Jaynes-Cummings model with quasiclassical fields: The effect of dissipation

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An approximate solution is given for the Jaynes-Cummings model with cavity losses, i.e., the problem of a two-level atom interacting with a single mode of the quantized radiation field, in the rotating-wave approximation, when the field is damped by a reservoir at zero temperature. The approximate solution is derived for initial coherent field states with moderately large numbers of photons. It is simpler in form than earlier results derived by other authors and, over the appropriate parameter range, substantially more accurate than some of them, as shown by direct numerical integration of the master equation. In particular, it is found that an earlier treatment of this problem based on a secular approximation is seriously flawed, in that the conditions for its validity are much more restrictive than was previously believed. Among the results derived it is shown that, just as for the lossless case, when the atom is initially prepared in one of the semiclassical eigenstates the evolution is very simple, with the field and the atomic dipole drifting together in phase. For moderate losses this leads, as in the lossless case, to a "state preparation"; i.e., to a good approximation, the state of the atom at a specific time can be made independent of its initial state. The effect of losses on the recently discovered "Schrödinger cat" state of the field is also analyzed. It is found that, although the dissipation destroys the coherence of the macroscopic superposition very rapidly, preparation and observation of the "cat" should be possible with the cavity quality factors reported in recent micromaser experiments.

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I. INTRODUCTION

This paper presents an approximate solution to the Jaynes-Cummings model (JCM) [1] with dissipation, that is, to the problem of a two-level atom interacting with a single mode of the quantized radiation field, in the rotating-wave approximation, when the field is damped by a reservoir at zero temperature. This dissipation may be thought of as "cavity losses" since, in practice, a resonant cavity of some sort may be used to select only one mode of the field for the atom to interact with.

The solution is derived along the lines of the "quasiclassical" or "asymptotic" solutions for the lossless JCM which have been presented elsewhere [2,3]. These solutions were derived for "quasiclassical" initial field states, that is, states which have a small phase and amplitude uncertainty. The case of an initial coherent-state field with a large average number of photons is the one treated explicitly in this paper. Since the losses eventually reduce the number of photons to zero, the approximation clearly ceases to be valid for sufficiently long times. With only this restriction, however, the solution presented here is, to the best of this author's knowledge, the most complete and arguably the simplest yet given for this problem.

The JCM is a model of fundamental theoretical importance, as the simplest nontrivial model of two coupled quantum systems (formally, a spin-$\frac{1}{2}$ system, to which the two-level atom is equivalent, coupled to a harmonic oscillator in an appropriate way). Without dissipation, it is exactly solvable (see [1] for the solution in the Schrödinger picture and [4] for the solutions in the Heisenberg picture). The solution in the presence of dissipative processes is not only of theoretical interest, but also important from a practical point of view since such processes would always be present in any experimental realization of the model. The "micromaser" setup, involving Rydberg atoms interacting with very high-Q resonant cavities in the microwave domain [5], comes closest to a realization of the ideal, lossless JCM, but even here the effects of finite cavity losses must be taken into account. In the optical domain, where great progress has been made in the past few years in the design and construction of very small, very high finesse cavities [6], spontaneous emission would also play a role, and this has motivated a recent numerical study of this problem by Tran Quang, Knight, and Bužek [7]. Spontaneous emission will not be considered in the present paper, which focuses on the effect of cavity losses only.

Over the years, a number of authors have treated the JCM with cavity losses in various ways, including analytical approximations [8,9] as well as numerical calculations [7,10]. Most of these publications dealt only with the population inversion "collapses and revivals." Eiselt and Risken [10] carried out a comprehensive numerical study of the state of the field, although again the only atomic variable they considered was the population inversion. Puri and Agarwal [9] did present formal solutions for the expectation values of a number of quantities of interest. It will be shown here, however, that their results are seriously flawed, in that the conditions for their applicability are much more restrictive than they realized. This failure of the "secular approximation" is perhaps one of the most noteworthy results of the present paper; it is illustrated in Sec. III and discussed at length in Appendix A.

The solution to be presented here is a closed-form solu-

tion for the total state of system, i.e., for the atom-field density operator. It is extremely simple in form and provides considerable physical insight into the problem. In spite of this simplicity, comparison with the numerical solutions shows it to be remarkably accurate.

Over the past couple of years, several effects have been discovered in the lossless JCM. Two of the most interesting are the possibility of state preparation, whereby the atom could be made to be, at a given time, in a state independent of its initial state [11,2,12,13], and the possibility of preparing the field, at the same time, in a coherent superposition of macroscopically distinct states [2,14,15] (a macroscopic quantum superposition, also known as a Schrödinger cat [16,17]). A natural question that is asked and answered in this article is how well these effects stand in the presence of dissipation. The conclusion is that observation of both effects lies within the range of current experimental capabilities, in particular in micromaser cavities. (The question of to what extent these ideas may be carried over to optical cavities has not been answered yet.)

The outline of the paper is as follows. In Sec. II the master equation is presented and the basic results of earlier work in the lossless case are reintroduced. The approximate solutions to the master equation are derived in Sec. III, where they are compared to the results of the numerical calculations and of the secular-approximation based work of Puri and Agarwal. A discussion of some of the main results follows, in Sec. IV, where the issue of the effect of losses on the atomic state preparation and the field Schrödinger cat is also addressed. Appendix A discusses and explains what is wrong with the secular approximation. Appendix B shows how to reduce the master equation to a relatively smaller set of ordinary differential equations for the purpose of the numerical calculations.

II. MASTER EQUATION AND APPROXIMATE SOLUTIONS WITHOUT DISSIPATION

The JCM involves a two-level atom, whose energy eigenstates are written here as $|a\rangle$ (upper) and $|b\rangle$ (lower), interacting with a mode of the quantized electromagnetic field, in the rotating-wave approximation. The field annihilation operator is denoted here by a; the coupling constant is written as g. The interaction Hamiltonian, on resonance, is

$$H_I=\hbar g(|a\rangle\langle b|a+a^\dagger|b\rangle\langle a|) . \tag{1}$$

It will be assumed that the field mode is damped, at a rate κ, into a reservoir (typically, the cavity walls or the outside world) which is taken to be at zero temperature. The resulting master equation for the atom-field density operator ρ is (for a derivation, see e.g., [18]):

$$\frac{d\rho}{dt}=-\frac{i}{\hbar}[H_I,\rho]+\kappa L_f(\rho) \tag{2}$$

where the field decay operator is

$$L_f(\rho)=2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a . \tag{3}$$

By taking the matrix elements of ρ between the atomic states $|a\rangle$ and $|b\rangle$, Eq. (2) becomes a set of four equations for the operators (in the field Hilbert space) ρ_{ij} $(i,j=a,b)$:

$$\dot\rho_{aa}=ig(\rho_{ab}a^\dagger-a\rho_{ba})+\kappa L_f(\rho_{aa}) , \tag{4a}$$

$$\dot\rho_{bb}=ig(\rho_{ba}a-a^\dagger\rho_{ab})+\kappa L_f(\rho_{bb}) , \tag{4b}$$

$$\dot\rho_{ab}=ig(\rho_{aa}a-a\rho_{bb})+\kappa L_f(\rho_{ab}) , \tag{4c}$$

$$\dot\rho_{ba}=ig(\rho_{bb}a^\dagger-a^\dagger\rho_{aa})+\kappa L_f(\rho_{ba}) . \tag{4d}$$

The exact solution for the Hamiltonian part of these equations (i.e., for the lossless JCM) has been known for a long time [1,4]. Recently, an approximate solution to the lossless JCM has been derived [2,3] which is much simpler in form and yields considerable insight into the evolution of the field-atom system when the initial state of the field is "quasiclassical," i.e., has small phase and amplitude uncertainties. This approximate solution will be used here as the starting point for the solution when losses are included.

The key observation is that the evolution of the field and atom in the lossless case is extremely simple when the initial state of the field is quasiclassical and the initial state of the atom is one of either

$$|+\rangle=\frac{1}{\sqrt{2}}[|a\rangle+|b\rangle] \tag{5a}$$

or

$$|-\rangle=\frac{1}{\sqrt{2}}[|a\rangle-|b\rangle] \tag{5b}$$

[to simplify the notation, the initial phase of the field, which should come into the definition (5), has been taken to be zero throughout]. The states $|+\rangle$ and $|-\rangle$ are the eigenstates of the semiclassical interaction Hamiltonian, i.e., of Eq. (1) with the field operators a and $a^\dagger$ replaced by a constant c number. They are not to be mistaken for the "bare" energy eigenstates $|a\rangle$ and $|b\rangle$. In the states $|+\rangle$ and $|-\rangle$ the atomic energy is not well defined, but the dipole moment has a nonvanishing expectation value which oscillates either in phase or 180° out of phase with the field [3].

As shown in [2] and [3], if the atom is initially prepared in $|+\rangle$, and the initial state of the field is $|\Phi(0)\rangle=\sum_n C_n|n\rangle$, with average number of photons $\bar n$, the subsequent evolution is approximately given by

$$|\Psi_+(t)\rangle=\frac{1}{\sqrt{2}}(e^{-igt/2\sqrt{\bar n}}|a\rangle+|b\rangle)\otimes\sum_{n=0}^{\infty}e^{-igt\sqrt{n}}C_n|n\rangle \tag{6a}$$

and, correspondingly, the initial state $|-\rangle|\Phi(0)\rangle$ evolves as

$$|\Psi_-(t)\rangle=\frac{1}{\sqrt{2}}(e^{igt/2\sqrt{\bar n}}|a\rangle-|b\rangle)\otimes\sum_{n=0}^{\infty}e^{igt\sqrt{n}}C_n|n\rangle . \tag{6b}$$

These expressions hold for times up to the order of the

"breakdown time" $T_b=\pi\bar{n}/g$ [19]. The most important thing about them, for the present purposes, is not immediately apparent but has been discussed at length in [2] and [3]; namely, Eqs. (6) describe states in which the field and atom *remain in phase* as they evolve in time. It can be shown from (6) that both the average phase of the field and that of the atomic dipole grow in time as $\pm gt/2\bar{n}^{1/2}$, so that the field and the atom remain at all times either in phase or in opposition, depending on whether the initial state is $|+\rangle$ or $|-\rangle$.

For the approximate solutions (6) to hold, it is not essential that the initial state of the field be exactly a coherent state; as shown in [3], the essential point is that the phase and amplitude should be relatively well defined, i.e., $\Delta\phi \ll 1$ and $\Delta n/\bar{n} \ll 1$. The solutions (6) do predict a growing phase uncertainty for the field states; for an initial coherent state, one has roughly $\Delta\phi \simeq (1/\bar{n}+g^2t^2/16\bar{n}^2)^{1/2}$, which is why one should expect the approximation to break down when gt is of the order of $\bar{n}$ [19]. For large $\bar{n}$, this may allow the approximation to hold over many collapses and revivals, since the revival period is given by $t_R=2\pi\sqrt{\bar{n}}/g$.

The states $|+\rangle$ and $|-\rangle$ are orthogonal, and hence a basis of the two-dimensional space of states of the atom; therefore, a knowledge of the solutions (6) is enough to predict the evolution of the system for any initial atomic state in the lossless case, in which one only has to solve the Schrödinger equation for the state vector, or wave function, of the total system of atom plus field. When dissipation is included, things become a little more complicated, since in general one now has to solve the master equation (2) for the atom-field density operator ρ. If the initial state of the atom is the general superposition $\gamma|+\rangle+\delta|-\rangle$ (with arbitrary γ and δ, only such that $|\gamma|^2+|\delta|^2=1$), the initial density operator may be written as

$$\rho(0)=[|\gamma|^2|+\rangle\langle+|+|\delta|^2|-\rangle\langle-| +(\gamma\delta^*|+\rangle\langle-|+\text{H.c.})]\otimes\rho_f(0)\ , \tag{7}$$

where $\rho_f(0)$ is the density operator for the initial state of the field, taken here to be a pure coherent state of average photon number n_0 and phase $\phi=0$. Because of the linearity of the master equation, the solution corresponding to the initial condition (7) can be written as

$$\rho(t)=|\gamma|^2\rho^{(++)}(t)+|\delta|^2\rho^{(--)}(t) +[\gamma\delta^*\rho^{(+-)}(t)+\text{H.c.}]\ , \tag{8}$$

where $\rho^{(++)}(t)$ is the solution to the master equation with initial condition $|+\rangle\langle+|\otimes\rho_f(0)$, i.e., with the atom initially in state $|+\rangle$, and analogously for $\rho^{(--)}(t)$; and $\rho^{(+-)}(t)$ is the (non-Hermitian) solution to the master equation with the initial condition $|+\rangle\langle-|\otimes\rho_f(0)$. In the lossless case it is a trivial matter to write down expressions for $\rho^{(++)}(t)$, $\rho^{(--)}(t)$, and $\rho^{(+-)}(t)$ from the solutions (6); when dissipation is included, however, $\rho^{(+-)}$ has to be considered separately from $\rho^{(++)}$ and $\rho^{(--)}$. This shall be done in the following section.

Before proceeding, it maybe useful to note the physical significance of the various pieces of the solution, namely, $\rho^{(++)}(t)$, $\rho^{(--)}(t)$, and $\rho^{(+-)}(t)$. The first two, $\rho^{(++)}(t)$ and $\rho^{(--)}(t)$, are actually the full solutions to the master equation when the initial state of the atom is one of the semiclassical eigenstates $|+\rangle$ or $|-\rangle$. On the other hand, $\rho^{(+-)}(t)$ comes into play whenever the initial state of the atom is any coherent superposition of $|+\rangle$ and $|-\rangle$, such as, for instance, the "bare" energy eigenstates $|a\rangle$ and $|b\rangle$. Accordingly (see also [3]), it is $\rho^{(+-)}(t)$ which, in the limit of large photon numbers, is responsible for the population inversion oscillations, and their collapses and revivals, for which the JCM is best known.

III. APPROXIMATE SOLUTION WITH CAVITY LOSSES

A. Evolution of $\rho^{(++)}(t)$

When the initial state of the atom is $|+\rangle$, the atomic dipole and the field are in phase and, in the absence of dissipation, as long as the solutions (6) are valid, their average phases grow together at the rate $g/2\bar{n}^{1/2}$ [while the phase uncertainty of the field slowly grows as well, as indicated in the discussion following Eq. (6)]. Since cavity losses are phase insensitive, they should not alter this picture substantially. That is, one still should expect the field and the atomic dipole to be locked together in phase. The main change brought about by the losses is a gradual reduction of the average number of photons, which should alter the rate at which the phase grows; one might conjecture that the phase should grow now at a rate given instantaneously by $g/2\bar{n}(t)^{1/2}$, and it will be shown in a moment [see Eq. (20) below] that this conjecture is indeed essentially correct. The decrease of the average photon number also increases the phase uncertainty, which has the added effect that the approximate solutions will lose validity earlier than they would in the absence of losses. This will be compensated here to some extent by the fact that the solution to be derived is an improvement over the earlier ones derived for the lossless case [2,3], in that it allows for some entanglement between the atom and field.

It was shown in [3] how, for the lossless case, the crucial assumption of a small phase uncertainty in the field state led to the following expression:

$$|\Psi_+(t)\rangle=\frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}C_n e^{i\phi_n(t)}|n\rangle \times(e^{i[\phi_{n+1}(t)-\phi_n(t)]}|a\rangle+|b\rangle)\ . \tag{9}$$

This is Eq. (9) of [3] in slightly rewritten form; here, the phase $\phi_n(t)$ is just equal to $-gt\sqrt{n}$. One can obtain Eq. (6a) from (9) simply by replacing $\phi_{n+1}(t)-\phi_n(t)$ in the atomic state by the approximate value $-1/2\sqrt{\bar{n}}$ (obtained by expanding around the mean of the photon number distribution). Equation (9), which is therefore somewhat more general than (6a), is a good starting point for an "ansatz" to substitute in the master equation (3) when dissipation is included, since it is "almost" a solution to the Hamiltonian part. Accordingly, let

$$\rho^{(++)}(t)=\tfrac{1}{2}\sum_{n,m} r_{nm}(t)e^{i[\phi_n(t)-\phi_m(t)]}|n\rangle\langle m| \otimes(e^{i[\phi_{n+1}(t)-\phi_n(t)]}|a\rangle+|b\rangle) \times(e^{-i[\phi_{m+1}(t)-\phi_m(t)]}\langle a|+\langle b|) , \quad (10)$$

where the phases $\phi_n(t)$ are to be determined. The coefficients $r_{nm}(t)$, also to be determined, would, in the lossless case, be constant and equal to C_nC_m; here we are allowing for the possibility that they may not be of the product form, i.e., that the combined atom-field state may not be a pure state anymore, because of the losses. They are taken to be real in what follows.

When the ansatz (10) is substituted in the master equation (4), and the real and imaginary parts are taken, one obtains four different equations for r_{nm} and another four for $\phi_n-\phi_m$. These are all similar, except for small differences, and the approximation basically consists in neglecting the slight differences. Specifically, for r_{nm} one finds

$$\dot{r}_{nm}=-\kappa(n+m)r_{nm} +2\kappa\sqrt{(n+1)(m+1)} \times\cos(\phi_{n+i+1}-\phi_{n+i}-\phi_{m+j+1}+\phi_{m+j})r_{n+1,m+1}, \quad (11)$$

where i and j can be 0 or 1. Under the assumption that $\phi_{n+1}-\phi_n$ is approximately independent of n, the cosine in Eq. (11) can be replaced by unity and the four different equations reduce to a single one. The crucial assumption, as discussed in [3], is that the field state has a fairly well-defined phase. To see this, consider the effect of acting on an arbitrary field state $|\Phi\rangle=\sum_n C_n\exp(i\phi_n)|n\rangle$ with the "phase operator" [20] $\langle\langle\exp(i\phi)\rangle\rangle=\sum_n|n-1\rangle\langle n|$:

$$\langle\langle\exp(i\phi)\rangle\rangle\sum_{n=0}^{\infty}C_ne^{i\phi_n}|n\rangle =\sum_{n=0}^{\infty}C_{n+1}e^{i\phi_{n+1}}|n\rangle =\sum_{n=0}^{\infty}\frac{C_{n+1}}{C_n}e^{i(\phi_{n+1}-\phi_n)}C_ne^{i\phi_n}|n\rangle . \quad (12)$$

This is approximately equal to a phase factor times the original state if the two following conditions hold: (1) $C_{n+1}/C_n\simeq 1$ and (2) $\phi_{n+1}-\phi_n\simeq\phi$, independent of n. These two properties of "quasiclassical" states will be used repeatedly in what follows.

For as long as the field state has a relatively well-defined phase, the four equations (11) for r_{nm} reduce to the single one

$$\dot{r}_{nm}=-\kappa(n+m)r_{nm}+2\kappa\sqrt{(n+1)(m+1)}r_{n+1,m+1} . \quad (13)$$

As for the phases $\phi_n(t)$, the four equations for them may be put in the form

$$\dot{\phi}_{n+i}-\dot{\phi}_{m+j}=-g(\eta_{ij}\sqrt{n+i}-\eta'_{ij}\sqrt{m+j})-2\frac{r_{n+1,m+1}}{r_{nm}}\sqrt{(n+1)(m+1)}\sin(\phi_{n+i+1}-\phi_{n+i}-\phi_{m+j+1}+\phi_{m+j}) , \quad (14)$$

where again i and j can be 1 or 0, and the coefficients η_{ij} and η'_{ij} are of the form $r_{n\pm1,m}/r_{nm}$, $r_{n,m\pm1}/r_{nm}$. In accordance with the well-defined phase assumption, such ratios will be set equal to 1 (note the similarity of this approximation to the original derivation of the asymptotic solutions for large $\bar{n}$ in Appendix A of [2]). Also, the sine function in (14) may be expanded to first order. The factor in front of the sine function turns out to be independent of n and m when Eq. (13) is solved under the assumption that the initial field state is a coherent state; indeed, under this condition, the exact solution to (13) is

$$r_{nm}(t)=e^{-\bar{n}(t)}\frac{\bar{n}(t)^{(n+m)/2}}{\sqrt{n!m!}} \quad (15a)$$

with

$$\bar{n}(t)=n_0e^{-2\kappa t} . \quad (15b)$$

This is the only point at which the assumption that the initial state is a coherent state is used explicitly. If r_{nm} were the total field density operator, the solution (15) would just be a coherent state again, with an average photon number $\bar{n}(t)$. It is well known, in fact, that the usual linear-loss master equation (at zero temperature) preserves a coherent state. In this case, the state is not coherent because of the phases $\phi_n(t)$, but it is sufficiently similar to a coherent state for its decay to be, in this approximation, very simple. In particular, the factorization of r_{nm} implies that the total atom-field state represented by (10) is pure, in spite of the cavity losses.

When the solution (15) for r_{nm} is used in Eq. (14) for the phases, after setting $\eta_{ij}\simeq\eta'_{ij}\simeq1$, and expanding the sine function to first order, the four equations for $\phi_n-\phi_m$ simplify to just the following equation for ϕ_n:

$$\dot{\phi}_n=-g\sqrt{n}+2\kappa n_0e^{-2\kappa t}(\phi_{n+1}-\phi_n) . \quad (16)$$

The easiest way to solve Eq. (16), always within the assumption that the numbers of photons are fairly large and the phase is well defined, is to treat n as a continuous variable and replace

$$\phi_{n+1}-\phi_n\simeq\frac{\partial\phi(n,t)}{\partial n} . \quad (17)$$

The resulting first-order partial differential equation is trivially solved [with the initial condition $\phi(n,0)=0$]. The result is somewhat complicated: when $n>\bar{n}(t)$, one has

$$\phi(n,t)=-\frac{g}{\kappa}\left[\sqrt{n_0+n-\bar{n}(t)}-\sqrt{n}+\tfrac{1}{2}\sqrt{n-\bar{n}(t)}\ln\left[\frac{\sqrt{n_0+n-\bar{n}(t)}-\sqrt{n-\bar{n}(t)}}{\sqrt{n_0+n-\bar{n}(t)}+\sqrt{n-\bar{n}(t)}}\frac{\sqrt{n}+\sqrt{n-\bar{n}(t)}}{\sqrt{n}-\sqrt{n-\bar{n}(t)}}\right]\right]. \tag{18a}$$

When $n<\bar{n}(t)$, one has

$$\phi(n,t)=-\frac{g}{\kappa}\left\{\sqrt{n_0+n-\bar{n}(t)}-\sqrt{n}+\tfrac{1}{2}\sqrt{\bar{n}(t)-n}\left[\tan^{-1}\left[\frac{n_0+n-\bar{n}(t)}{\bar{n}(t)-n}\right]^{1/2}-\tan^{-1}\left[\frac{n}{\bar{n}(t)-n}\right]^{1/2}\right]\right\}, \tag{18b}$$

and when $n=\bar{n}(t)$, one has simply

$$\phi(n,t)=-\frac{g}{\kappa}[\sqrt{n_0}-\sqrt{\bar{n}(t)}] \tag{18c}$$

with $\bar{n}(t)$ given by (15b).

Equations (10), (15), and (18) form the complete solution for the JCM in a lossy cavity if the initial state of the atom is $|+\rangle$. Equation (10) shows that the field and atom phases evolve together, just as they do in the absence of losses; Eq. (15a) implies that, in this approximation, the total state of the field and atom remains pure in spite of the cavity losses, which cause the average number of photons to decay as given by Eq. (15b). In this approximation, therefore, one may actually replace the density operator (10) by the state vector

$$|\Psi_+(t)\rangle=\frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}e^{-\bar{n}(t)/2}\frac{\bar{n}(t)^{n/2}}{\sqrt{n!}}e^{i\phi_n(t)}|n\rangle \times(e^{i[\phi_{n+1}(t)-\phi_n(t)]}|a\rangle+|b\rangle). \tag{19}$$

The overall phase of the field and the atom may be derived from Eq. (18): to do so, note that the states which at any given time have greater weights in (10) and (19) are those corresponding to numbers of photons close to the instantaneous average $\bar{n}(t)$. One may then expand the n dependence of $\phi_{n+1}-\phi_n$ in a power series in $[n-\bar{n}(t)]/n_0$, and the result, to lowest order, is simply

$$\phi_{n+1}(t)-\phi_n(t)\simeq-\frac{g}{2\kappa\sqrt{n_0}}(e^{\kappa t}-1) =-\int_0^t\frac{g}{2\sqrt{\bar{n}(t')}}\equiv\phi(t). \tag{20}$$

Hence the phase grows at an instantaneous rate $-g/2\sqrt{\bar{n}}$, just as in the absence of losses, only $\bar{n}$ is the instantaneous average number of photons. It is tempting to make the replacement $\phi_{n+1}(t)-\phi_n(t)\simeq\phi(t)$ in Eq. (19), which allows one to write the state of the field and the atom as a product of pure states, in complete analogy to (6a) for the lossless case. This should be a good approximation for as long as $\bar{n}$ remains large, but if the losses are substantial it is better to stick to the more exact form (19), which (somewhat surprisingly) remains fairly accurate even for rather low values of $\bar{n}$, as will be shown presently.

Two quantities of interest which can be easily calculated from (19) are the expectation value of the field amplitude $\langle a\rangle$ and of the atomic dipole moment, which is proportional to $-\langle\sigma\rangle$, where $\sigma\equiv|b\rangle\langle a|$. The results are

$$\langle a(t)\rangle=\tfrac{1}{2}\sum_{n=0}^{\infty}e^{-\bar{n}(t)}\frac{\bar{n}(t)^n}{n!}\sqrt{n}\,(e^{i[\phi_{n+2}(t)-\phi_{n+1}(t)]}+e^{i[\phi_{n+1}(t)-\phi_n(t)]}) \tag{21}$$

and

$$\langle\sigma(t)\rangle=\tfrac{1}{2}\sum_{n=1}^{\infty}e^{-\bar{n}(t)}\frac{\bar{n}(t)^n}{n!}e^{i[\phi_{n+1}(t)-\phi_n(t)]}, \tag{22}$$

with $\bar{n}(t)$ given by Eq. (15b) and $\phi_n(t)$ given by Eq. (18). Equations (21) and (22) have the simple form of averages over the Poisson distribution corresponding to the instantaneous number of photons.

Figure 1 shows the evolution of the real parts of $\langle a(t)\rangle$ and $\langle\sigma(t)\rangle$, calculated from a numerical integration of the master equation (solid lines), compared to the prediction from Eqs. (21) and (22) (dashed lines), for the moderately large initial number of photons $\bar{n}_0=25$ and the also moderately large losses $\kappa/g=0.01$. The initial state of the atom is $|+\rangle$. The agreement with the numerical calculation is excellent for times up to around $gt=50$, for which the average number of photons left in the cavity is only about 9.2. By the end of the time interval shown, $\bar{n}$ is down to only about 4.1 photons; in spite of this, the approximate solution still exhibits good qualitative agreement with the numerical calculation.

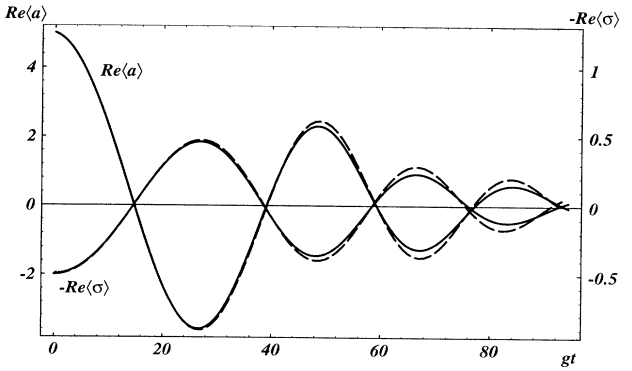

FIG. 1. Real parts of the expectation values of the field amplitude and the atomic dipole moment, calculated numerically for the case $n_0=25$, $\kappa=0.01g$ (solid lines), and calculated from the analytical approximation presented here [Eqs. (21) and (22), dashed lines]. The initial state of the atom is $|+\rangle$.

The total time shown in Fig. 1 is three times the lossless revival period $2\pi\sqrt{n_0}$. In fact, however, Eq. (20) shows that the frequency of the oscillations in the atomic dipole (or the field) amplitude grows as $\bar{n}$ decreases, and Fig. 1 actually shows four "revivals" (i.e., maxima and minima) of the real part of the atomic dipole amplitude. This is seen more clearly in Fig. 2 (to be compared to similar figures in [3]), which displays the evolution of the atomic state in the Bloch sphere, as given by the numerical integration of the master equation, over the time range of Fig. 1, both with and without cavity losses. The $\kappa=0$ case shows three revivals, i.e., one and a half turns around the Bloch sphere, whereas the $\kappa/g=0.01$ case, corresponding to Fig. 1, shows more than two complete turns around the Bloch sphere [for a thorough discussion of these Bloch sphere characterizations of the state of the atom, see [3]; note that the real and imaginary parts of $\langle\sigma(t)\rangle$ are, except for a factor of 2, the components R_1 and R_2 of the Bloch vector]. In spite of this shortening of the revival period with time, Fig. 1 shows that the field and the atom stay "in step" throughout (i.e., always in opposition), as in the lossless case.

It is instructive to compare the solution (19) to other possible (in general, less accurate) approximations. As discussed in [3], the atomic state is pure if its Bloch vector has unit length, that is, if it lies on the surface of the sphere. From Fig. 2 it may be seen that the state is indeed approximately pure for early times. This means that for these times one can in fact carry out the approximation (20) and factorize the total state into a field part and an atom part,

$$|\Psi_+(t)\rangle \simeq \frac{1}{\sqrt{2}}(e^{i\phi(t)}|a\rangle+|b\rangle) \otimes \sum_{n=0}^{\infty} e^{-\bar{n}(t)/2}\frac{\bar{n}(t)^{n/2}}{\sqrt{n!}}e^{i\phi_n(t)}|n\rangle , \tag{23}$$

in complete analogy with the lossless case [Eq. (6a)]. The time over which (23) is a good approximation increases for larger numbers of photons and/or smaller losses. The atomic state purity is lost through entanglement with the field, which takes place over a time scale proportional to the average number of photons; the cavity losses hasten this entanglement merely by reducing the number of photons. It is still a good rule of thumb to say that Eq. (23) is approximately valid as long as $gt \ll \pi\bar{n}$, where now, however, $\bar{n}$ depends on t. For the case illustrated in Figs. 1 and 2, one has $gt=\pi\bar{n}$ at $gt\simeq 37$; Fig. 1 shows that the better approximation (19) is valid well beyond that point.

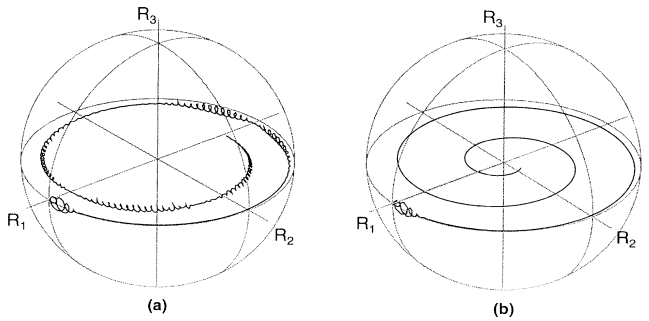

FIG. 2. Evolution of the atomic state in the Bloch sphere for the initial condition $n_0=25$ and initial atomic state $|+\rangle$, for (a) no losses and (b) $\kappa=0.01g$. The initial state is (1,0,0); the total time shown is $gt=30\pi\simeq 94$.

Closely related to (23) is the "neoclassical" approximation. As discussed in [3], as long as the factorization (23) holds approximately, the expectation values of products of field and atom operators in the Heisenberg equations of motion could be factorized as well. Doing this in the Heisenberg-Langevin equations for the lossy cavity, one obtains the system

$$\frac{d\langle a\rangle}{dt}=-\kappa\langle a\rangle-ig\langle\sigma\rangle , \tag{24a}$$

$$\frac{d\langle\sigma\rangle}{dt}=ig\langle a\rangle\langle\sigma_3\rangle , \tag{24b}$$

$$\frac{d\langle\sigma_3\rangle}{dt}=-2ig(\langle\sigma\rangle^*\langle a\rangle-\langle a\rangle^*\langle\sigma\rangle) , \tag{24c}$$

where $\sigma_3\equiv|a\rangle\langle a|-|b\rangle\langle b|$ is the population inversion.

It is a fairly straightforward matter to verify that the numerical solutions of (24) agree indeed with the results shown in Figs. 1 and 2 as long as $gt\ll\pi\bar{n}$ is satisfied, whereas for longer times both $\langle a\rangle$ and $\langle\sigma\rangle$ decay rather faster than (24) predicts, because of field-atom entanglement. In particular, Eqs. (24) lead to a conserved Bloch vector length (as was the case when $\kappa=0$ [3]), which Fig. 2 shows is approximately the case only for early times. The conclusion, therefore, as in [3], is that the initial condition $|+\rangle$ (or equivalently $|-\rangle$) leads to an evolution which is well described by a "neoclassical" approximation for times which can be rather long if n_0 is large.

By contrast, one may consider the so-called "secular approximation" which has been used by other authors to treat the JCM with cavity losses. According to Puri and Agarwal [9], the terms neglected in making the secular approximation are of the order of $(\kappa/g)^2$, which for the present case is 10^{-4}; hence the approximation would seem to be quite justified. Puri and Agarwal have given expressions [Eqs. (4.5) and (6.4) of [9]] which allow one to calculate $\langle a(t)\rangle$ and $\langle\sigma(t)\rangle$. For the present case ($\kappa=0.01, n_0=25$, initial atomic state $|+\rangle$) the result is shown in Fig. 3. Comparing it to Fig. 1, one must conclude that the secular approximation is, in fact, totally inadequate for this problem. The reason for this somewhat surprising result is discussed in Appendix B.

B. Evolution of $\rho^{(--)}(t)$

It is straightforward to verify that, just as in the lossless case, the evolution of $\rho^{(--)}(t)$ is identical to that of $\rho^{(++)}(t)$ except for the sign of the phases, and, of course, the relative sign between $|a\rangle$ and $|b\rangle$ in the wave function. Thus one can write

$$\rho^{(--)}(t)=\tfrac{1}{2}\sum_{n,m} r_{nm}(t)e^{-i[\phi_n(t)-\phi_m(t)]}|n\rangle\langle m| \otimes(e^{-i[\phi_{n+1}(t)-\phi_n(t)]}|a\rangle-|b\rangle) \times(e^{i[\phi_{m+1}(t)-\phi_m(t)]}\langle a|-\langle b|) , \tag{25}$$

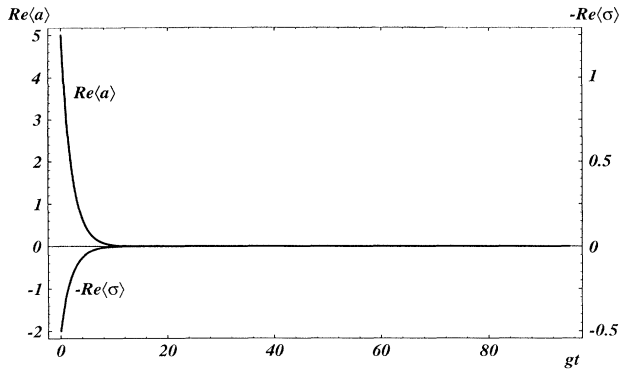

FIG. 3. The prediction of the secular approximation [9] for the quantities shown in Fig. 1.

with $r_{nm}(t)$ and $\phi_n(t)$ given by the same expressions, (15) and (18), as for $\rho^{(++)}(t)$. To the same approximation, one may alternatively use the atom-field wave function

$$|\Psi_-(t)\rangle=\frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}e^{-\bar{n}(t)/2}\frac{\bar{n}(t)^{n/2}}{\sqrt{n!}}e^{-i\phi_n(t)}|n\rangle \times(e^{-i[\phi_{n+1}(t)-\phi_n(t)]}|a\rangle-|b\rangle) \quad (26)$$

in complete analogy with (19).

C. Evolution of $\rho^{(+-)}(t)$

The part of the density operator that evolves from the term proportional to $|+\rangle\langle-|$ (and it complex conjugate) is responsible for the mutual coherence between the solutions which evolve from the initial conditions $|+\rangle$ and $|-\rangle$. Hence this term is important whenever the initial state of the atom is a coherent superposition of $|+\rangle$ and $|-\rangle$. In the limit of large photon numbers, this is the term essentially responsible for the population inversion oscillations and the famous collapses and revivals, as has been shown in [2] and [3].

The discussion in [2] provides some physical insight into the nature of this term and allows one to predict qualitatively how it is likely to be affected by the cavity losses. Essentially, $\rho^{(+-)}(t)$ represents a coherence or interference term between two "paths"—the one evolving from $|+\rangle$, and the one evolving from $|-\rangle$—which, in the limit of a large number of photons, become macroscopically distinct over the collapse time scale. This is because the field is potentially a "large" system (in energy, or number of quanta) whose phase evolves in opposite ways along the two paths. It is well known how sensitive such macroscopic quantum superpositions are to coupling to the environment, i.e., to "dissipation" in general [21]. By coupling the field to a loss reservoir, one may expect the coherence between the two macroscopically distinct parts of the wave function to disappear very rapidly, in a time scale inversely proportional to some power of the size of the system. Hence one expects the term $\rho^{(+-)}(t)$ to be damped away much faster than the photons leave the cavity and to become negligible very rapidly even for very small values of κ.

These considerations are borne out by the numerical calculations. An approximate analytical derivation follows along lines similar to those for $\rho^{(++)}(t)$. One may assume a form

$$\rho^{(+-)}(t)=\tfrac{1}{2}\sum_{n,m}f_{nm}(t)e^{i[\xi_n(t)+\xi_m(t)]}|n\rangle\langle m| \otimes(e^{i[\xi_{n+1}(t)-\xi_n(t)]}|a\rangle+|b\rangle) \times(e^{i[\xi_{m+}(t)-\xi_m(t)]}\langle a|-\langle b|) \quad (27)$$

by analogy with the product $|\Psi_+\rangle\langle\Psi_-|$ [see Eqs. (19) and (26)]. One might consider making the phases $\xi_n(t)$ identical to the $\phi_n(t)$ derived earlier [Eq. (18)], but these are too complicated for the present purposes. If n_0 is reasonably large, $\rho^{(+-)}(t)$ may be expected to become negligible very rapidly, for all but the smallest values of the loss coefficient κ. Hence it is simplest to let the ξ's have the value for the lossless case

$$\xi_n(t)=-\sqrt{n}\,gt \quad (28)$$

and to allow for the f_{nm} to be complex if necessary.

When (27) and (28) are substituted in the master equation (4), the lossless part of the equation (i.e., the part proportional to g) is satisfied provided that, as was done in Sec. III A for the r_{nm}, the ratios $f_{n\pm1,m}/f_{nm}$ and $f_{n,m\pm1}/f_{nm}$ are set equal to 1. The result is the following equation for the f_{nm}:

$$\dot{f}_{nm}=-\kappa(n+m)f_{nm} +2\kappa\sqrt{(n+1)(m+1)} \times e^{-i(\sqrt{n+i+1}-\sqrt{n+i}+\sqrt{m+j+1}-\sqrt{m+j})gt} \times f_{n+1,m+1} \quad (29)$$

again with i,j equal to zero or one. Always assuming that the losses are small, and therefore that the average number of photons does not change much from the initial value n_0 before the term $\rho^{(+-)}(t)$ becomes completely negligible, one may simplify the exponent in Eq. (29) rather drastically to

$$-igt(\sqrt{n+i+1}-\sqrt{n+i}+\sqrt{m+j+1}-\sqrt{m+j}) \simeq -i\frac{gt}{\sqrt{n_0}}\,. \quad (30)$$

With this, it is easy to solve for the f_{nm} by assuming the coherent-state-like form [compare Eq. (15)]

$$f_{nm}(t)=F(t)e^{-\bar{n}(t)}\frac{\bar{n}(t)^{(n+m)/2}}{\sqrt{n!m!}}\,. \quad (31)$$

Substitution of (30) and (31) into (29) yields, for the factor $F(t)$,

$$F(t)=\exp\left[\frac{n_0}{1-ig/2\kappa\sqrt{n_0}}(1-e^{-(2\kappa-ig/\sqrt{n_0})t}) -n_0(1-e^{-2\kappa t})\right]. \quad (32)$$

Equations (27), (28), (31), and (32) constitute a solution

for $\rho^{(+-)}(t)$ valid for as long as the losses do not significantly deplete the field. The factor $F(t)$ [Eq. (32)] accounts for the loss of coherence between the $|+\rangle$ branch and the $|-\rangle$ branch. Essentially, $\rho^{(+-)}(t)$ is of the form $F(t)|\Psi_+\rangle\langle\Psi_-|$, with $|\Psi_+\rangle$ given by the small-loss limit of Eq. (19) and $|\Psi_-\rangle$ by the small-loss limit of its mirror image Eq. (26). If $F(t)$ has magnitude close to 1, the total state of atom and field remains approximately pure and a coherent superposition of the paths $|\Psi_+\rangle$ and $|\Psi_-\rangle$, but, even for very small losses, $F(t)$ decays very rapidly. At the revival time $t_R=2\pi\sqrt{n_0}/g$, one has

$$|F(t_R)|\simeq e^{-4\pi\kappa n_0^{3/2}/g} \tag{33}$$

to lowest order in κ in the exponent. For the values considered in the preceding subsection, $\kappa/g=-0.01$, $n_0=25$, and Eq. (33) predicts $|F(t)|\simeq e^{-5\pi}=1.5\times10^{-7}$, which is already quite negligible.

As mentioned earlier, when the number of photons is large the population inversion oscillations come mostly from $\rho^{(+-)}(t)$. In the approximation represented by the expressions (10), (25), and (27), in fact, only (27) leads to a nonvanishing population inversion at any time. Hence, if these expressions are used in (8) to calculate the evolution of the generic initial coherent superposition (7), the result, for the population inversion, is

$$\begin{aligned}\langle\sigma_3\rangle&=\mathrm{Tr}[(|a\rangle\langle a|-|b\rangle\langle b|)\rho]\\&=\frac{\gamma\delta^*}{2}F(t)\sum_{n=0}^{\infty}e^{-\bar{n}(t)}\frac{\bar{n}(t)^n}{n!}(e^{-2igt\sqrt{n+1}}\\&\qquad+e^{-2igt\sqrt{n}})+\text{c.c.}\ ,\end{aligned} \tag{34}$$

coming entirely from $\rho^{(+-)}(t)$.

If the initial state of the atom is the upper energy eigenstate $|a\rangle=(|+\rangle+|-\rangle)/\sqrt{2}$, one has $\gamma=\delta=1/\sqrt{2}$ and Eq. (34) predicts, for the lossless case, the usual collapses and revivals of the population inversion. There is a small difference with the exact formulas in that the phase factor being averaged in Eq. (34) is neither $\exp(-ig\sqrt{n+1})$ nor $\exp(-ig\sqrt{n})$, but the average of the two; this difference, however, is not very significant for large n, and in particular it is largely negligible around the revival times, which are the only times when (34) is substantially different from zero [see, in this connection, the discussion of Eq. (28) of [3]]. When $\kappa\neq0$, these collapses and revivals are multiplied by $F(t)$, which introduces both damping and a time-dependent phase shift.

Figure 4 shows the average value of the population inversion, according to the numerical integration of the master equation (4), for $n_0=25$ and $\kappa=0.002$; the initial atomic state is now $|a\rangle$. The disappearance of the revivals has of course been observed by a number of previous authors [8–10]. The inset shows an enlarged view of the revival region and the comparison between the numerical calculation (solid line) and the prediction of the approximation (34) (dashed line). The agreement is quite good, in spite of the small size of the revivals. Smaller losses lead to similar or better agreement; for larger losses the agreement is not so good, but then the revivals are vanishingly small anyway.

What about the secular approximation? Use of Eqs. (3.1) and (3.3) of [9] leads to an excellent agreement with the numerical results for this case, so good, in fact, that they look indistinguishable at the resolution of the figure. Thus it would seem that the secular approximation is quite good for the diagonal elements, in the basis of $|a\rangle, |b\rangle$, and $|n\rangle$, needed to calculate $\langle\sigma_3\rangle$. On the other hand, the results from Sec. III A indicate that it is terrible for the off-diagonal elements (in the same basis) needed to calculate $\langle a\rangle$ or $\langle\sigma\rangle$.

This, however, is not the whole story, for, as discussed in Appendix A, the secular approximation also breaks down eventually for the diagonal elements, for sufficiently large values of κ, even though one may still have $(\kappa/g)^2\ll1$. Figure 5 shows the collapse of the population inversion oscillations for $\kappa=0.05g$ and $n_0=25$, along with the predictions of Eq. (34) (dashed line) and of the secular approximation, Eqs. (3.1) and (3.3) of [9] (dash-dotted line). The secular approximation does poorly, as explained in Appendix A, because for these parameters one no longer has $\kappa\ll g/2\sqrt{n_0}$. The present approximation, Eq. (34), is significantly better for the short times shown in the figure. This is because the main approximation made in its derivation, Eq. (30), amounts to the requirement that the losses should not significantly change the number of photons over the time considered, which in this case would mean $gt\ll g/2\kappa=10$ or so.

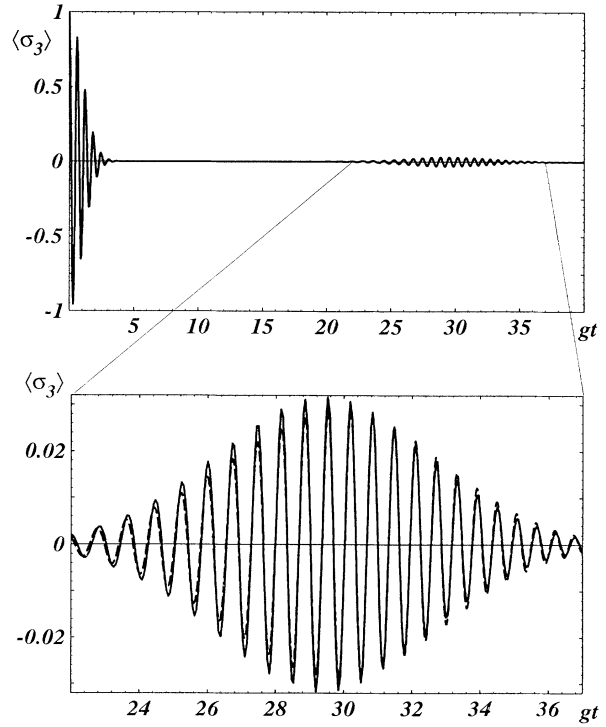

FIG. 4. The population inversion as a function of time for the case $n_0=25$, $\kappa=0.002g$, as calculated numerically (solid line) and from the analytic approximation (34) (dashed line). The enlargement shows the region of the first revival. The initial atomic state is $|a\rangle$.

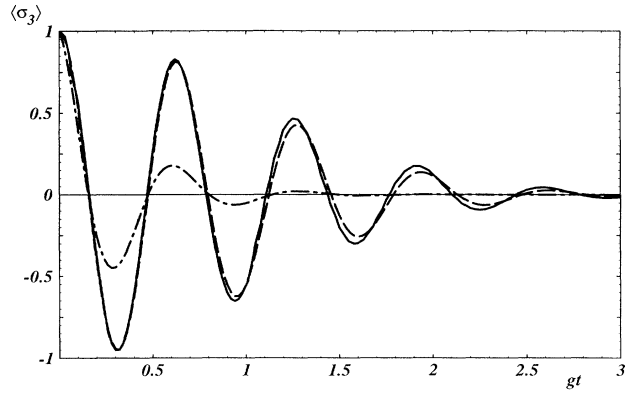

FIG. 5. The population inversion as a function of time for the case $n_0=25$, $\kappa=0.05g$, as calculated numerically (solid line), from the analytic approximation (34) (dashed line), and from the secular approximation [9] (dashed-dotted line). The initial atomics state is $|a\rangle$.

The conclusion, then, as regards the secular approximation of [9], is that it is better for the diagonal elements than for the off-diagonal ones, but it still breaks down for moderately large losses, even if they still satisfy $(\kappa/g)^2 \ll 1$. The reasons for this, and the actual conditions for the secular approximation of [9] to be valid, are set forth in detail in Appendix A.

To summarize, in this section a complete solution, for the field and atom, and arbitrary initial atomic state, has been presented for the Jaynes-Cummings model with cavity losses. The solution agrees very well with the results of the direct numerical integration of the master equation, provided that the number of photons be large enough; $n_0=25$ has been illustrated as a typical case. Some of the main results are discussed in the next section, which also addresses the question of how two of the most important recent predictions for the lossless JCM, namely, the atomic state preparation and the field Schrödinger cat, are modified by the presence of cavity losses.

IV. DISCUSSION

The approximate solution developed here provides a fairly complete picture of the evolution of the atom-field system for an initial quasiclassical field state and for not-too-large cavity losses. The first phase of the evolution, up to the collapse time, involves the usual damped Rabi oscillations. By the time of the collapse, the two branches of the wave function which evolve from each one of the states $|+\rangle$ and $|-\rangle$, in the superposition representing the initial state of the atom, have evolved to be associated with macroscopically distinct states of the field, with the same amplitude but opposite phases [2]. At that point, it is best to think of the system's wave function as being split into two alternate "paths." It was pointed out in [3] that the macroscopic nature of the field makes it very hard (in the limit as $\bar{n}\to\infty$, impossible) to observe interference between these two branches after the collapse, until the revival time, when the phases of the two field states again approach each other.

The main result of Sec. III C, not altogether surprising, is that for any but the smallest cavity losses and very small numbers of photons, the coherence between the two branches is in fact all but lost by the revival time [see Eq. (33)], and hence the interference is most likely to be unobservable. The conclusion is that, for most practical purposes, for large number of photons and moderate losses it is a good approximation to regard the two branches of the wave function after the collapse time as mutually *incoherent;* i.e., to neglect $\rho^{(+-)}(t)$ completely for all times after the collapse time. If this is done, expectation values of system operators are given merely by a weighted sum of the expectation values calculated with $|\Psi_+(t)\rangle$ and $|\Psi_-(t)\rangle$ [given by Eqs. (19) and (16)]. The population inversion revivals disappear in this approximation. Interestingly, as has been pointed out by other authors, other features, such as field squeezing, are not affected so much by moderate losses [10]. In the present picture, this may be understood by noting that both the field states associated with $|\Psi_+(t)\rangle$ and $|\Psi_-(t)\rangle$ are squeezed [2], along approximately the same direction, and by the same amount, at the revival time; hence, even if there is no interference left between the two branches at the revival time, the underlying squeezing of each branch results in an overall squeezing of essentially the same magnitude.

As mentioned in the Introduction, two important results have recently been derived for the lossless JCM. One is the "Schrödinger cat" state for the field, to be discussed in a moment; the other, closely related, is the "state preparation" for the atom. This is the result, first pointed out in [11], that at half-revival time, $t_0=t_R/2=\pi\sqrt{\bar{n}}/g$, the state of the atom is almost pure and independent of its initial state. On the Bloch sphere (Fig. 2) this state is the point (0,1,0).

Figure 2 shows that losses as large as $\kappa=0.01g$ do not substantially alter this result, except for the time at which it occurs. The point (0,1,0) is reached when the phase of the atomic dipole is equal to $-\pi/2$, and hence, from Eq. (20), the equation for the time t_0 is

$$\phi(t_0)=-\frac{g}{2\kappa\sqrt{n_0}}(e^{\kappa t_0}-1)=-\frac{\pi}{2} \tag{35}$$

so that

$$t_0=\frac{1}{\kappa}\ln\left[1+\frac{\pi\kappa\sqrt{n_0}}{g}\right]. \tag{36}$$

The evolution of $\rho^{(--)}$, being the mirror image of that shown in Fig. 2, would also lead to the same "almost" pure state at the same time, and hence so would any initial coherent or incoherent superposition of the states $|+\rangle$ and $|-\rangle$, i.e., any general initial state of the atom.

The loss of state purity by the time t_0, relative to the lossless case (compare [19]), can be estimated from (10). One has, for the trace of the square of the reduced atomic density operator,

$$\mathrm{Tr}(\rho_{\mathrm{at}}^2)=\frac{1}{2}+\frac{1}{2}\left|\sum_n e^{-\bar{n}(t)}\frac{\bar{n}(t)^n}{n!}e^{i[\phi_{n+1}(t)-\phi_n(t)]}\right|^2. \tag{37}$$

To obtain a value different from 1 for the right-hand side of (37), the difference $\phi_{n+1}(t)-\phi_n(t)$ must be evaluated to one order higher in $n-\bar{n}(t)$ than that shown in Eq. (20). Expansion of Eq. (18) around $n=\bar{n}$ leads to

$$\phi_n(t)\simeq -\frac{g}{\kappa}\sqrt{n_0}\left[1-e^{-\kappa t}+\frac{1}{2}(e^{\kappa t}-1)\frac{n-\bar{n}}{n_0} -\frac{1}{24}(e^{3\kappa t}-1)\left[\frac{n-\bar{n}}{n_0}\right]^2\right], \tag{38}$$

and hence

$$\mathrm{Tr}(\rho_{\mathrm{at}}^2)\simeq\frac{1}{2}+\frac{1}{2}\left|\sum_n e^{-\bar{n}(t)}\frac{\bar{n}(t)^n}{n!}e^{in\xi}\right|^2 =\frac{1}{2}+\frac{1}{2}e^{-4\bar{n}(t)\sin^2(\xi/2)}, \tag{39a}$$

where

$$\xi=\frac{g}{12\kappa n_0^{3/2}}(e^{3\kappa t}-1), \tag{39b}$$

which goes as $gt/4\bar{n}^{3/2}$ for small κ.

For the example shown in Fig. 2, with $\kappa=0.01g$ and $n_0=25$, the "half-revival time" t_0 is equal to $14.6/g$, somewhat less than the lossless value $5\pi/g$. Equation (39) predicts a state purity at this time, as measured by $\mathrm{Tr}(\rho_{\mathrm{at}}^2)$, of 0.988. This is a little off, according to the numerical integration, which yields instead $\mathrm{Tr}(\rho_{\mathrm{at}}^2)=0.979$. Using the latter figure, the conclusion is that an atom interacting with the field for this time, regardless of its initial state, has a probability $\{1+[2\,\mathrm{Tr}(\rho_{\mathrm{at}}^2)-1]^{1/2}\}/2=0.989$ of emerging in the state (0,1,0), that is, the state

$$|\psi_0\rangle=\frac{1}{\sqrt{2}}(-i|a\rangle+|b\rangle). \tag{40}$$

This is practically the same as if there were no losses. (Compare with the similar discussion in [3]; note that because of an error in [3], the discussion of state purity on p. 541 goes with Fig. 4, not Fig. 3.)

The conclusion, then, is that the state preparation at the half-revival time in the JCM should be readily observable even in lossy cavities. This could be an extremely interesting effect to observe (see the discussion in [2] for more details). For one thing, the atomic evolution leading to the unique state $|\psi_0\rangle$ at $t=t_0$ is incompatible with the semiclassical theory, in which the atom is assumed to interact with a nonquantized field, since under such conditions unitary evolution requires initially orthogonal atomic states to evolve into states which remain orthogonal at all times. This state preparation is therefore an example of the unique effects that may arise when a quantum system is driven by another quantum system. The importance of such systems has recently been pointed out also by Ballentine [22], who has studied another solvable example.

As discussed above, losses are much more destructive towards the other effect predicted in the JCM, the macroscopic quantum superposition state (or Schrödinger cat state) of the field at the time t_0. It is easy to see how such a state arises [2,14]. Because the state of the atom at t_0 is to a good approximation $|\psi_0\rangle$, regardless of the initial condition, the total density operator at that time can be written

$$\rho(t_0)\simeq|\psi_0\rangle\langle\psi_0|\otimes[|\gamma|^2|\Phi_+\rangle\langle\Phi_+|+|\delta|^2|\Phi_-\rangle\langle\Phi_-| +\gamma\delta^*F(t)|\Phi_+\rangle\langle\Phi_-| +\gamma^*\delta F^*(t)|\Phi_-\rangle\langle\Phi_+|], \tag{41}$$

where the field states $|\Phi_+\rangle$ and $|\Phi_-\rangle$ are given by

$$|\Phi_\pm\rangle=e^{-\bar{n}(t_0)/2}\sum_n\frac{\bar{n}(t_0)^{n/2}}{\sqrt{n!}}e^{\pm i\phi_n(t_0)}|n\rangle \tag{42}$$

[this is in the very-small-loss limit, where $\phi_n(t)\simeq -gt\sqrt{n}$]. These two states represent fields which have opposite phases and a relatively large amplitude. The field density operator, according to (41), would be a coherent superposition of both, with coefficients γ and δ, respectively, but for the factor $F(t)$ not being equal to 1. As $F(t)$ goes to zero (which, as explained in Sec. III C, happens very rapidly even for very small losses) the state of the field becomes an *incoherent* mixture of $|\Phi_+\rangle$ and $|\Phi_-\rangle$.

It was pointed out in [2], and should be clear from all the foregoing, that, when the initial state of the field is a coherent state with a well-defined phase, the revival of the population inversion oscillations is an indication of the mutual coherence between the two branches of the wave function. Thus it may be regarded as a signature of the Schrödinger cat realized at the earlier time t_0. Equation (33) may be used to estimate how large the losses can be if one is to see still, at the revival time, any trace of the cat. Figure 4 suggests that, if $n_0=25$, κ/g should not exceed 0.002. This is not entirely beyond reach. In [23], Rempe, Schmidt-Kaler, and Walther report on a micromaser cavity with $\kappa=2.3\ \mathrm{s}^{-1}$ and a coupling constant, for the Rydberg atom considered, $g/2\pi=20$ kHz, so $\kappa/g\sim 3.7\times10^{-5}$. For such small losses, and $n_0=25$, Eq. (33) predicts $F(t_R)=0.94$, that is, almost perfect coherence. Hence the Schrödinger cat should be observable in such a system.

It is not clear at present whether the very few residual thermal photons in current micromaser experiments ($n_{\mathrm{th}}=0.15$ was reported in [23]) might pose a problem in the detection of the cat. A recent study by Vourdas [24] on the effect of thermal noise on coherent superpositions of the form $|\alpha\rangle\pm|-\alpha\rangle$ suggests that one would need $n_{\mathrm{th}}\ll 1/n_0$ to be able to tell a coherent from an incoherent superposition of coherent states, but this would apply only at the time t_0, when the states being superposed have opposite phases; the detection technique based on waiting until the revival time to see the oscillations in $\langle\sigma_3\rangle$ should be able to get around this problem. The atomic state preparation at the time t_0 also should not be very much affected by a number of thermal photons as small as those reported, although here also a detailed calculation remains to be made.

Perhaps the most difficult part of any of these experiments would be to prepare the initial state of the field. It

has to be a quasiclassical state with a fairly well-defined phase and amplitude. It does not necessarily have to be a coherent state; it could be a cotangent state such as those studied by Meystre and co-workers [25], which could be prepared in a micromaster cavity by pumping it with atoms in a coherent superposition of $|a\rangle$ and $|b\rangle$. Such a coherent superposition is a small radiating dipole, and the trick of the preparation is to phase these dipoles carefully as they enter the cavity. This would require, among other things, very careful control of the incoming atom velocity.

It should be pointed out that the generation of a cotangent state as proposed by Meystre and co-workers is in itself an example of state preparation of a quantum system (the field, in this case) by another (a stream of atoms). Not only that, but some cotangent states may also be regarded as Schrödinger cats in and of themselves. Hence the realization of such states would already be of considerable interest. On the other hand, the state preparation and Schrödinger cat discussed here and in [2,11,14] have enough unique features of their own to justify the comparatively small additional effort necessary to produce them: the state preparation leads to a unique state completely independent of the initial state ("ideal atom polarizer"; see [2]), and the Schrödinger cat is in fact produced by the interaction of a *single* atom with the field, just as Schrödinger's original cat was supposed to be placed in a superposition state by just a single photon.

In conclusion, the results of this paper show the usefulness of the approach to the study of the atom-field dynamics developed in [2] and [3]. It is very likely that the same approach will be useful to deal with other related problems of current experimental interest, such as optical-cavity quantum electrodynamics [26].

Note added in proof. The review of earlier work in the Introduction should have included the very recent paper by Daeubler, Risken, and Schoendorff [29], who have derived a formal exact solution for the intensity and population inversion of the JCM with losses in the form of an infinite series.

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APPENDIX A: LIMITATIONS OF THE SECULAR APPROXIMATION

The use of a secular approximation to simplify the JCM equations goes back at least to Haroche and Raimond [27], but it was Puri and Agarwal who applied it most systematically to the study of the JCM with cavity losses in their classic 1987 paper [9]. The only condition they expressly state for the validity of their results is that κ/g be small; specifically, they claim that the terms they neglect are of the order of κ^2/g^2. In view of the results presented in this paper, this cannot be the whole story, and more serious restrictions must apply if the secular approximation is to be trusted. The purpose of this appendix is to determine more precisely the size of the terms neglected by Puri and Agarwal, and hence what the actual restrictions are on the applicability of their results.

Following Haroche and Raimond, Puri and Agarwal work with eigenstates of the JCM Hamiltonian ("dressed states"). Following their notation, the excited states will be denoted here by $|\Psi_n^{\pm}\rangle$, whereas the ground state, which they write as $|0,\frac{1}{2}\rangle$, will be denoted here by $|b,0\rangle$ (atom in lower state $|b\rangle$, zero photons in field). One has, in the interaction picture in which the free atom and field evolution has been absorbed in a redefinition of the operators and the states,

$$H_I|b,0\rangle=0\ , \tag{A1a}$$

$$H_I|\Psi_n^{\pm}\rangle=\hbar g\sqrt{n+1}\,|\Psi_n^{\pm}\rangle\ , \tag{A1b}$$

with

$$|\Psi_n^{\pm}\rangle=\frac{1}{\sqrt{2}}(|a,n\rangle\pm|b,n+1\rangle)\ . \tag{A2}$$

These $|\Psi_n^{\pm}\rangle$ are not to be mistaken for the states $|\Psi_{\pm}(t)\rangle$ used elsewhere in this paper. The connection between the present work and the eigenstates $|\Psi_n^{\pm}\rangle$ is actually rather simple: if the initial state of the atom is $|+\rangle$, and the field is in a quasiclassical state, the total initial state is mostly a superposition of the $|\Psi_n^{+}\rangle$, i.e, the coefficients of the $|\Psi_n^{-}\rangle$ are then vanishingly small; and conversely if the initial atomic state is $|-\rangle$.

Next, Puri and Agarwal write the annihilation operator in term of these states, and transform away the Hamiltonian part of the evolution by defining

$$W(t)=e^{iH_It}\rho e^{-iH_It}\ . \tag{A3}$$

Here, the interaction Hamiltonian instead of the full Hamiltonian has been used, since everything is already in the interaction picture. As a result, the master equation for $W(t)$ becomes

$$\dot{W}(t)=\kappa(2\tilde{a}W\tilde{a}^{\dagger}-\tilde{a}^{\dagger}\tilde{a}W-W\tilde{a}^{\dagger}\tilde{a})\ , \tag{A4}$$

where

$$\tilde{a}=\exp(iH_It)a\exp(-iH_It)$$

and

$$\tilde{a}^{\dagger}=\exp(iH_It)a^{\dagger}\exp(-iH_It)$$

are transformed annihilation and creation operators, given explicitly by

$$\begin{aligned}\tilde{a}=&\frac{1}{\sqrt{2}}(e^{-igt}|b,0\rangle\langle\Psi_0^{+}|-e^{igt}|b,0\rangle\langle\Psi_0^{-}|)\\&+\tfrac{1}{2}\sum_{n=1}^{\infty}(\sqrt{n+1}+\sqrt{n})(e^{-ig(\sqrt{n+1}-\sqrt{n})t}|\Psi_{n-1}^{+}\rangle\langle\Psi_n^{+}|+e^{ig(\sqrt{n+1}-\sqrt{n})t}|\Psi_{n-1}^{-}\rangle\langle\Psi_n^{-}|)\\&+\tfrac{1}{2}\sum_{n=1}^{\infty}(\sqrt{n+1}-\sqrt{n})(e^{ig(\sqrt{n+1}+\sqrt{n})t}|\Psi_{n-1}^{+}\rangle\langle\Psi_n^{-}|+e^{-ig(\sqrt{n+1}+\sqrt{n})t}|\Psi_{n-1}^{-}\rangle\langle\Psi_n^{+}|)\end{aligned} \tag{A5a}$$

and

$$\tilde{a}^{\dagger}=\frac{1}{\sqrt{2}}(e^{igt}|\Psi_0^+\rangle\langle b,0|-e^{-igt}|\Psi_0^-\rangle\langle b,0|)$$
$$+\tfrac{1}{2}\sum_{n=1}^{\infty}(\sqrt{n+1}+\sqrt{n})(e^{ig(\sqrt{n+1}-\sqrt{n})t}|\Psi_n^+\rangle\langle\Psi_{n-1}^+|+e^{-ig(\sqrt{n+1}-\sqrt{n})t}|\Psi_n^-\rangle\langle\Psi_{n-1}^-|)$$
$$+\tfrac{1}{2}\sum_{n=1}^{\infty}(\sqrt{n+1}-\sqrt{n})(e^{-ig(\sqrt{n+1}+\sqrt{n})t}|\Psi_n^-\rangle\langle\Psi_{n-1}^+|+e^{ig(\sqrt{n+1}+\sqrt{n})t}|\Psi_n^+\rangle\langle\Psi_{n-1}^-|)\ . \tag{A5b}$$

As for the transformed number operator $\tilde{a}^{\dagger}\tilde{a}$, it has the simpler expression

$$\tilde{a}^{\dagger}\tilde{a}=\tfrac{1}{2}\sum_{n=0}^{\infty}(2n+1)(|\Psi_n^+\rangle\langle\Psi_n^+|+|\Psi_n^-\rangle\langle\Psi_n^-|)-\sum_{n=0}^{\infty}(e^{igt\sqrt{n+1}}|\Psi_n^+\rangle\langle\Psi_n^-|+e^{-igt\sqrt{n+1}}|\Psi_n^-\rangle\langle\Psi_n^+|)\ . \tag{A6}$$

The way Agarwal and Puri arrive at their starting equation (2.8) is to substitute (A5) and (A6) into (A4) and throw away all the terms with explicit time dependence. The logic behind this is that all the frequencies appearing in them are at least as large as g, and, in the limit $\kappa \ll g$, such terms would have negligible effect driving the more slowly varying matrix elements of W. The flaw with this argument is that it ignores the fact that the *combination frequencies* in the $\tilde{a}W\tilde{a}^{\dagger}$ term can, for large n, be extremely small, and therefore some of the terms neglected can be very slowly varying themselves.

Looking at (A5a) and (A5b) one can see that the term $\tilde{a}W\tilde{a}^{\dagger}$ in (A4) involves the following combination frequencies:

$$g(\sqrt{n+1}-\sqrt{n}-\sqrt{m+1}+\sqrt{m})\simeq\frac{g}{2\sqrt{n}}-\frac{g}{2\sqrt{m}}\ , \tag{A7a}$$

$$g(\sqrt{n+1}-\sqrt{n}+\sqrt{m+1}-\sqrt{m})\simeq\frac{g}{2\sqrt{n}}+\frac{g}{2\sqrt{m}}\ , \tag{A7b}$$

$$g(\sqrt{n+1}-\sqrt{n}+\sqrt{m+1}+\sqrt{m})\simeq 2g\sqrt{m}\ , \tag{A7c}$$

$$g(\sqrt{n+1}-\sqrt{n}-\sqrt{m+1}-\sqrt{m})\simeq -2g\sqrt{m}\ , \tag{A7d}$$

$$g(\sqrt{n+1}+\sqrt{n}-\sqrt{m+1}-\sqrt{m})\simeq 2g\sqrt{n}-2g\sqrt{m}\ , \tag{A7e}$$

$$g(\sqrt{n+1}+\sqrt{n}+\sqrt{m+1}+\sqrt{m})\simeq 2g\sqrt{m}+2g\sqrt{n}\ , \tag{A7f}$$

The forms on the right-hand side are for large m and n (m and n are summation indices). Some of these frequencies are obviously large, but others can be quite small if the average number of photons is large, and some, like (A7a), can be even smaller if $n\sim m$.

The original condition for the secular approximation, $\kappa \ll g$, allows one in principle to neglect *only* the terms having the combination frequencies (A7c), (A7d),and (A7f). In addition, if

$$\kappa \ll \frac{g}{2\sqrt{\bar{n}}} \tag{A8}$$

holds, one may neglect also (A7b) and, if $n\neq m$, (A7e). This follows from the fact that, for n close to m (for instance, $m=n+1$), (A7e) goes essentially as $g/\sqrt{n}$, for large n. [Clearly, (A7e) must be kept when $n=m$, regardless of the size of κ; this is indeed done in [9]]. The condition (A8) amounts to the requirement that the losses should not deplete the photon number significantly over a revival time.

To be able to neglect (A7a) for $n\neq m$ requires even more stringent conditions, because for n close to m the combination frequency can be extremely small. For instance, if $m=n+1$ one has

$$g(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n})\simeq-\frac{g}{4n^{3/2}} \tag{A9}$$

and hence one cannot neglect this term unless

$$\kappa \ll \frac{g}{4\bar{n}^{3/2}}\ . \tag{A10}$$

As an example, when $n_0=10$, this condition means $\kappa \ll 0.008g$. More precisely, when $n=10$ one has $g(\sqrt{12}-2\sqrt{11}+\sqrt{10})=-0.007g$. This suggests that Fig. 14 of [9], calculated for the case $\kappa=0.005g$, $n_0=10$, is probably inaccurate.

It is clear, then, that for all but the smallest values of κ and $\bar{n}$, i.e., the values satisfying (A10), one must keep at least the terms having combination frequencies like (A7a) in the equation for dW/dt. This means that the single sum in the second line of Eq. (2.8) of [9] should be replaced by the double sum

$$\sum_{n,m=1}^{\infty}(\sqrt{n+1}+\sqrt{n})(\sqrt{m+1}+\sqrt{m})[e^{-ig(\sqrt{n+1}-\sqrt{n}-\sqrt{m+1}+\sqrt{m})t}|\Psi_{n-1}^+\rangle\langle\Psi_n^+|W(t)|\Psi_m^+\rangle\langle\Psi_{m-1}^+|$$
$$+e^{ig(\sqrt{n+1}-\sqrt{n}+\sqrt{m+1}-\sqrt{m})t}|\Psi_{n-1}^-\rangle\langle\Psi_n^-|W(t)|\Psi_m^-\rangle\langle\Psi_{m-1}^-|]\ . \tag{A11}$$

As for the third line of Eq. (2.8) of [9], it is all right to leave it as a single sum as long as the condition (A8) holds; otherwise it too has to be replaced by a double sum, whose terms have combination frequencies of the form (A7e). These terms are, however, very small in the limit of large n, because they are multiplied by $(\sqrt{n+1}-\sqrt{n})^2$, so in this limit this correction can probably be ignored.

The replacement indicated by (A11) does not change the equation of motion for any matrix elements of W between states having the same index n, but it changes the equations for $\langle \Psi_n^+|W|\Psi_m^+\rangle$ and $\langle \Psi_n^-|W|\Psi_m^-\rangle$ for $n\neq m$, so that their time evolution can no longer be as simple as that shown in Eqs. (2.9) of [9]. As shown in [9], to calculate the population inversion only the matrix elements with $n=m$ are needed, which means that the formulas for the population inversion derived in [9] should be valid, provided only that (A8) holds. The terms with $n\neq m$, however, are necessary to calculate the dipole moment and the field amplitude expectation values, and they will only be correctly given by the formulas in [9] if the very restrictive condition (A10) holds. In particular, for the parameters used in Fig. 1 of the present paper, $\kappa=0.01g$, and $n_0=25$ implies $1/4n_0^{3/2}=0.002$, which clearly violates (A11), and as a result the predictions based on the equations in [9] are, as Fig. 3 shows, completely wrong.

In conclusion, the condition quoted in [9], that is, $\kappa \ll g$, is not enough to justify the approximations made there, which require the much stronger condition (A10) to hold. If the losses do not significantly deplete the photon number over a revival time, that is, if the less restrictive condition (A8) holds, then the formulas derived in [9] for the population inversion are accurate, but in order to handle the atomic dipole moment and the field amplitude expectation values the master equation (2.8) of [9] must be modified, by replacing the single sum on the second line by the double sum (A11) above. Finally, if (A8) does not hold further terms need to be added to Eq. (2.8) of [9] if it is to predict accurately even the population inversion, regardless of whether $\kappa \ll g$ or not. This is illustrated by the failure of this equation to predict the results shown in Fig. 5.

It is interesting to note that Agarwal and Puri prove in their paper the equivalence of their result for the population inversion [Eq. (3.3) of [9]] to the earlier result of Barnett and Knight [8]. Barnett and Knight do mention explicitly the restriction (A10) on the validity of their approximate result, and refer to it as the "underdamped case." As shown here, their result for the population inversion in fact holds under the less restrictive condition (A8); it is not easy to tell whether Barnett and Knight were already aware of this (see footnote 11 of [8]).

Finally, it should be noted that most of the figures in [9] were calculated for such small values of κ and n_0 that the restriction (A10) almost certainly holds for most of them; hence they may all be essentially correct, except, probably, for Fig. 14, as remarked above.

APPENDIX B: THE NUMERICAL CALCULATION

The master equation (4) has the nice property (recently exploited by Tran Quang, Knight, and Bužek; see [7] for details) that, at any degree k of off-diagonality, it leads to a closed system of $4N$ equations, where N is the largest photon number considered. For instance, the equation of motion for $(\rho_{aa})_{n,n+k}$ involves only $(\rho_{bb})_{n+1,n+k+1}$, $(\rho_{ab})_{n,n+k+1}$, and $(\rho_{ba})_{n+1,n+k}$; the losses couple to similar terms with different values of n but the same value of k.

Introducing the vectors $\mathbf{A}^k,\mathbf{B}^k,\mathbf{C}^k,\mathbf{D}^k$, of components

$$A_n^k=(\rho_{aa})_{n,n+k}\ , \tag{B1a}$$

$$B_n^k=(\rho_{bb})_{n,n+k}\ , \tag{B1b}$$

$$C_n^k=(\rho_{ab})_{n,n+k+1}\ , \tag{B1c}$$

$$D_n^k=(\rho_{ba})_{n,n+k-1}\ , \tag{B1d}$$

with $n=0,\ldots,N$, one can easily see that Eq. (4) leads to the system

$$\dot{A}_n^k=ig(\sqrt{n+k+1}\,C_n^k-\sqrt{n+1}\,D_{n+1}^k) +\kappa[2\sqrt{(n+1)(n+1+k)}\,A_{n+1}^k-(2n+k)A_n^k]\ , \tag{B2a}$$

$$\dot{B}_n^k=ig(\sqrt{n+k}\,D_n^k-\sqrt{n}\,C_{n-1}^k) +\kappa[2\sqrt{(n+1)(n+1+k)}B_{n+1}^k-(2n+k)B_n^k]\ , \tag{B2b}$$

$$\dot{C}_n^k=ig(\sqrt{n+k+1}\,A_n^k-\sqrt{n+1}B_{n+1}^k) +\kappa[2\sqrt{(n+1)(n+2+k)}C_{n+1}^k-(2n+k+1)C_n^k]\ , \tag{B2c}$$

$$\dot{D}_n^k=ig(\sqrt{n+k}\,B_n^k-\sqrt{n}\,A_{n-1}^k) +\kappa[2\sqrt{(n+1)(n+k)}D_{n+1}^k-(2n+k-1)D_n^k]\ , \tag{B2d}$$

where it is understood that if a subscript is negative, the corresponding variable is zero, and likewise, for the purpose of the numerical calculation, if a subscript exceeds N.

To calculate the expectation value of the population inversion $\langle\sigma_3\rangle$, one needs the matrix elements $(\rho_{aa})_{nn}$ and $(\rho_{bb})_{nn}$, i.e., the system (B2) needs to be integrated with $k=0$. To calculate the expectation value of the field amplitude $\langle a\rangle$ one needs $(\rho_{aa})_{n,n+1}$ and $(\rho_{bb})_{n,n+1}$, whereas for the atomic dipole moment $\langle\sigma\rangle$, one needs $(\rho_{ba})_{n,n}$ (or its complex conjugate); all these can be obtained from (B2) with $k=1$. Expectation values of other quantities may be obtained from (B2) with other values of k; for instance, to calculate the squeezing $k=2$ would be required.

It seems, therefore, that to calculate all of the quantities considered in this paper, namely, $\langle\sigma\rangle$, $\langle a\rangle$, and $\langle\sigma_3\rangle$, one needs to integrate two sets ($k=0$ and 1) of $4N$ complex, first-order differential equations. This is equivalent to $16N$ real equations, where N can be large. The system may be reduced considerably by noting that when $k=0$ one has

$$(C_n^0)^* = D_{n+1}^0 \ . \tag{B3}$$

This makes it possible to rewrite the system (B2), when $k=0$, in a form involving only the real quantities A_n^0, B_n^0, $C_n^0+D_{n+1}^0$, and $i(C_n^0-D_{n+1}^0)$, with real coefficients. Then one has only a total of $12N$ real equations. When $k=1$ (B2) may also be simplified somewhat by noting that the system of $4N$ real equations for $\mathrm{Re}A_n^1$, $\mathrm{Re}B_n^1$, $\mathrm{Im}C_n^1$, and $\mathrm{Im}D_n^1$ decouples from the system of $4N$ real equations for $\mathrm{Im}A_n^1$, $\mathrm{Im}B_n^1$, $\mathrm{Re}C_n^1$, and $\mathrm{Re}D_n^1$. In fact, if the phase of the initial coherent state field is chosen to be zero (so that it coefficients in the $|n\rangle$ basis are real) the latter system is not needed whenever the initial state for the atom is such that $\mathrm{Re}\langle\sigma\rangle=0$, since in that case $\mathrm{Im}A_n^1=\mathrm{Im}B_n^1=\mathrm{Re}C_n^1=\mathrm{Re}D_n^1=0$ for all times.

To truncate the system, N was chosen to be five standard deviations above the initial mean number of photons n_0. Typically this leads to errors of the order of a few times 10^{-6}. For $n_0=25$, this means $N=50$. The total number of real, first-order, differential equations is then $3\times4\times50=600$. The numerical integration takes only a few minutes using a fourth-order Runge-Kutta [28] on a NeXT computer.

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