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Relation between ideal and feasible phase concepts

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The isometry $\hat{U}^\dagger, \hat{U}^\dagger\hat{U} \neq \hat{U}\hat{U}^\dagger = \hat{1}$, which relates the ideal phase measurement to the feasible phase measurement based on heterodyne detection, is specified here. Consequently, the Shapiro-Wagner phase measurement [IEEE J. Quantum Electron. **QE-20**, 803 (1984)] incorporates the ideal phase concepts of Susskind and Glogower [Physics **1**, 49 (1964)] or Pegg and Barnett [Phys. Rev. A **39**, 1665 (1989)].

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Quantum-phase investigations belong to the topic of quantum mechanics which attracts the attention of both theoreticians and experimentalists. Since the community of physicists still has not achieved a consensus, the possible treatments of the quantum-phase problem in quantum optics differ substantially. It is not the main aim of this contribution to give an overview of various methods applied to this problem. Instead of this, we will focus on the two particular models introduced recently [1,2] and we address the possibility to realize the ideal phase measurement in the framework of the feasible phase concept. Our motivation comes from the quantum estimation theory [3]. Addressing the problem of phase measurement on a single-mode field, the continuous phase-shift variable θ enters the displacement transformation of the input field as $|\psi(\theta)\rangle = e^{-i\theta\hat{n}}|\psi\rangle$, $\hat{n}$ being $\hat{a}^\dagger\hat{a}$. The purpose of a quantum-phase measurement is to get some piece of information about this induced phase shift θ by means of registration of the continuous phase variable ϕ. The feasible measurement of the continuous phase-shift variable may be treated in the framework of heterodyne (multimode homodyne) detection [1] as measurement of the phase of complex amplitude

$$\hat{Y}_{\rm SW} = \hat{a} + \hat{b}^\dagger.$$

This concept is known as Shapiro-Wagner (SW) phase detection. Assuming the performance measure of the phase resolution as dispersion [3,4]

$$D^2 = 1 - |\langle e^{i\phi}\rangle|^2,$$

the best measurement yielding the minimum dispersion may be characterized as an ideal phase measurement [2] associated with the measurement of the Susskind-Glogower (SG) operator $\hat{E} = (\hat{n}+1)^{-1/2}\hat{a}$. Nevertheless, this is the prediction of quantum estimation theory only and the experimental aspects of the possible realization of such a measurement are unclear. The fundamental question is this: is it possible to specify SW measurement in a real experiment, which could tend to the maximum performance allowed by quantum mechanics? We will show that the SW phase concept includes the SG one, i.e., that for every signal state it is possible, at least in principle, to specify the SW phase measurement yielding the minimum dispersion.

Let us remember for this purpose the known results associated with the ideal Susskind-Glogower phase concept in the representation of the relative-number state (RNS) basis introduced by Ban [5]. Let us consider the system composed of two independent and distinguishable subsystems A and B. The full Hilbert space can be written as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and the annihilation operators on both spaces will be designated as $\hat{a}$ and $\hat{b}$, respectively. The two subsystems can then be described by the complete orthogonal discrete bases of Fock states $|m\rangle_A \otimes |n\rangle_B, m, n = 0, 1, ...,$ or alternatively by the relative-number state basis generated by the states

$$\begin{aligned} |n,m\rangle\rangle = {} & \Theta(n)|m+n\rangle_A\,|m\rangle_B \\ & +\Theta(-n-1)|m\rangle_A|m-n\rangle_B, \end{aligned} \tag{1}$$

where $-\infty < n < \infty, m \geq 0$ and the function $\theta(n)$ is defined as

$$\Theta(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0. \end{cases}$$

Equivalently, these states may be associated with the product of Fock states on both Hilbert subspaces according to the rule

$$|n-m, \min(m,n)\rangle\rangle = |m\rangle_A\,|n\rangle_B$$

and the first quantum number of the state $|n,m\rangle\rangle$ predicts therefore the eigenvalue of the operator $\hat{N} = \hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}$ so that

$$\hat{N}|n,m\rangle\rangle = n|n,m\rangle\rangle. \tag{2}$$

Further, the basis of the relative-number states is complete and orthonormal, satisfying for $m \geq 0, -\infty < n < \infty$ the relations

$$\begin{aligned} & \langle\langle k,l|n,m\rangle\rangle = \delta_{ln}\delta_{km}, \\ & \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} |n,m\rangle\rangle\,\langle\langle m,n| = \hat{1}. \end{aligned} \tag{3}$$

Consequently, the unitary phase operator $\hat{D}$ may be defined on the full Hilbert space $\mathcal{H}$ as

$$\hat{D} = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} |n-1, m\rangle\rangle \, \langle\langle m, n|, \tag{4}$$

satisfying the relations

$$\hat{D}\hat{D}^\dagger = \hat{D}^\dagger \hat{D} = \hat{1} \tag{5}$$

and the commutation rule

$$[\hat{D}, \hat{N}] = \hat{D}. \tag{6}$$

Moreover, the reduction of the operator $\hat{D}$ on the subsystem A in the vacuum state $|0\rangle_B$ of the system B tends to the realization of the Susskind-Glogower exponential phase operator on the (signal) Hilbert subspace $\mathcal{H}_A$ as

$$\hat{E} = (\hat{a}^\dagger \hat{a} + 1)^{-1/2} \hat{a} = {}_B\langle 0|\hat{D}|0\rangle_B. \tag{7}$$

A similar treatment may be applied to the description of the realizable SW phase concept. The unitary operator may be formally treated as

$$\hat{R} = \hat{Y}_{\rm SW} (\hat{Y}_{\rm SW}^\dagger \hat{Y}_{\rm SW})^{-1/2} = \sqrt{\frac{\hat{a} + \hat{b}^\dagger}{\hat{a}^\dagger + \hat{b}}} \tag{8}$$

and fulfills again the commutation rule [6]

$$[\hat{R}, \hat{N}] = \hat{R}. \tag{9}$$

Assuming the operator

$$\hat{Y}_{\rm SW}^\dagger \hat{Y}_{\rm SW} = \hat{a}^\dagger \hat{a} + \hat{b}\hat{b}^\dagger + \hat{a}^\dagger \hat{b}^\dagger + \hat{a}\hat{b},$$

the algebra of SU(1,1) generators may be introduced [7] as

$$\begin{aligned} \hat{K}_1 &= \tfrac{1}{2}(\hat{a}^\dagger \hat{b}^\dagger + \hat{a}\hat{b}), \\ \hat{K}_2 &= \frac{1}{2i}(\hat{a}^\dagger \hat{b}^\dagger - \hat{a}\hat{b}), \\ \hat{K}_3 &= \tfrac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + 1), \end{aligned} \tag{10}$$

and the Casimir operator as

$$\hat{K}^2 = \hat{K}_3^2 - \hat{K}_1^2 + \hat{K}_2^2 = \tfrac{1}{2}\hat{N}(\tfrac{1}{2}\hat{N} + 1).$$

Consequently, the operators $\hat{N}$ and

$$\tfrac{1}{2}\hat{Y}_{\rm SW}^\dagger \hat{Y}_{\rm SW} = \hat{K}_3 + \hat{K}_1$$

may be diagonalized simultaneously on the subspaces characterized by the discrete index $n, -\infty < n < \infty$ yielding the eigenstates $|n,\eta\rangle$, $\eta \geq 0$ being the continuous variable

$$\begin{aligned} \hat{N}|n,\eta\rangle &= n|n,\eta\rangle, \\ (\hat{K}_3 + \hat{K}_1)|n,\eta\rangle &= \eta|n,\eta\rangle. \end{aligned} \tag{11}$$

These eigenvectors have already been investigated as the so-called Linblad-Nagel (LN) basis [8] described by the discrete parameter n and the continuous variable η. These states are orthogonal,

$$\langle \eta, n|m, \eta'\rangle = \delta_{nm}\delta(\eta - \eta'), \tag{12}$$

and provide the resolution of the identity operator in the full Hilbert space

$$\int_0^\infty d\eta \sum_{n=-\infty}^{\infty} |n,\eta\rangle \, \langle \eta, n| = \hat{1}. \tag{13}$$

Moreover, the relation

$$|n-1,\eta\rangle = \hat{R}|n,\eta\rangle \tag{14}$$

may be easily concluded as the consequence of the definition of the states $|n,\eta\rangle$ and the commutation rule (9). Using this and the completeness of the LN basis (13), the unitary exponential operator $\hat{R}$ may be decomposed as

$$\hat{R} = \int_0^\infty d\eta \sum_{n=-\infty}^{\infty} |n-1,\eta\rangle \, \langle \eta, n|. \tag{15}$$

This form represents the desired result analogous to the decomposition (4) of the operator $\hat{D}$ in the relative-number state basis.

Let us specify the relation between the two unitary operators $\hat{R}$ and $\hat{D}$. Let us define the operator $\hat{U}$ as

$$\hat{U} = \int_0^\infty d\eta \sum_{n=-\infty}^{\infty} |n, [\eta]\rangle\rangle \, \langle \eta, n|. \tag{16}$$

Let us emphasize that in this notation the bra states belong to the LN basis, whereas the ket states are the relative-number states, [ ] being the integer part of a non-negative number. One can easily verify that such an operator fulfills the relation

$$\hat{U}\hat{U}^\dagger = \int_0^\infty d\,\eta \sum_{n=-\infty}^{\infty} |n, [\eta]\rangle\rangle \, \langle\langle [\eta], n| \tag{17}$$

$$= \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} |n, m\rangle\rangle \, \langle\langle m, n| = \hat{1}. \tag{18}$$

The orthogonality of the LN basis (12) was used in the derivation of the equality (17) and the completeness of the RNS basis (3) in derivation of the equality (18). Let us emphasize that $\hat{U}$ is not a unitary operator, since the relation

$$\hat{U}^\dagger \hat{U} = \int d\eta \int d\eta' \sum_{n=-\infty}^{\infty} |n,\eta\rangle \, \langle \eta', n| \neq \hat{1} \tag{19}$$

is valid. The integration over η, η' is restricted by the conditions

$$0 < \eta,\ \eta' < \infty \ \text{ and } \ [\eta] = [\eta'].$$

The operator $\hat{U}^\dagger$ is therefore isometry ("one-sided unitary"), since the norm of the state $\hat{U}^\dagger|\psi\rangle$ is preserved.

The relation between the unitary operators $\hat{R}$ and $\hat{D}$ is now simply given as the transformation

$$\hat{U}\hat{R}\hat{U}^\dagger = \hat{D} \tag{20}$$

representing the main result of this contribution. Let us emphasize that this relation cannot be simply inverted, since $\hat{U}$ is not an unitary operator. Particularly, assuming the relation (7), the Susskind-Glogower phase measurement performed in the state $|\psi\rangle$ may be treated as the feasible measurement of Shapiro-Wagner performed in the correlated state

$$|\Psi\rangle = \hat{U}^\dagger|\psi\rangle_A|0\rangle_B \tag{21}$$

$$= \int_0^1 d\eta \sum_{n=0}^{\infty} \psi_n|n,\eta\rangle, \tag{22}$$

where we used the decomposition on the Hilbert subspace $\mathcal{H}_A$ as $|\psi\rangle = \sum \psi_n|n\rangle$ and the definition of the RNS (1). We can also give a more explicit form of this state by taking into account the known decomposition of the LN basis [8]. The scalar product may be specified as

$$_A\langle n+k|\ _B\langle k|n,\eta\rangle = \frac{1}{\sqrt{\eta\Gamma(n+k+\frac{1}{2})\Gamma(k+\frac{1}{2})}} \times W_{k+\frac{n+1}{2},\frac{n+1}{2}}(2\eta), \tag{23}$$

where $W_{k+\frac{n+1}{2},\frac{n+1}{2}}$ is the Whittaker function and $k,n \geq 0$. The decomposition of the correlated state $|\Psi\rangle$ on the full Hilbert space is then simply given as

$$|\Psi\rangle = \sum_{n=0}^{\infty} \psi_n \sum_{k=0}^{\infty} Z_{n,k}|n+k\rangle_A|k\rangle_B, \tag{24}$$

where

$$Z_{n,k} = \frac{1}{\sqrt{\Gamma(k+n+\frac{1}{2})\Gamma(k+\frac{1}{2})}} \times \int_0^1 d\eta\, \frac{1}{\sqrt{\eta}} W_{k+\frac{n+1}{2},\frac{n+1}{2}}(2\eta).$$

Our conclusion explicitly demonstrates the ability of the feasible phase concept based on the annihilation of photons to incorporate also the ideal phase measurement yielding the minimum dispersion. Of course, this result cannot be interpreted as a suggestion of how to perform the ideal phase measurement. The experimental aspects are beyond the scope of this contribution and represent a more complex problem. Let us mention some open questions concerning this topic: (i) It is not clear whether the proposed solution is unique. (ii) The fields entering the SW phase concept are not mutually independent but strongly correlated. It is not clear how the transformation $\hat{U}$ can be achieved experimentally.

It might seem that the fundamental problems of experimental realization of the ideal phase concept are only replaced by similar problems of different (in principle feasible) phase measurement. Nevertheless, this is not the main purpose of this contribution. The experimental motivation for ideal phase measurement is questionable. Frankly, the only reason why the ideal phase concept seems to be interesting is its (relative) mathematical simplicity; there is no physical justification for the experimental realization of such a measurement. Particularly in the case of strong signal field, the SG phase measurement is equivalent to the SW phase measurement with the infinitely squeezed field on auxiliary input port [6]. On the other hand the $\sqrt{2}$ times worse resolution may be achieved for limited energy if the auxiliary field is suitable matched to the signal in the SW model. Consequently, the ultimate resolution predicted by the ideal concept does not represent the optimum measurement.

The purpose of this contribution is therefore to support the role of the Shapiro-Wagner phase measurement in quantum optics. This concept could be accepted as the desired quantum-phase operator by theoreticians as well as experimentalists. The measurement is feasible and the accuracy is not far from the ultimate quantum resolution. The ideal phase concept may in principle also be incorporated into this framework, even if it is not sufficiently clear why and how to do it experimentally.

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