(q',t) , \tag{2.11}$$

where $\Phi'_A(q',t)$, $\Phi'_B(q',t)$, and $\hat{F}'(q',\hat{p}',t)$ are the wave functions and operator under the new coordinate (q') and weight factor $W'(q',t)$. This relation forms the content of the covariance condition and imposes a certain restriction on the form of operator $\hat{F}$. This condition can also be stated as follows: Under an invariant volume element $W(dq)$ (accordingly, the wave functions are scalar functions), the integrand

$$\Phi^*_A(q,t)\hat{F}(q,\hat{p},t)\Phi_B(q,t)$$

and, therefore, $\hat{F}(q,\hat{p},t)\Phi_B$ and $\hat{F}(q,\hat{p},t)\Phi_A$ must be scalar functions.

Let us give a simple application of the condition by searching for the operator $\hat{F}_1$ of the physical quantity represented by

$$F_1(q,p)=\sum_k f^k(q)p_k . \tag{2.12}$$

Since the coefficients $f^k(q)$ and the weight factor W are real, the imaginary number (i) can only occur in the operator $\hat{F}_1$ through $[-i\hbar(\partial/\partial q_l)]$. It follows that $\hat{F}_1$ is proportional to $(-i\hbar)$. Under an invariant volume element $W(dq)$, the scalar $\hat{F}_1\Phi$ can only be formed by coupling $\partial\Phi/\partial q_k$ to a vector or multiplying Φ by a scalar, and the part containing $\partial\Phi/\partial q_k$ must be

$$\sum_k f^k(q)\left[-i\hbar\frac{\partial}{\partial q_k}\right]\Phi .$$

One therefore has

$$\hat{F}_1\Phi=\sum_k f^k(q)\left[-i\hbar\frac{\partial}{\partial q_k}\right]\Phi-i\hbar B(q)\Phi ,$$

where $B(q)$ is a real scalar function. Taking into account the Hermiticity of $\hat{F}_1$, one gets

$$\hat{F}_1=\tfrac{1}{2}\sum_k f^k(q)\left[-i\hbar\frac{\partial}{\partial q_k}\right] +\frac{1}{2}\left\{\sum_k f^k(q)\left[-i\hbar\frac{\partial}{\partial q_k}\right]\right\}^\dagger . \tag{2.13}$$

Due to the special structure of this formula, it is valid for an arbitrary volume element.

III. QUANTIZATION OF H_2 AND F_2

In this section we will follow the generalized covariance condition as well as the other standard conditions to implement the canonical quantization method and determine the operator of H_2 and that of an arbitrary quadratic quantity F_2:

$$H_2(q,p)=\tfrac{1}{2}\sum_{k,k'} g^{kk'}(q)p_kp_{k'} , \tag{3.1}$$

$$F_2(q,p)=\tfrac{1}{2}\sum_{k,k'} f^{kk'}(q)p_kp_{k'} . \tag{3.2}$$

The coefficients $g^{kk'}$ constitute a contravariant symmetric tensor and can also depend on the time and so do $f^{kk'}$. The matrix $(g^{kk'})$ is assumed to be nonsingular. We will denote by g the determinant of the inverse of this matrix and by $|g|$ the absolute value of g.

For the convenience of applying the covariance condition and reflecting the relationship between the operator structure of an arbitrary F_2 and that of H_2, we choose as $W(dq)$ the invariant volume element $|g|^{1/2}(dq)$. So an arbitrary wave function Φ is a scalar function and so is the result of operation of $\hat{F}_2$ on it. Since the coefficients $f^{kk'}(q)$ and the weight factor are real, the imaginary number (i) can only occur in the operator $\hat{F}_2$ through $[-i\hbar(\partial/\partial q_l)]$. It follows that $\hat{F}_2$ is proportional to $(-i\hbar)^2$. $\hat{F}_2\Phi$ contains the derivatives of Φ no higher than second order, and the part containing the second derivatives must be

$$\frac{(-i\hbar)^2}{2}\sum_{k,k'} f^{kk'}(q)\frac{\partial^2}{\partial q_k\partial q_{k'}}\Phi .$$

One can therefore write

$$\hat{F}_2\Phi=\frac{(-i\hbar)^2}{2}\sum_{k,k'} f^{kk'}(q)\frac{\partial^2}{\partial q_k\partial q_{k'}}\Phi +(-i\hbar)\sum_k A_k(q)\left[-i\hbar\frac{\partial\Phi}{\partial q_k}\right]+(-i\hbar)^2B_0(f)\Phi , \tag{3.3}$$

where $A_k(q)$ and $B_0(f)$ are real functions. Due to the Hermiticity of $\hat{F}_2$, (3.3) can be rewritten as

$$\hat{F}_2\Phi=\tfrac{1}{2}\sum_{k,k'}\left[-i\hbar\frac{\partial}{\partial q_k}\right]^\dagger f^{kk'}(q)\left[-i\hbar\frac{\partial}{\partial q_{k'}}\Phi\right] +(-i\hbar)^2B_0(f)\Phi , \tag{3.4}$$

where

$$\left[-i\hbar\frac{\partial}{\partial q_k}\right]^\dagger=|g|^{-1/2}\left[-i\hbar\frac{\partial}{\partial q_k}\right]|g|^{1/2} . \tag{3.5}$$

The first part in (3.4) is a scalar function and can be expressed as

$$\tfrac{1}{2}\sum_{k,k'}\left[-i\hbar\frac{\partial}{\partial q_k}\right]^\dagger f^{kk'}(q)\left[-i\hbar\frac{\partial}{\partial q_{k'}}\Phi\right] =\frac{(-i\hbar)^2}{2}\sum_{k,k'} f^{kk'}\Phi_{k;k'}+\frac{(-i\hbar)^2}{2}\sum_{k,k'}(f^{kk'})_{;k'}\Phi_k , \tag{3.6}$$

where Φ_k stands for the covariant vector $\partial/\partial q_k\Phi$; $\Phi_{k;k'}$

and $(f^{kk'})_{;l}$ are the covariant derivatives of the vector Φ_k and tensor $f^{kk'}$ in the Riemannian space with the metric $g_{kk'}^{-1}$. In order to conform to the covariance condition and the additive principle, $B_0(f)$ must be a scalar function and depend linearly on $f^{kk'}$. The fact that $\hat{F}_2 \propto (-i\hbar)^2$ also implies that B_0 consists of terms operated by two of the differential operators $(\partial/\partial q_1, \partial/\partial q_2, \ldots)$.

Since $B_0(g)$ vanishes for such a kind of H_2 that the Riemannian space with the metric $g_{kk'}^{-1}$, is flat, i.e., the Riemann-Christoffel tensor $R^l_{kl'k'}$ vanishes and $R^l_{kl'k'}$ has the dimension of $(\partial/\partial q)^2$, $B_0(g)$ can only contain the curvature term in the form

$$\alpha \sum_{k,k'} g^{kk'} R_{kk'} ,$$

with

$$R_{kk'} = \sum_l R^l_{klk'} , \tag{3.7}$$

where α is a numerical constant, $\sum_{k,k'} g^{kk'} R_{kk'}$ is the scalar curvature R of the Riemannian space with the matrix $g_{kk'}^{-1}$, and R is the unique scalar that depends linearly on the second derivatives of $g^{kk'}$ and contains the first derivatives not higher than the second power. Consequently, one has

$$B_0(g) = \alpha R \tag{3.8}$$

or

$$\hat{H}_2 = \tfrac{1}{2} \sum_{k,k'} \left[-i\hbar \frac{\partial}{\partial q_k} \right]^\dagger g^{kk'} \left[-i\hbar \frac{\partial}{\partial q_{k'}} \right] + \alpha(-i\hbar)^2 R . \tag{3.9}$$

As for $B_0(f)$, it can at most contain some covariant derivative terms of $f^{kk'}$ besides $\alpha \sum_{k,k'} f^{kk'} R_{kk'}$, i.e.,

$$B_0(f) = \alpha \sum_{k,k'} f^{kk'} R_{kk'} + \beta \sum_{k,k'} (f^{kk'})_{;k';k} + \sum_{k,k'} (f^{kk'})_{;k'} G_k(g) ,$$

where β is a constant and $G_k(g)$ depend on $g^{kk'}$ and constitute a covariant vector with the dimension of $(\partial/\partial q)$. In fact, a vector cannot be formed with the first derivatives of $g^{kk'}$ so one has

$$\hat{F}_2 = \tfrac{1}{2} \sum_{k,k'} \left[-i\hbar \frac{\partial}{\partial q_k} \right]^\dagger f^{kk'} \left[-i\hbar \frac{\partial}{\partial q_{k'}} \right] + (-i\hbar)^2 \alpha \sum_{k,k'} f^{kk'} R_{kk'} + (-i\hbar)^2 \beta \sum_{k,k'} (f^{kk'})_{;k';k} . \tag{3.10}$$

In order to determine α and β, it is enough to give the following additional rule based on the very idea in the canonical quantization approach: For a free particle with the Hamiltonian

$$h_2 = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) ,$$

the operator of l_z^2 is

$$(l_z^2)_Q = (\hat{l}_z)^2 , \tag{3.11}$$

where p_x, p_y, and p_z are the canonical momenta conjugate to the Cartesian coordinates x, y, and z, respectively, and l_z is the z component of the angular momentum,

$$l_z = x p_y - y p_x .$$

The operator of l_z under the polar coordinate (r, θ, φ) and the volume element

$$m^{3/2} dx\, dy\, dz = |g(r,\theta,\varphi)|^{1/2} dr\, d\theta\, d\varphi = m^{3/2} r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

takes the form

$$\hat{l}_z = \hat{p}_\varphi = |g|^{-1/2} \left[i\hbar \frac{\partial}{\partial \varphi} \right] |g|^{1/2} = -i\hbar \frac{\partial}{\partial \varphi} ,$$

which is Hermitian. Eq. (3.11) can be regarded as a result of the following assumption: If $-i\hbar(\partial/\partial q_j)$ is Hermitian with respect to the volume element $|g|^{1/2}(dq)$, then the operator of p_j^2 is $(\hat{p}_j)^2$. However, we treat as our additional rule (3.11) itself and not a stronger assumption. Applying (3.10) to $(l_z^2)_Q$ one gets

$$(l_z^2)_Q = \hat{l}_z^2 + 4\beta\hbar^2 ,$$

which shows that $\beta = 0$ for a free particle. Clearly, β vanishes for an arbitrary system with three degrees of freedom because it is independent of the special structure of H_2 and F_2. Let systems A and A' have three and s' degrees of freedom, respectively, and A' have a constant matrix (g'^{kl}) and consider a quantity F' of A' such that (f'^{kl}) is also a constant matrix. Denote by $\hat{F}_2$ the operator of F_2 of system A under the volume element $|g|^{1/2}(dq)$ and by $\hat{F}'_2$ the operator of F'_2 under $|g'|^{1/2}(dq')$. The operator of $F_2 + F'_2$ of the whole system $A + A'$ under $|g|^{1/2}(dq)|g'|^{1/2}(dq')$ is $\hat{F}_2 + \hat{F}'_2$ and cannot contain a β term, while

$$\sum_{k,l} (f^{kl} + f'^{kl})_{;l;k} \neq 0 .$$

That implies $\beta = 0$ for the whole system $A + A'$ and therefore for an arbitrary system with more than three degrees of freedom. Next assume A' is an arbitrary system with $s' < 3$. Since $s' + 3 > 3$, the operator of a quadratic quantity of the whole system $A + A'$ does not contain a β term and so the system A' cannot contain a β term. Consequently, β vanishes for any system and (3.10) becomes

$$\hat{F}_2 = \tfrac{1}{2} \sum_{k,k'} \left[-i\hbar \frac{\partial}{\partial q_k} \right]^\dagger f^{kk'} \left[-i\hbar \frac{\partial}{\partial q_{k'}} \right] + (-i\hbar)^2 \alpha \sum_{k,k'} f^{kk'} R_{kk'} . \tag{3.12}$$

We shall argue that $\alpha = 0$. Applying (3.9) and (3.12) to a particle moving on a unit spherical surface with the Hamiltonian h_2 given in the Introduction, one has

$$\hat{h}_2=\tfrac{1}{2}(\hat{l}_x^2+\hat{l}_y^2+\hat{l}_z^2)+4\alpha\hbar^2 ,$$

$$(l_z^2)_Q=\hat{l}_z^2+2\alpha\hbar^2\sin^2\theta .$$

The condition that $\hat{h}_2$ conserve the angular momentum implies

$$[\hat{h}_2,(l_z^2)_Q]=0 ,$$

which means

$$\alpha=0 .$$

With the similar argument that indicates $\beta=0$, one can conclude that α vanishes for any system.

To sum up under the volume element $|g|^{1/2}(dq)$, the operators $\hat{H}_2$ and $\hat{F}_2$ are

$$\hat{H}_2=\tfrac{1}{2}\sum_{k,k'}\left[-i\hbar\frac{\partial}{\partial q_k}\right]^\dagger g^{kk'}\left[-i\hbar\frac{\partial}{\partial q_{k'}}\right] , \tag{3.13}$$

$$\hat{F}_2=\tfrac{1}{2}\sum_{k,k'}\left[-i\hbar\frac{\partial}{\partial q_k}\right]^\dagger f^{kk'}\left[-i\hbar\frac{\partial}{\partial q_{k'}}\right] . \tag{3.14}$$

Rewriting $\hat{F}_2$ in the form independent of the choice of the volume element, one has

$$\hat{F}_2=\tfrac{1}{2}\sum_{k,k'}|g|^{-1/4}\hat{p}_k|g|^{1/2}f^{kk'}\hat{p}_{k'}|g|^{-1/4} , \tag{3.15}$$

which clearly shows the dependence of $\hat{F}_2$ and H_2.

We notice that (3.6) indicates a rule to construct $\hat{F}_2\Phi$ by the covariant derivative method based on the special metric tensor $g_{kk'}^{-1}$. Another rule for F_2 can be stated as follows: In constructing the integral expression for the matrix element $(\Phi_A,\hat{F}_2\Phi_B)$ under the coordinate (q) and the volume element $|g|^{1/2}(dq)$, the integrand can be obtained by replacing the vector $p_k\Phi_A$ and $p_{k'}\Phi_B$ in $\Phi_A^*F_2\Phi_B$ with the vector $-i\hbar(\partial/\partial q_k)\Phi_A$ and $-i\hbar(\partial/\partial q_{k'})\Phi_B$, respectively. Namely,

$$(\Phi_A,\hat{F}_2\Phi_B)=\int|g|^{1/2}(dq)\tfrac{1}{2}\times\sum_{k,k'}\left[-i\hbar\frac{\partial\Phi_A}{\partial q_k}\right]^* f^{kk'}(q)\left[-i\hbar\frac{\partial\Phi_B}{\partial q_{k'}}\right] . \tag{3.16}$$

Both types of rules can also be applied to a linear function $F_1(q,p)$, as given in (2.12). Thus the covariant derivative formula for $\hat{F}_1\Phi$ takes the form

$$\hat{F}_1\Phi=(-i\hbar)\sum_k f^k\Phi_k+\frac{(-i\hbar)}{2}\sum_k (f^k)_{;k}\Phi .$$

In using the second kind of rule, one should combine $(-i\hbar(\partial/\partial q_k)\Phi_A)^*\Phi_B$ and $\Phi_A^*(-i\hbar(\partial/\partial q_{k'})\Phi_B)$ to reflect the Hermiticity of $\hat{F}_1$. Namely,

$$(\Phi_A,\hat{F}_1\Phi_B)=\int|g|^{1/2}(dq)\tfrac{1}{2}\sum_k\left\{f^k(q)\left[\left[-i\hbar\frac{\partial\Phi_A}{\partial q_k}\right]^*\Phi_B+\Phi_A^*\left[-i\hbar\frac{\partial\Phi_B}{\partial q_k'}\right]\right]\right\} . \tag{3.17}$$

IV. A MODIFIED ORDERING METHOD

According to the usual ordering method, it is assumed that a classical quantity $F(q,p)$, expressed in terms of momenta and some coefficient functions can be rewritten by arranging the ordering of the momenta and the coefficient functions so that the operator $\hat{F}$ can be obtained by replacing p_k with the operator $\hat{p}_k$. Such a method is applicable for linear functions of momenta, but often contradicts the additive principle and covariance condition for H_2 and F_2. We now present another ordering method which can be expressed as follows: If a quadratic function of momenta is written in the form

$$F_2(q,p)=\tfrac{1}{2}\sum_{l,l'}\Pi_l F_{ll'}(q)\Pi_{l'} , \tag{4.1}$$

with

$$\Pi_l=\sum_k B_{lk}p_k , \tag{4.2}$$

$$F_{ll'}(q)=F_{l'l}(q) , \tag{4.3}$$

so that the operators $\sum_k B_{lk}[-i\hbar(\partial/\partial q_k)]$ are all Hermitian with respect to the volume element $|g|^{1/2}(dq)$, then the operator of F_2 under this volume element can be written as

$$\hat{F}_2=\tfrac{1}{2}\sum_{l,l'}\hat{\Pi}_l F_{ll'}(q)\hat{\Pi}_{l'} . \tag{4.4}$$

In fact the operator of Π_l takes the form

$$\hat{\Pi}_l=\tfrac{1}{2}\sum_k B_{lk}\left[-i\hbar\frac{\partial}{\partial q_k}\right]+\frac{1}{2}\left\{\sum_k B_{lk}\left[-i\hbar\frac{\partial}{\partial q_k}\right]\right\}^\dagger .$$

Under the Hermiticity condition

$$\left\{\sum_k B_{lk}\left[-i\hbar\frac{\partial}{\partial q_k}\right]\right\}^\dagger=\sum_k B_{lk}\left[-i\hbar\frac{\partial}{\partial q_k}\right] ,$$

one has

$$\hat{\Pi}_l=\sum_k B_{lk}\left[-i\hbar\frac{\partial}{\partial q_k}\right]$$

and

$$\tfrac{1}{2}\sum_{l,l'}\hat{\Pi}_l F_{ll'}(q)\hat{\Pi}_{l'}=\tfrac{1}{2}\sum_{l,l'}\sum_k B_{lk}\left[-i\hbar\frac{\partial}{\partial q_k}\right]^\dagger F_{ll'}(q)\sum_{k'}B_{l'k'}\left[-i\hbar\frac{\partial}{\partial q_{k'}}\right]$$

$$=\tfrac{1}{2}\sum_{k,k'}\left[-i\hbar\frac{\partial}{\partial q_k}\right]^\dagger\sum_{l,l'}B_{lk}F_{ll'}B_{l'k'}\left[-i\hbar\frac{\partial}{\partial q_k}\right] .$$

The method is thus correct.

Let us illustrate the method with some applications. First, we write F_2 in the form

$$F_2=\frac{1}{2}\sum_{k,k'}|g|^{-1/2}p_k|g|f^{kk'}|g|^{-1/2}p_{k'} .$$

Since the operator

$$|g|^{-1/2}[-i\hbar(\partial/\partial q_k)]$$

is Hermitian with respect to the volume element $|g|^{1/2}(dg)$, one can regard $|g|^{-1/2}p_k$ as Π_k and get

$$\hat{F}_2=\frac{1}{2}\sum_{k,k'}\hat{\Pi}_k|g|f^{kk'}\hat{\Pi}_{k'} .$$

For the second example consider the rigid-body rotation. One can write H_2 as

$$H_2=\sum_{l=1}^{3}\frac{1}{2I_l}L_l^2 ,$$

where L_1, L_2, and L_3 are the components of the angular momentum in the body system formed by the principle axes of the inertia tensor and I_1, I_2, and I_3 are the moments of inertia about the body axes. Denoting by (q_1,q_2,q_3) and (p_1,p_2,p_3) the Euler angles and the corresponding momenta and defining a matrix (B_{lk}) by

$$L_l=\sum_k B_{lk}p_k ,$$

one has

$$H_2=\frac{1}{2}\sum_{k,k'}g^{kk'}p_kp_{k'} ,$$

where

$$g^{kk'}=\sum_l\frac{1}{I_l}B_{lk}B_{l'k'} .$$

The operator of L_l can be written as

$$\hat{L}_l=\frac{1}{2}\sum_k B_{lk}\left[-i\hbar\frac{\partial}{\partial q_k}\right]+\frac{1}{2}\left\{\sum_k B_{lk}\left[-i\hbar\frac{\partial}{\partial q_k}\right]\right\}^{\dagger} .$$

Under the volume element $|g|^{1/2}dq_1dq_2dq_3$, the operators $\sum_k B_{lk}[-i\hbar(\partial/\partial q_k)]$ can be checked to be Hermitian and the operators of L_l^2 and H_2 are

$$(L_l^2)_Q=(\hat{L}_l)^2 ,$$

$$\hat{H}_2=\sum_l\frac{1}{2I_l}(\hat{L}_l)^2 .$$

The same argument is tenable for the angular momentum components L_x, L_y, and L_z along the spatial axes and the operators of L_x^2, L_y^2, and L_z^2 are $(\hat{L}_x)^2$, $(\hat{L}_y)^2$, and $(\hat{L}_z)^2$, respectively.

V. PATH-INTEGRATION FORMULA FOR DERIVING $\hat{H}_2$

Without any loss of generality let us consider a system with the Hamiltonian H_2 given in (3.1). The Lagrangian thus takes the form

$$L(q,\dot{q},t)=\frac{1}{2}\sum_{k,k'}M_{kk'}(q,t)\dot{q}_k\dot{q}_{k'} , \tag{5.1}$$

where q_k and $\dot{q}_k$ are coordinates and velocities and $(M_{kk'})$ is the inverse matrix of $(g^{kk'})$ and is assumed to be positive definite. We define $A(q,t,\epsilon)$ and $B(q,t,\epsilon)$ by

$$A(q,t,\epsilon)=\int|M|(q',t)|^{1/2}dq'_1\cdots dq'_s \times\exp\left\{\frac{i\epsilon}{\hbar}L\left[\frac{q+q'}{2},\frac{q-q'}{\epsilon},t\right]\right\} , \tag{5.2}$$

$$B(q,t,\epsilon)=\int|M(q',t)|^{1/2}dq'_1\cdots dq'_s\Phi(q',t) \times\exp\left\{\frac{i\epsilon}{\hbar}L\left[\frac{q+q'}{2},\frac{q-q'}{\epsilon},t\right]\right\} , \tag{5.3}$$

where ϵ is a small real number, $|M(q',t)|$ stands for the determinant of $(M_{kk'})$, and $\Phi(q',t)$ is the wave function under the weight factor $|M(q',t)|^{1/2}$.

$$L((q+q')/2,(q-q')/\epsilon,t)$$

is obtained from $L(q,\dot{q},t)$ by replacing q_l and $\dot{q}_l$ with $(q_l+q_l')/2$ and $(q_l-q_l')/\epsilon$, respectively.

It is easy to prove that

$$\frac{B(q,t,\epsilon)}{A(q,t,\epsilon)}\approx\Phi(q,t)+\frac{\epsilon}{i\hbar}\hat{H}_2\Phi(q,t) \quad \text{for } \epsilon\sim 0 , \tag{5.4}$$

where $\hat{H}_2$ is the operator derived in Sec. III. As mentioned in Sec. II, since the weight factor $|M(q,t)|^{1/2}$ depends on the time, the Schrödinger equation takes the form

$$i\hbar|M(q,t)|^{-1/4}\frac{\partial}{\partial t}|M(q,t)|^{1/4}\Phi(q,t)=\hat{H}_2\Phi(q,t) .$$

Consequently, relation (5.4) is equivalent to the following time-evolution formula of the wave function:

$$|M(q,t)|^{1/4}\Phi(q,t+\epsilon)\approx|M(q,t)|^{1/2}\frac{B(q,t,\epsilon)}{A(q,t,\epsilon)} \quad \text{for } \epsilon\sim 0 . \tag{5.5}$$

This is the path-integration formula which yields the Hamiltonian operator $\hat{H}_2$ derived in Sec. III.

VI. CONCLUDING REMARKS

We have expounded the generalized covariance condition in quantization and studied the quantization problem of the quadratic quantities H_2 and F_2. In searching for the operators of H_2 and F_2, we have used the following standard conditions:

(1) The quantum-mechanical operator of a physical quantity is Hermitian.

(2) The additive principle: The operator of the sum of the quantities F and G is equal to the sum of the operators of F and G.

(3) The generalized covariance condition: The matrix element of the operator for a given quantity and two given states is independent of the choice of coordinates.

(4) For a free particle with the Hamiltonian

$$h_2=\frac{1}{2m}(p_x^2+p_y^2+p_z^2)\ ,$$

the operator of l_z^2 is equal to $(\hat{l}_z)^2$, where $l_z=xp_y-yp_x$.

(5) For a particle moving on a fixed spherical surface, the kinetic-energy operator commutes with the operator of l_z^2 where $l_z=p_\phi$ and stands for the z component of the angular momentum about the center of the sphere, which is also the origin of the coordinate system.

Based on these conditions as well as some general considerations, we have determined the operators of H_2 and F_2. The quantization rule for these quantities has also been given. For the convenience of some applications we have developed a modified ordering method. We have also formed a path-integration formula with which one can derive the expression of $\hat{H}_2$ given in Sec. III. It is worthwhile emphasizing again that the operator of F_2 depends also on H_2 and that $\hat{H}_2$ or $\hat{F}_2$ cannot contain a curvature term.

A special feature of our method to search for the operators of H_2 and F_2 is to use only the easily prehensile arguments. This has already been seen clearly. The situation is different for the other methods. For instance, among the different path-integration formulas used in Ref. [1,2] and the present paper (Sec. V), it is difficult to choose unless one tests them with other standard conditions. Similarly, without reference criteria one does not know whether some more complicated assumptions used in the literature are convincing. Another feature of our method is to study H_2 together with F_2. In this way we have been able to make use of the standard conditions effectively and clarify how the operator expression of F_2 depends on H_2.

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