Steady-state analysis of a single-mode laser with correlations between additive and multiplicative noise

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The statistical fluctuations of a single-mode laser with correlations between additive and multiplicative white-noise terms are investigated theoretically. The mean, variance, and skewness of the steady-state laser intensity are calculated through a one-dimensional laser equation. Compared with a laser model of independent noises, the fluctuation appearing in the laser field is much larger.

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I. INTRODUCTION

The experimental measurements and theoretical analyses of the statistical properties of a single-mode laser that contains both additive and multiplicative noise showed anomalously large fluctuations in the system [1–6]. These fluctuations have been interpreted in terms of the pump noise due to external random disturbances coupled to the system. Moreover, the pump noise is treated either as multiplicative colored noise in a dye laser [1(c),2,3(b),4,5(a)] or as multiplicative white noise in certain theoretical analyses [1(d),3(a),4(b),5(b)–5(d),6]. The quantum fluctuation representing spontaneous emission is usually treated as additive white noise. Though the additive and multiplicative noise are presented simultaneously in some real processes, both types of noise are assumed to have different origins and are treated as independent random variables in most of the previous analyses [1–6]. In certain situations both additive and multiplicative noise may have a common origin and thus may be correlated as well.

In this paper, the steady-state fluctuations of a single-mode laser with correlations between additive and multiplicative white noise are investigated theoretically. In Sec. II the analytic expression of the steady-state laser-intensity distribution function is derived through a one-dimensional laser equation and the mean, variance, and skewness of the laser intensity are calculated. In Sec. III a comparison of the laser model containing two correlated noise sources and the one containing two independent noise sources is presented. A discussion of the results concludes the paper.

II. STEADY-STATE DISTRIBUTION FUNCTION

If only the intensity fluctuation is concerned, the phase variable of the laser field can be eliminated [2(b),4,5]. Thus, the single-mode laser model containing two types of correlated white-noise terms can be described by a one-dimensional Langevin equation [2(b)]:

$$\frac{dx}{dt}=a_0x-Ax^3+\frac{P}{2x}+xp(t)+q(t)\ , \qquad (1)$$

where all the variables and parameters are real. The laser intensity $I=x^2$ is dimensionless and the parameters a_0 and A stand for net gain and self-saturation coefficients. The random variables $q(t)$ and $p(t)$ represent the quantum and pump noise. The statistical properties of the noise terms are characterized by their first and second moments:

$$\langle q(t)\rangle=\langle p(t)\rangle=0\ , \qquad (2)$$

$$\langle q(t)q(t')\rangle=P\delta(t-t')\ , \qquad (3)$$

$$\langle p(t)p(t')\rangle=P'\delta(t-t')\ , \qquad (4)$$

$$\langle q(t)p(t')\rangle=\lambda\sqrt{PP'}\delta(t-t')\quad(0\le\lambda\le1)\ , \qquad (5)$$

where P and P' are the quantum- and pump-noise strengths, respectively. The parameter λ measures the strength of correlations between additive and multiplicative noise terms. If the random variables are changed to

$$q(t)=\eta_1(t)\ , \qquad (6)$$

$$p(t)=\lambda\sqrt{P'/P}\,\eta_1(t)+\sqrt{1-\lambda^2}\eta_2(t)\quad(0\le\lambda\le1)\ , \qquad (7)$$

with

$$\langle\eta_1(t)\eta_1(t')\rangle=P\delta(t-t')\ , \qquad (8)$$

$$\langle\eta_2(t)\eta_2(t')\rangle=P'\delta(t-t')\ , \qquad (9)$$

$$\langle\eta_1(t)\eta_2(t')\rangle=0\ , \qquad (10)$$

Eq. (5) is still satisfied. Then Eq. (1) can be written as follows [7]:

$$\frac{dx}{dt}=a_0x-Ax^3+\frac{P}{2x}+(1+\lambda\sqrt{P'/P}x)\eta_1(t)+\sqrt{1-\lambda^2}x\eta_2(t)\quad(0\le\lambda\le1)\ . \qquad (11)$$

The corresponding Fokker-Planck equation for the probability density function $Q(x,t)$ of the amplitude of the laser field $x=\sqrt{I}$ is given by [8]

$$\frac{\partial Q(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[\left[a_0 x - Ax^3 + \frac{P}{2x} + \frac{\lambda}{2}\sqrt{PP'} + \frac{P'}{2}x\right]Q(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[(P + 2\lambda\sqrt{PP'}x + P'x^2)Q(x,t)] . \tag{12}$$

If the additive and multiplicative noise terms are independent random variables with $\lambda=0$, Eq. (12) reduces to the Fokker-Planck equation discussed in great detail in Refs. [2(b),4–6].

The steady-state distribution function $Q(x)$ can be obtained directly from Eq. (12) and is given by

$$Q(x) = Nx(P + 2\lambda\sqrt{PP'}x + P'x^2)^{\alpha} \times \exp\left[-\frac{Ax^2}{P'} + \frac{4A\lambda\sqrt{PP'}x}{(P')^2} - \beta \arctan\left[\frac{\lambda + \sqrt{P'/P}x}{\sqrt{1-\lambda^2}}\right]\right] \tag{13}$$

for $0 \le \lambda < 1$ and

$$Q(x) = N_0 x(\sqrt{P} + \sqrt{P'}x)^{2\alpha_0} \times \exp\left[-\frac{Ax^2}{P'} + \frac{4A\sqrt{PP'}x}{(P')^2} + \frac{\beta_0}{1+\sqrt{P'/P}x}\right] \tag{14}$$

for $\lambda=1$, where

$$\alpha = \frac{a_0}{P'} - \frac{AP(4\lambda^2-1)}{(P')^2} - 1,$$
$$\beta = \frac{\lambda}{\sqrt{1-\lambda^2}}\left[\frac{2a_0}{P'} - \frac{2AP(4\lambda^2-3)}{(P')^2} + 1\right] \quad (0 \le \lambda < 1) , \tag{15}$$

$$\alpha_0 = \frac{a_0}{P'} - \frac{3AP}{(P')^2} - 1, \quad \beta_0 = \frac{2a_0}{P'} - \frac{2AP}{(P')^2} + 1 \quad (\lambda = 1) , \tag{16}$$

and N and N_0 are the normalization constants for Eqs. (13) and (14), respectively. For maximum coupling between the two types of noise sources, the coupling constant $\lambda=1$. Then the expectation values of the nth power of the laser intensity I are given by

$$\langle I^n \rangle = \langle x^{2n} \rangle = \int_0^{\infty} x^{2n} Q(x) dx , \tag{17}$$

where $Q(x)$ is given by Eq. (14). The normalization constant N_0 is given by the equation

$$\int_0^{\infty} Q(x) dx = 1 .$$

The mean, normalized variance, and skewness of the laser intensity are given by the numerical integrations of Eq. (17). The mean laser intensity is

$$\langle I \rangle = \int_0^{\infty} x^2 Q(x) dx . \tag{18}$$

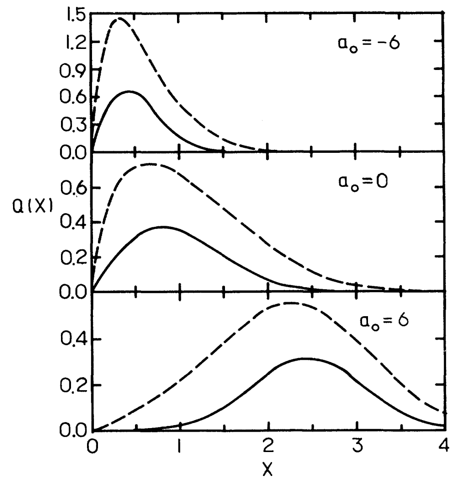

FIG. 1. The distribution function $Q(x)$ vs variable x for $P'=1.32$ with $A=1$ and $P=2$: ——, $\lambda=0$; – – –, $\lambda=1$.

The normalized variance of the intensity is

$$\lambda_2(0) = \frac{\langle I^2 \rangle}{\langle I \rangle^2} - 1 , \tag{19}$$

and the normalized skewness is

$$\lambda_3(0) = \frac{\langle I^3 \rangle}{\langle I \rangle^3} - 3\lambda_2(0) - 1 . \tag{20}$$

For independent noise sources, i.e., $\lambda=0$ in Eqs. (13) and (15), the mean, normalized variance, and skewness of the laser intensity have already been calculated explicitly in Refs. [4–6] and will not be reproduced here. These two extreme cases ($\lambda=1$ and $\lambda=0$) will be discussed in Sec. III.

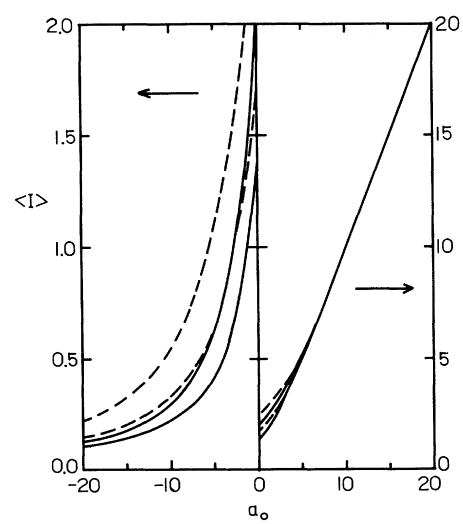

FIG. 2. The mean steady-state laser intensity $\langle I \rangle$ as a function of the pump parameter a_0 with $A=1$ and $P=2$: ——, $\lambda=0$ with (from bottom to top) $P'=1.32$ and 4.26; – – –, $\lambda=1$ with (from bottom to top) $P'=1.32$ and 4.26.

III. COMPARISON OF LASER MODELS WITH CORRELATED AND INDEPENDENT NOISE SOURCES

To see the effect of coupling between additive and multiplicative white noise, it is necessary to compare the laser model with correlated noise terms ($\lambda=1$) and the one with independent noise sources ($\lambda=0$). The distribution function $Q(x)$ is plotted in Fig. 1 as a function of the amplitude x of the laser field for three different values of a_0. It is seen that the height of the peak of $Q(x)$ goes up as a_0 becomes small. There is a long tail in $Q(x)$ as a_0 becomes large. The peak of $Q(x)$ shifts to a small value of x as λ changes from zero to 1. The value of $Q(x)$ increases as λ increases from zero to 1.

The mean laser intensity $\langle I \rangle$ is plotted in Fig. 2 as a function of the pump parameter a_0. It is obvious that the mean laser intensity $\langle I \rangle$ increases as λ increases when the laser is operated below or slightly above threshold. However, when the laser is operated well above threshold, the mean intensity $\langle I \rangle$ increases linearly with a_0 and there is almost no difference between the curves of $\lambda=0$ and $\lambda=1$.

The normalized variance $\lambda_2(0)$ and skewness $\lambda_3(0)$ of the laser intensity are plotted against the pump parameter a_0 in Figs. 3(a) and 3(b) for two different values of P'. The curves of $\lambda_2(0)$ and $\lambda_3(0)$ exhibit a peak before decreasing to zero. The magnitude of $\lambda_2(0)$ and $\lambda_3(0)$ increases and the peak position shifts to smaller values of a_0 as the coupling strength λ is increased. Large deviations in $\lambda_2(0)$ and $\lambda_3(0)$ between different values of λ occur when the laser is operated well below threshold. However, the differences become quite small when the laser is operated well above threshold. If the coupling strength λ is greater than zero but less than 1, the curves of $\lambda_2(0)$ and $\lambda_3(0)$ should lie between the curves for the two extreme cases of $\lambda=1$ and $\lambda=0$.

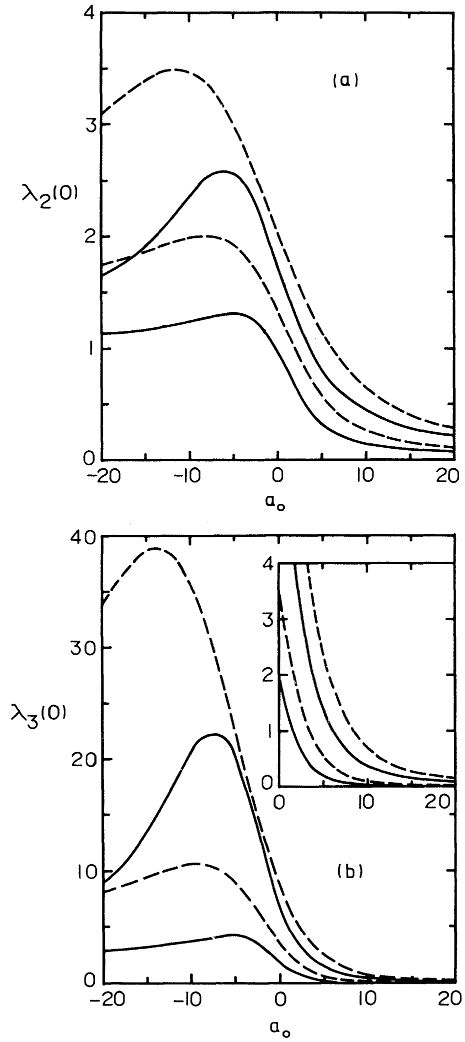

FIG. 3. (a) The normalized variance $\lambda_2(0)$ of the laser intensity as a function of the pump parameter a_0 and (b) the normalized skewness $\lambda_3(0)$ of the laser intensity as a function of a_0. The parameters are $A=1$ and $P=2$. ——, $\lambda=0$ with (from bottom to top) $P'=1.32$ and 4.26; — — —, $\lambda=1$ with (from bottom to top) $P'=1.32$ and 4.26.

IV. DISCUSSION

The statistical fluctuations of a single-mode laser that includes correlations between additive and multiplicative white noise are investigated theoretically through a one-dimensional laser model. The mean, normalized variance, and skewness of the steady-state laser intensity are calculated through an analytic expression of the distribution function $Q(x)$ for the coupling strength $\lambda=1$. The fluctuations appearing in $\lambda_2(0)$ and $\lambda_3(0)$ for $\lambda=1$ are much larger than that for $\lambda=0$ especially when the laser is operated well below threshold. The deviations in $\lambda_2(0)$ and $\lambda_3(0)$ for $\lambda=1$ and $\lambda=0$ are quite small but noticeable when the laser is operated near and above threshold. The effect of coupling between additive and multiplicative white noise can produce larger fluctuations in $\lambda_2(0)$ and $\lambda_3(0)$.

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