Anharmonic and nonclassical effects of the quantum-deformed harmonic oscillator

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An isomorphism is realized between the (q)-deformed harmonic oscillator and a particular anharmonic oscillator model: it permits insight into the problem of the physics behind the deformation parameter q. In our model, the anharmonic coupling strength is shown to be proportional to q. The isomorphism permits us also to analyze and to interpret the appearance of various nonclassical features induced by a q deformation during the time evoluton of an su(2) coherent state.

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I. INTRODUCTION

Deformations of groups and corresponding algebras have been considered for some time [1]. In particular, the q deformation of the Lie algebras, also called quantum groups, has recently attracted much attention since its introduction by Jimbo [2–4] and Drinfeld [5]. Quantum groups are the natural algebraic setting for the inverse-scattering problem, and a great deal of interest has been paid to their relevance to problems of either physical or mathematical nature. This deformation of the Lie algebras is indeed strictly connected on one hand with rational conformal field theory [6–9], exactly solvable statistical models [10], and inverse-scattering theory applied to integrable models in quantum field theories [11,12], and on the other hand with commutative geometry [13], knot theory in three dimensions, and in general with various areas of mathematical physics where the Yang-Baxter equation plays an essential role [12,14–17]. In this paper our concern will be addressed to the q deformation of the Lie algebra of su(2), usually denoted by su$(2)_q$, which has already been extensively studied by many authors after the work of Kulish and Reshetikhin [14]. In particular, we will be concerned with the realization of this quantum group in terms of a q-deformed quantum harmonic oscillator [18] as discussed by Biedenharn [19], and independently by MacFarlane [20].

In recent years, although many aspects of the q deformation of the Bose harmonic oscillator algebra have been investigated, one of the most interesting problems still at issue is the physical meaning of the deformation parameter q in the realization of su$(2)_q$ as a q analog of the harmonic oscillator [21]. This is the problem addressed here.

In this paper we show that the q-harmonic oscillator can be used to describe a specific anharmonic oscillator model; in particular, we analyze the conditions for which coherent states of the anharmonic oscillator and coherent states of the q-harmonic oscillator are equivalent. This procedure allows us to give q a definite physical meaning: it is a measure of the anharmonicity. This is discussed in Sec. III, whereas a brief review of our anharmonic oscillator model is given in Sec. II. The equivalence between the two oscillator models is a very helpful one and is in turn used to investigate the effect of a q deformation during the time development of a coherent state of the conventional harmonic oscillator. This reveals effects of self-squeezing that depend on the amplitude of the q deformation: i.e., the reduction of the uncertainty expectations of the two orthogonal components (quadratures) of the harmonic oscillator field below their vacuum values varies with q. Furthermore, the q deformation does alter also the minimality properties of the initial minimum uncertainty coherent state, but not its Poissonian counting statistics. The equivalence between the two oscillator models again provides a straightforward way for a sound physical interpretation for the occurrence of these phenomena.

II. ANHARMONIC OSCILLATOR MODEL

We take as our oscillator model a system described by the anharmonic Hamiltonian [22]

$$\hat{H}_\lambda=\hat{H}_0+\frac{\lambda}{\omega_0}\hat{N}^3\equiv\hat{H}_0+\hat{H}_1 \ , \tag{2.1}$$

where $\hat{H}_0=\hat{b}^\dagger\hat{b}+\frac{1}{2}$ is the Hamiltonian of the harmonic part of the oscillator, and the anharmonic term is taken to be proportional to $\hat{N}^3$. $\hat{N}=\hat{b}^\dagger\hat{b}$ is the number operator, whereas $\hat{b}$ and $\hat{b}^\dagger$ are, respectively, the lowering and raising operators, satisfying standard Bose commutation rules. Here $\hbar=1$ so $\hat{H}_\lambda$ is in units of ω_0 when $\hat{H}_0$ is in units of ω_0. ω_0 is the fundamental frequency of the harmonic part of the oscillator. The anharmonic coupling strength λ is positive and conveniently taken as $\lambda\equiv\omega_0\gamma^2/6$, where γ will be discussed further below.

Within the domain of quantum optics or solid-state physics, typical anharmonic deformations [23] of the relevant fields are extremely small at ordinary energies and the Hamiltonian in Eq. (2.1) can be rewritten in the representation [24]

$$\hat{H}_\gamma=\Omega_\gamma \hat{a}_\gamma^\dagger \hat{a}_\gamma \ , \quad \Omega_\gamma=\gamma^{-1}\sinh\gamma \ , \tag{2.2}$$

where

$$\hat{a}_\gamma=(\Omega_\gamma^{-1})^{1/2}\left[1+\gamma^2\frac{(\hat{b}^\dagger\hat{b}+1)^2}{2(3!)}\right]\hat{b} \ . \tag{2.3}$$

Quantum states for this Hamiltonian can be constructed from the vacuum defined as $\hat{a}_\gamma|0\rangle=0$. $\hat{a}_\gamma$ and $\hat{a}_\gamma^\dagger$ can be shown [25] to satisfy commutation rules different from the standard ones, and to be the lowering and raising operators for normalized energy eigenstates (number states) $|n\rangle_\gamma$ of the Hamiltonian $\hat{H}_\gamma$. In particular, we will be concerned in this paper with coherent states (CS). They are defined as solutions for the equation $\hat{a}_\gamma|\alpha\rangle_\gamma=\alpha|\alpha\rangle_\gamma$ and can be expressed [25] as a superposition of number states, i.e. [26],

$$\begin{aligned}|\alpha\rangle_\gamma&=C_\gamma\sum_{n=0}^{\infty}\frac{\alpha^n}{(c_{n,\gamma})^{1/2}}|n\rangle_\gamma\\&\equiv\sum_{n=0}^{\infty}c_n^\gamma(\hat{a}_\gamma^\dagger)^n|0\rangle, \quad {}_\gamma\langle\alpha|\alpha\rangle_\gamma=1 \ .\end{aligned}\tag{2.4}$$

Here

$$C_\gamma^{-2}=\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{c_{n,\gamma}} \tag{2.5}$$

and

$$\begin{aligned}c_{n,\gamma}&=n!\Omega_\gamma^{-n}\prod_{k=1}^{n}\left[1+\frac{\gamma^2k^2}{2(3!)}\right]^2\\&=n!\Omega_\gamma^{-n}\left[\left[1+\frac{\gamma^2n^2}{2(3!)}\right]^2\right]! \quad (c_{0,\gamma}=1) \ .\end{aligned}\tag{2.6}$$

The resemblance of the $|\alpha\rangle_\gamma$'s with CS's for the harmonic part of the oscillator is readily seen: however, unlike the latter, the $|\alpha\rangle_\gamma$'s are a linear combination of number states whose squared coefficients do not represent the probability of finding n quanta of the oscillator field in a Poisson distribution [25]. Coherent states for the anharmonic oscillator model outlined here exhibit quite a number of typical nonclassical features which are discussed elsewhere [25].

III. q-DEFORMED HARMONIC OSCILLATOR AND ANHARMONICITY

Let us recall the $(\hat{b},\hat{b}^\dagger)$ bose operators for the conventional harmonic oscillator. They have also been introduced in the preceding section as the creation and annihilation operators for the free part of the anharmonic oscillator Hamiltonian in Eq. (2.1), and they satisfy the Weyl-Heisenberg (WH) algebra

$$[\hat{b},\hat{b}^\dagger]=1 \ , \quad [\hat{N},\hat{b}^\dagger]=\hat{b}^\dagger \ , \quad \hat{N}\equiv\hat{b}^\dagger\hat{b} \ . \tag{3.1}$$

MacFarlane [20] and Biedenharn [19] have discussed a deformation of the WH algebra characterized by a parameter q such that

$$\hat{a}_q\hat{a}_q^\dagger-q\hat{a}_q^\dagger\hat{a}_q=q^{-\hat{N}} \ , \quad [\hat{N},\hat{a}_q^\dagger]=\hat{a}_q^\dagger \ . \tag{3.2}$$

q is in general a complex number although here it is taken to be real and greater than 1. In the limit $q\to1$ the q-deformed Weyl-Heisenberg (q-WH) algebra reduces to the conventional WH algebra. It is the purpose of this section to investigate the link between q deformations and anharmonic deformations of the conventional harmonic oscillator.

One can start by characterizing the connection between the q operators $(\hat{a}_q,\hat{a}_q^\dagger)$ and the Bose operators $(\hat{b},\hat{b}^\dagger)$. The former have been shown to be realizable in terms of conventional Bose operators of the form [27]

$$\hat{a}_q=\left[\frac{[\hat{N}+1]_q}{\hat{N}+1^{1/2}}\right]^{1/2}\hat{b} \ , \quad \hat{a}_q^\dagger=\hat{b}^\dagger\left[\frac{[\hat{N}+1]_q}{\hat{N}+1}\right]^{1/2} \ , \tag{3.3}$$

where

$$[x]_q\equiv\frac{q^x-q^{-x}}{q-q^{-1}} \quad\text{or}\quad [x]_q=\frac{e^{sx}-e^{-sx}}{e^s-e^{-s}}=\frac{\sinh sx}{\sinh s} \qquad (s\equiv\ln q) \tag{3.4}$$

is so defined for a c number as well as for an operator. Again, it is clear that in the limit $q\to1$, or $s\to0$, the q operators reduce to the conventional Bose operators.

Quantum states for the q-harmonic oscillator are constructed from the q-deformed vacuum defined as $\hat{a}_q|0\rangle_q=0$. From Eq. (3.3) note that $|0\rangle_q$ and the vacuum for the ordinary harmonic oscillator are the same. On the usual Fock space, one has

$$|n\rangle_q=\frac{(\hat{a}_q^\dagger)^n}{([n]_q!)^{1/2}}|0\rangle \ , \quad {}_q\langle m|n\rangle_q=\delta_{m,n} \ , \tag{3.5}$$

where $[n]_q!=[n]_q[n-1]_q\cdots[1]_q$. Also, $[0]_q$ is defined to be equal to one. From the above it follows that

$$\hat{a}_q^\dagger|n\rangle_q=([n+1]_q)^{1/2}|n+1\rangle_q \ , \tag{3.6}$$

$$\hat{a}_q|n\rangle_q=([n]_q)^{1/2}|n-1\rangle_q \ . \tag{3.7}$$

Further using Eq. (3.3) one finds that

$$\begin{aligned}\hat{N}_q&\equiv\hat{a}_q^\dagger\hat{a}_q=[\hat{b}^\dagger\hat{b}]_q=[\hat{N}]_q \ ,\\ \hat{a}_q\hat{a}_q^\dagger&=[\hat{b}^\dagger\hat{b}+1]_q=[\hat{N}+1]_q \ ,\end{aligned}\tag{3.8}$$

where the operator $\hat{N}_q$ is such that

$$\hat{N}_q|n\rangle_q=[n]_q|n\rangle_q \ , \quad \hat{N}_q|n\rangle=[n]_q|n\rangle \ , \tag{3.9}$$

and similarly,

$$\hat{N}|n\rangle_q=n|n\rangle_q \ , \quad \hat{N}|n\rangle=n|n\rangle \ . \tag{3.10}$$

The same set of eigenvectors $|n\rangle$ and $|n\rangle_q$ expand the whole Hilbert space both for the harmonic oscillator and for its q analog. Note now that the q analog $\exp(\alpha\hat{a}_q^\dagger)|0\rangle$ to the coherent states of the conventional harmonic oscillation is not available in the present case for the q operators, since it is not normalizable for all $q\neq1$ ($\alpha\neq0$). Following Biedenharn [19] one can define coherent states (q-CS) for the q-harmonic oscillator via the alternative definition, i.e.,

$$\hat{a}_q|\alpha\rangle_q=\alpha|\alpha\rangle_q \ . \tag{3.11}$$

This yields [26]

$$|\alpha\rangle_q=C_q^{-1}\exp_q(\alpha\hat{a}_q^\dagger)|0\rangle \ , \quad \text{with } C_q^{-2}=\exp_q\alpha^2 \ . \tag{3.12}$$

Here the q analog of the exponential function has been introduced,

$$\exp_q\chi\equiv\sum_{n=0}^{\infty}\frac{\chi^n}{[n]_q!} \ . \tag{3.13}$$

Using the standard procedure one can more conveniently decompose $|\alpha\rangle_q$ on the basis $|n\rangle_q$,

$$|\alpha\rangle_q=C_q\sum_{n=0}^{\infty}\frac{\alpha^n}{([n]_q!)^{1/2}}|n\rangle_q$$

$$\equiv\sum_{n=0}^{\infty}c_n^q(\hat{a}_q^\dagger)^n|0\rangle \ , \quad {}_q\langle\alpha|\alpha\rangle_q=1 \ . \tag{3.14}$$

These states contain a nearly classical distribution of quanta. The density probability distribution of finding n quanta of the conventional harmonic oscillator in a q-CS is given by

$$P_n^q(\alpha)=|\langle n|\alpha\rangle_q|^2=C_q^2\frac{\alpha^{2n}}{[n]_q!} \ . \tag{3.15}$$

This function, known from the theory of orthogonal polynomials of discrete variables, coincides, with the weight function of the so-called Charlier q polynomials as discussed in [28]. The effect of a q deformation on the classical density probability distribution of a conventional CS consists in deforming the associated Poisson distribution ($q=1$) by shifting its average value toward n smaller than $|\alpha|^2$ with a corresponding increase of its most probable value. This is clearly seen in Fig. 1 where the distribution $P_n^q(\alpha)$ is reported for various values of q. Note that for any $q\neq1$ its width (root-mean-square deviation) becomes less than the average value, that is, a sub-Poissonian distribution. The states (3.14) exhibit also effects of quadrature squeezing by which we mean that in the $|\alpha\rangle_q$'s the quantum fluctuations of one or the other of the two orthogonal components *(quadratures)* of the conventional harmonic oscillator field are smaller than those associated with the vacuum state [29]. The two quadratures are denoted by

$$\hat{x}=\frac{\hat{b}+\hat{b}^\dagger}{\sqrt{2}} \ , \quad \hat{p}=\frac{\hat{b}-\hat{b}^\dagger}{i\sqrt{2}} \ , \quad [\hat{x},\hat{p}]=i \tag{3.16}$$

and physically they describe the *in*-phase and *out-of*-phase components of the harmonic oscillator field. The squeezing condition can be written as $\langle(\Delta\hat{x})^2\rangle$ or $\langle(\Delta\hat{p})^2\rangle<\frac{1}{2}$, where $\frac{1}{2}$ is the dispersion for the vacuum [29]. This squeezing has been investigated in [30] and demonstrated via a numerical computation for some values of the parameters α and the whole range of q's. On the contrary, q-CS's do not display any squeezing with respect to the quadratures *(q quadratures)* of the q-harmonic oscillator [31]. q-CS's, however, are minimum-uncertainty states (MUS) [32] for both quadratures [33]. In the next section the study of the time evolution of these phenomena will reveal a different type of squeezing associated with a q deformation of the conventional harmonic oscillator.

Here, instead, we proceed to examine a condition for which coherent states $|\alpha\rangle_\gamma$ for the anharmonic oscillator of Sec. III are equivalent to q-CS. This can indeed be demonstrated [cf., Eqs. (2.4) and (3.14)] provided

$$[n]_q!\to c_{n,\gamma} \quad \text{and} \quad \hat{a}_q\to\hat{a}_\gamma \ . \tag{3.17}$$

Using $[n]_q$ in Eq. (3.4) in terms of s and the infinite-product expansion [34]

$$\sinh x=x\prod_{k=1}^{\infty}\left[1+\frac{x^2}{\pi^2k^2}\right] \tag{3.18}$$

one has

$$[n]_q!=n!\Omega_\gamma^{-n}\prod_{k=1}^{\infty}\left[1+\frac{s^2n^2}{\pi^2k^2}\right]\prod_{k=1}^{\infty}\left[1+\frac{s^2(n-1)^2}{\pi^2k^2}\right]$$

$$\times\prod_{k=1}^{\infty}\left[1+\frac{s^2(n-2)^2}{\pi^2k^2}\right]\cdots$$

$$=n!\Omega_\gamma^{-n}\left[\prod_{k=1}^{\infty}\left[1+\frac{s^2n^2}{\pi^2k^2}\right]\right]! \ . \tag{3.19}$$

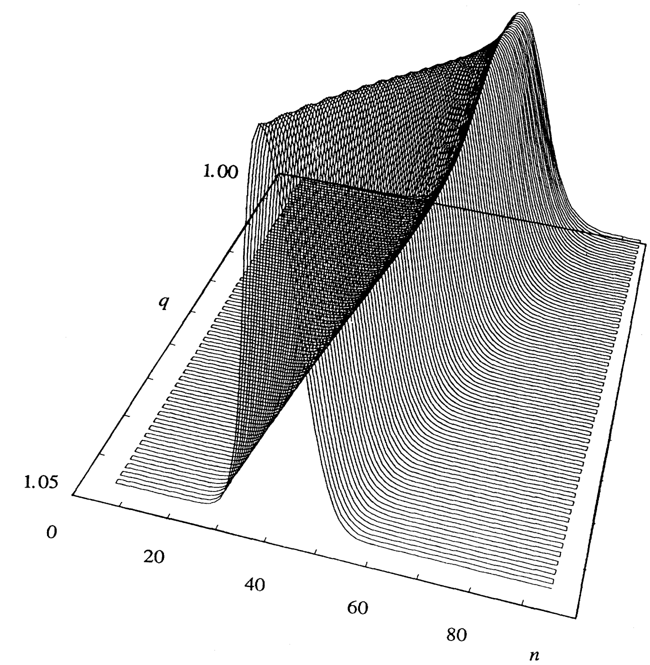

FIG. 1. Density probability distribution (vertical axis) of finding n quanta in a coherent state of a q-deformed oscillator with $\alpha=7$. The deformation, measured by q, with respect to the Poisson distribution ($q=1$) of a coherent state for the conventional harmonic oscillator having the same α, consists in a change of the counting statistics (Poissonian$\to$sub-Poissonian) and in a shift of the average value toward n smaller than α^2 with a corresponding increase of the distribution most probable value. As we recede away from limit case $q=1$ these features become more and more noticeable.

The infinite product can now be approximated by

$$\prod_{k=1}^{\infty}\left[1+\frac{s^2n^2}{\pi^2k^2}\right]=1+\frac{s^2n^2}{3!}+\cdots\cong\left[1+\frac{s^2n^2}{2(3!)}\right]^2 \quad (3.20)$$

provided

$$sn<1\ . \quad (3.21)$$

Thus for n and s that satisfy this condition, $[n]_q!$ exactly gives $c_{n,\gamma}$ upon the identification $s\equiv\gamma$. Similarly [cf. Eqs. (2.3) and (3.3)], one can show that in the same limit $\hat{a}_q\rightarrow\hat{a}_\gamma$. Note, on the other hand, that when n and s (or γ) do not satisfy Eq. (3.21) the coefficients c_n^q (or c_n^γ) in the expansion (3.14) [or (2.4)] vanish, e.g., for displacements α and parameters s (or γ) such that [35]

$$\alpha(\tfrac{1}{4}\alpha+2)<s^{-1}=(\ln q)^{-1}\ , \quad (3.22)$$

which ultimately establishes a condition for the equivalence $|\alpha\rangle_q\leftrightarrow|\alpha\rangle_\gamma$.

It is here further instructive to compare the density probability distribution in Eq. (3.15) with the analogous probability but for coherent states of the anharmonic oscillator of Sec. II, that is,

$$P_n^\gamma(\alpha)=|\langle n|\alpha\rangle_\gamma|^2=C_\gamma^2\frac{\alpha^{2n}}{c_{n,\gamma}}\ . \quad (3.23)$$

A numerical evaluation is reported in Fig. 2 for values of s and α, respectively, conforming and not conforming with the condition (3.22). In the latter case $P_n^{\gamma_2}(\alpha_2)$ is strongly shifted with respect to $P_n^{q_2}(\alpha_2)$, whereas in the former case the two distributions are nearly the same with a relative difference as small as one part in 10^5 (inset of Fig. 2). Owing to the definition (3.15) and (3.23) of the probability in terms of overlap over the same state $|n\rangle$, the remarkable agreement between $P_n^{\gamma_1}(\alpha_1)$ and $P_n^{q_1}(\alpha_1)$ implies the equivalence of the states $|\alpha\rangle_q$ and $|\alpha\rangle_\gamma$ under the condition (3.22). This equivalence will be particularly useful in discussing the physical origin of certain nonclassical properties associated with a q deformation. Incidentally, as anticipated above, Fig. 2 is an instance of the vanishing of the distribution for values of $n>s^{-1}$, depending on whether the condition (3.22) is satisfied or not.

In conclusion, for appropriate displacements (α) and anharmonic couplings (λ) coherent states for the anharmonic oscillator model of Sec. II are correctly described in terms of coherent states of the q-deformed Lie algebra of SU(2), with $q\cong\exp(\lambda/\omega_0)^{1/2}$. This result is particularly important because the q-WH algebra deformation parameter (q) can be given a direct physical meaning: its logarithm is directly proportional to the square root of the anharmonic coupling strength for the anharmonic oscillation model of Sec. II.

IV. q DEFORMATION AND SELF-SQUEEZING

Here squeezing, a well-known manifestation of certain nonclassical states, will be related, in our model, to a q deformation. We adopt the usual definition of squeezing [29] as introduced in Sec. III. Specifically in this section we will demonstrate that because of a q deformation the state generated during the time evolution of an SU(2) coherent state is self-squeezed. The self-squeezing associated with a q deformation is best illustrated through an analysis of the time development of the harmonic oscillator [18] under the nonlinear effective Hamiltonian that governs a q deformation. Taking advantage of the equivalence $|\alpha\rangle_q\leftrightarrow|\alpha\rangle_\gamma$ (Sec. III), the discussion is carried out by using the $|\alpha\rangle_\gamma$'s. This also provides insight into the squeezing mechanism. Use of the $|\alpha\rangle_q$'s, as they appear in Eq. (3.14), would only tend to obscure the physics behind the mechanism of squeezing. For deformations q of the WH algebra and oscillator displacements α satisfying Eq. (3.22) the $|\alpha\rangle_q$'s are the corresponding coherent states of the oscillator Hamiltonian in Eq. (2.1) [36]. The Heisenberg equation of motion for the field an-

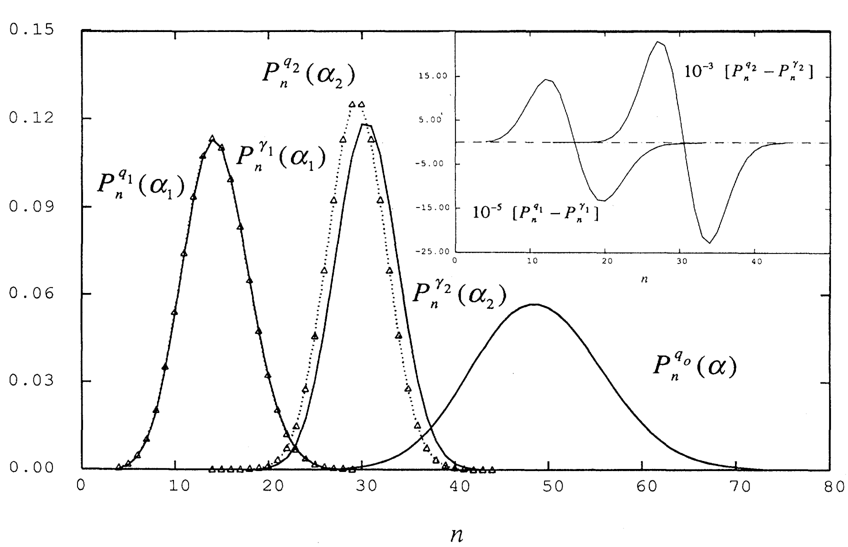

FIG. 2. Coherent states ($|\alpha_1\rangle_{q_1}$, $|\alpha_2\rangle_{q_2}$) of a q-deformed oscillator and coherent states ($|\alpha_1\rangle_{\gamma_1}$, $|\alpha_2\rangle_{\gamma_2}$) of an oscillator with a third-order anharmonicity in the number of quanta. Their equivalence, inferred from the equivalence between the corresponding probability distributions, holds depending on whether the oscillator parameters satisfy ($\alpha_1=4$, $\gamma_1=0.05$) or do not satisfy ($\alpha_2=10$, $\gamma_2=0.1$) the condition (3.22), respectively. Here $q=e^\gamma$. The relative difference between the former and the latter distributions is reported in the inset. $P_n^{q_0}(\alpha)$ is a Poisson ($q_0=1$) reference distribution with $\alpha=7$.

nihilation operator is

$$\dot{\hat{b}}=-i[\hat{b},\hat{H}_{\lambda}]=-i\left[1+\frac{\lambda}{\omega_0}(3\hat{N}^2+3\hat{N}+1)\right]\hat{b}(0)\ . \qquad (4.1)$$

Since $\hat{H}_{\lambda}$ is a function of $\hat{N}$ the latter is a constant of motion, reflecting the conservation of the number of quanta in the oscillator, so that the solution of Eq. (4.1) is simply an exponential [37]

$$\hat{b}(t)=\exp\left[-i\left[1+\frac{\gamma^2}{6}(3\hat{N}^2+3\hat{N}+1)\right]t\right]\hat{b}(0)\ . \qquad (4.2)$$

This result lends itself to some interesting considerations.

First, if $|\beta\rangle$ is the initial state of the harmonic oscillator at $t=0$, that is, a CS with mean number of quanta $|\beta|^2$ [$\hat{b}(0)|\beta\rangle=\beta|\beta\rangle$], a q deformation induces squeezing and antisqueezing in each quadrature of the harmonic oscillator during different intervals of time. Using Eq. (4.2) one can calculate the expectation values $\langle[\Delta\hat{x}(t)]^2\rangle$ and $\langle[\Delta\hat{p}(t)]^2\rangle$ to obtain

$$\begin{aligned}\langle[\Delta\hat{x}(t)]^2\rangle&=\tfrac{1}{2}+C(\hat{b}^{\dagger}\hat{b})+\mathrm{Re}[\hat{b}(t)]^2\\&=\tfrac{1}{2}+|\beta|^2\left[1-\frac{|S_1(\beta,\gamma,t)|^2}{e^{2|\beta|^2}}\right]+\mathrm{Re}\left[\frac{\beta^2}{e^{\theta_1(\beta,\gamma,t)}}\left[S_2(\beta,\gamma,t)-\frac{S_1^2(\beta,\gamma,t)}{e^{\theta_1(\beta,\gamma,t)}}\right]\right]\end{aligned} \qquad (4.3)$$

and

$$\begin{aligned}\langle[\Delta\hat{p}(t)]^2\rangle&=\tfrac{1}{2}+C(\hat{b}^{\dagger}\hat{b})-\mathrm{Re}[\hat{b}(t)]^2\\&=\tfrac{1}{2}+|\beta|^2\left[1-\frac{|S_1(\beta,\gamma,t)|^2}{e^{2|\beta|^2}}\right]-\mathrm{Re}\left[\frac{\beta^2}{e^{\theta_1(\beta,\gamma,t)}}\left[S_2(\beta,\gamma,t)-\frac{S_1^2(\beta,\gamma,t)}{e^{\theta_1(\beta,\gamma,t)}}\right]\right],\end{aligned} \qquad (4.4)$$

where we have introduced the notations $C(\hat{A}\hat{B})\equiv\langle\hat{A}\hat{B}\rangle-\langle\hat{A}\rangle\langle\hat{B}\rangle$, $\theta_1(\beta,\gamma,t)\equiv|\beta|^2+2it(1+2\gamma^2/3)$, $\theta_2(\beta,\gamma,t)\equiv|\beta|^2-i\gamma t$, and the sums

$$S_1(\beta,\gamma,t)=\sum_{n=0}^{\infty}\frac{|\beta|^{2n}}{n!}\exp\left[-i\frac{\gamma^2 t}{2}(n^2+n)\right] \qquad (4.5)$$

and

$$S_2(\beta,\gamma,t)=\sum_{n=0}^{\infty}\frac{|\beta|^{2n}}{n!}\exp[-i\gamma^2 t(n^2+2n)]\ . \qquad (4.6)$$

Physically the variances (4.3) and (4.4) describe how the dispersions of the conventional harmonic oscillator, initially in a coherent state of the WH algebra, evolve in time owing to a q deformation. Since the quadratic dependence on n prevents us from expressing S_1 and S_2 in a closed form in terms of elementary functions, both dispersions are evaluated numerically and reported in Fig. 3 as a function of time for different values of $q\neq1$. Reduction of these variances below $\frac{1}{2}$ is a signature of squeezing in the *in*-phase or *out-of*-phase quadrature components of the harmonic oscillator. Note that squeezing is quite sensitive to the magnitude of the q deformation; fairly small deviations from the undeformed case $q=1$ are sufficient to produce remarkable amounts of squeezing for the time intervals of interest here.

Second, the periodicity with which squeezing and antisqueezing alternate in each variance reminds one of the dynamics of ordinary squeezed states for which the $\hat{x}$ and $\hat{p}$ uncertainties oscillate in time in and out of the vacuum level at twice the oscillator characteristic frequency [38]. However, note that the evolution of the dispersions of a conventional harmonic oscillator induced by ordinary squeezing and that induced by a q deformation are not quite the same. This difference results from the difference in the type of interaction. The former is caused by a two-particle interaction represented by a quadratic Hamiltonian [29], whereas the latter is caused by a six-particle interaction represented by the Hamiltonian $\hat{H}_1$ in Eq. (2.1). The difference is particularly evident when studying the minimality properties associated with the two types of squeezing. In our case the uncertainty product

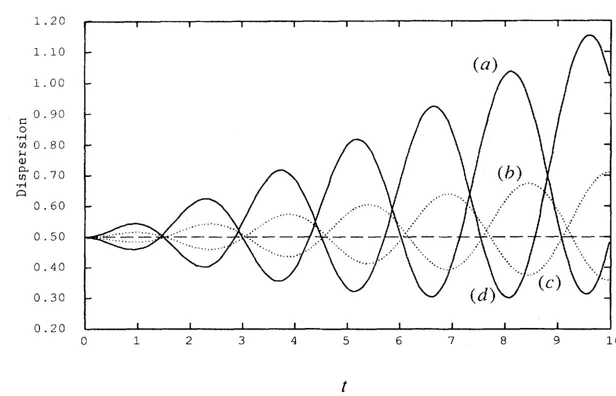

FIG. 3. Oscillations, due to a q deformation, in the time [37] evolution of the *in*-phase [curves (a) and (b)] and *out-of*-phase [curves (c) and (d)] quadrature fluctuations for a conventional harmonic oscillator initially in a coherent state of the Weyl-Heisenberg algebra. The dispersions are evaluated for $q=1.05$ (solid line) and 1.03 (dotted line), and compared to the dispersion of the vacuum state (dashed line). The oscillator displacement is $\alpha=2$. Even fairly small q deformations can produce considerable squeezing that occurs when the dispersion becomes less than that for the vacuum.

for $\hat{x}$ and $\hat{p}$ evaluated from Eqs. (4.3) and (4.4) can be expressed as

$$\langle[\Delta\hat{x}(t)]^2\rangle\langle[\Delta\hat{p}(t)]^2\rangle=\tfrac{1}{4}+C^2(\hat{b}^\dagger\hat{b})+C(\hat{b}^\dagger\hat{b}) -\{\mathrm{Re}C[\hat{b}^{\dagger 2}(t)]\}^2 \geq \frac{|\langle[\hat{x},\hat{p}]\rangle|^2}{4}=\tfrac{1}{4}\,, \qquad (4.7)$$

and depends of course on the relative weights of the different correlations C. The states in this inequality are MUS's when the equal sign holds; this is indeed the case when $s\rightarrow 0$ ($q\rightarrow 1$) since in the absence of interaction the initial CS remains a minimal state. In Fig.4 we demonstrate the absence of minimal behavior due to a q deformation during the time evolution of a coherent state of the WH algebra. Again this is very sensitive to the magnitude of q, particularly during the very first periods of the evolution. On the contrary, because of ordinary squeezing an initial MUS oscillates in time and unlike in our case, in fact, the oscillator will be periodically in a MUS [38].

Third, Eq. (4.2) is the exact operator solution describing the dynamics of the q-harmonic oscillator for deformations and displacements conforming with the condition (3.22). This result clearly shows that the physical effect associated with a q deformation is an *intensity-dependent* change in the phase of the field. This is ultimately what generates squeezing through a q deformation. Since such squeezing is intensity dependent it can be referred to as *self-squeezing* [39], and the state evolving because of a q deformation as self-squeezed states. It is the field of the harmonic oscillator itself that during its evolution squeezes its own quantum fluctuation via a q deformation.

A coherent state of the WH algebra displays quadrature self-squeezing during its evolution because of a q deformation, and yet is not a MUS for the quadratures $\hat{x}$ and $\hat{p}$. Because of this distinctly nonclassical behavior, it is also of interest to examine the relevant counting statistics. This is done by examining the Mandel Q factor [40] $Q=\langle[\Delta(\hat{b}^\dagger\hat{b})]^2\rangle-\langle\hat{b}^\dagger\hat{b}\rangle/\langle\hat{b}^\dagger\hat{b}\rangle$, $[\Delta(\hat{b}^\dagger\hat{b})]^2=(\hat{b}^\dagger\hat{b})^2-\langle\hat{b}^\dagger\hat{b}\rangle^2$ being the variance of the number of oscillator quanta. Namely, $Q\neq 0$ and $Q=0$ indicate, respectively, non-Poissonian or Poissonian statistics. From Eq. (4.2) it is readily seen that Q is identically 0 if the initial state of the harmonic oscillator is a coherent state of the WH algebra. A q deformation does not produce any change in the quantum statistics during the time evolution of the harmonic oscillator. As expected, the quantum statistics, which is phase insensitive [41], is not affected by a change in the phase of the field (4.2) produced by a q deformation.

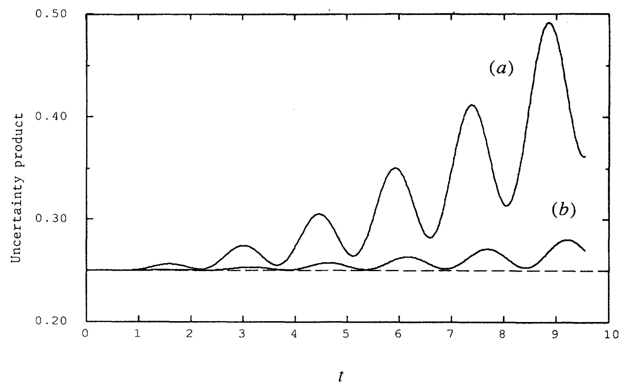

FIG. 4. Oscillations, due to a q deformation, in the time [37] evolution of the minimality properties of a coherent state of a conventional harmonic oscillator. Deformations for which $q=1.05$ [curve (a)] and 1.03 [curve (b)] are evaluated and compared to the case $q=1$ (no deformation) for which the uncertainty product remains constant in time and equal to $\frac{1}{4}$ (dashed line). The oscillator displacement is $\alpha=2$. The uncertainty product remains fairly close to that for a MUS only during the very first few periods, especially for the case of the smaller deformation (curve b).

V. CONCLUSION

In the present paper an attempt has been made to study the physics behind the q structures in order to get some insight into the physical implications of these deformations. We have shown that for appropriate displacements and anharmonic couplings the quantum states of an oscillator with anharmonicity $\sim\hat{N}^3$ are correctly described in terms of the quantum states for $\mathrm{su}(2)_q$. This seems to indicate that a q deformation of a harmonic oscillator can be understood as an effective anharmonic deformation, where q is proportional to the strength of the anharmonicity. Within this framework a number of interesting nonclassical features that are induced by a q deformation during the time evolution of an SU(2) coherent state can be examined. These include quadrature self-squeezing, loss of minimality, and preservation of the quantum statistics. The physical origin of these effects associated with a q deformation has been discussed here. Other studies on nonclassical properties of q deformed states are given in [30,31].

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