Supersymmetry aspects of the Dirac equation in one dimension with a Lorentz scalar potential

Y. Nogami
Department of Physics and Astronomy, McMaster University, Hamilton, Ontario, Canada L8S 4M1

F. M. Toyama
Institute of Computer Sciences, Kyoto Sangyo University, Kyoto 603, Japan
(Received 17 September 1992)

The Dirac equation in one dimension with a Lorentz scalar potential is associated with a supersymmetric pair of Schrödinger Hamiltonians H_1 and H_2. The H_1 and H_2 share the same energy spectrum and scattering phases. The shared spectrum includes the lowest states unless the Dirac equation allows a zero mode (a zero-energy bound state). This situation is unlike the common examples of supersymmetric quantum mechanics. The Dirac equation admits a zero mode only if the scalar potential has certain "topology." Various such features are illustrated through explicit examples. In particular, the phase equivalence between H_1 and H_2 is exploited to construct transparent potentials for the Dirac equation in one dimension.

PACS number(s): 03.65.Ge, 11.10.Qr

I. INTRODUCTION

Cooper *et al.* [1] pointed out that the Dirac equation in one dimension with a Lorentz scalar potential exhibits supersymmetry (SUSY) and is associated with a SUSY pair of Schrödinger Hamiltonians H_1 and H_2. However, the SUSY Dirac equation has some interesting aspects which Copper *et al.* left to be uncovered. The H_1 and H_2 have identical energy spectra including the lowest levels unless the Dirac equation allows a zero mode (a normalizable solution for $E=0$). Many examples of nonrelativistic SUSY quantum mechanics have been discussed in the literature [2,3]. In almost all of those examples the SUSY pair of Schrödinger Hamiltonians have the same energy spectra except for the ground state of one of the pairs of Hamiltonians. In this sense the H_1 and H_2 associated with the Dirac equation are unusual. The Dirac equation admits a zero mode only if the scalar potential has certain "topology," namely, if it has different limits for $x\rightarrow\infty$ and $x\rightarrow-\infty$ [4]. The purpose of this paper is to examine such various aspects of the SUSY Dirac equation through explicit examples. In particular, the feature that H_1 and H_2 are "phase equivalent" is exploited to construct transparent (reflectionless) potentials for the one-dimensional Dirac equation.

In Sec. II we examine the SUSY structure of the one-dimensional Dirac equation with a Lorentz scalar potential. In Sec. III we illustrate various SUSY features which was set out in Sec. II. Discussions are given in Sec. IV.

II. SUPERSYMMETRIC STRUCTURE OF THE DIRAC EQUATION

Let us consider the Dirac equation in one space dimension,

$$H_D\psi(x)=E\psi(x)\ ,\quad H_D=\alpha p+\beta m+\beta S(x)\ , \qquad (2.1)$$

where $c=\hbar=1$, $p=-id/dx$, $m(>0)$ is the mass, and $S(x)$ is a Lorentz scalar. The ψ is a two-component spinor; $\psi=\binom{\psi_1}{\psi_2}$. The α and β are 2×2 Pauli matrices; we use $\alpha=\sigma_y$ and $\beta=\sigma_x$. Then Eq. (2.1) reads

$$(ip+m+S)\psi_1=E\psi_2\ , \qquad (2.2)$$

$$(-ip+m+S)\psi_2=E\psi_1\ . \qquad (2.3)$$

Equations (2.2) and (2.3) can be reduced to two uncoupled Schrödinger equations [1],

$$H_i\psi_i=\left[\frac{p^2}{2m}+U_i\right]\psi_i=\varepsilon\psi_i\ ,\quad \varepsilon=\frac{E^2-m^2}{2m}\ , \qquad (2.4)$$

where $i=1$ or 2, and

$$U_i(x)=\frac{1}{2m}\left[(m+S)^2-m^2\mp\frac{dS}{dx}\right]\ . \qquad (2.5)$$

The double sign in Eq. (2.5) is $-$ $(+)$ for $i=1$ (2). Note the distinction between E and ε; ε is the Schrödinger counterpart of E. The ε is common between H_1 and H_2. If we define operators $A^{\pm}$ by

$$A^{\pm}=\frac{1}{\sqrt{2m}}\left[\pm\frac{d}{dx}+m+S\right]\ , \qquad (2.6)$$

H_i can be expressed as

$$H_1=A^-A^+-\frac{m}{2}\ ,\quad H_2=A^+A^--\frac{m}{2}\ . \qquad (2.7)$$

Clearly H_1 and H_2 are a SUSY pair of Schrödinger Hamiltonians. The "superpotential" is $m+S$ and the "factorization energy" is $-m/2$ [3].

We are interested in constructing solvable examples of the SUSY Dirac equation. We do this by starting with a solvable model for one of the Schrödinger Hamiltonians, say, H_1 [5]. In this scheme a solution of the Dirac equation for $E=0$, which corresponds to $\varepsilon=-m/2$, plays a key role. Let this solution be $\phi=\binom{\phi_1}{\phi_2}$. It then follows from Eqs. (2.2) and (2.3) that

$$\phi_1\phi_2=\text{const} . \tag{2.8}$$

Hence ϕ_1 and ϕ_2 cannot be *both* normalizable. For satisfying Eqs. (2.2) and (2.3) for $E=0$, the constant of Eq. (2.8) can be anything, but it must be zero if the spinor ϕ is to be normalizable [4].

The ϕ_1 satisfies Eq. (2.4) for $i=1$ and $E=0$ ($\varepsilon=-m/2$), i.e.,

$$\left[-\frac{1}{2m}\frac{d^2}{dx^2}+U_1\right]\phi_1=-\frac{m}{2}\phi_1 . \tag{2.9}$$

We will assume U_1 such that Eq. (2.9) is solvable for ϕ_1. We also assume that the ground state for U_1 is above the factorization energy $-m/2$. Then we can choose ϕ_1 with no node; this is understood henceforth. For the assumed U_1, Eq. (2.5) with $i=1$ can be regarded as a differential equation for the unknown S. If we define S by

$$m+S=-\frac{1}{\phi_1}\frac{d\phi_1}{dx} , \tag{2.10}$$

then Eq. (2.5) is satisfied. Actually Eq. (2.10) is a special case of the general solution which we will discuss in Sec. IV [3,6]. For the S determined above, the partner potential U_2 can be determined by Eq. (2.5). Solution ϕ_2 of Eq. (2.4) for $i=2$ is related to ϕ_1 through Eq. (2.8) apart from a constant factor, $\phi_2=1/\phi_1$.

For the asymptotic behavior of ϕ_1 we consider three cases:

$$\phi_1\to 0 \quad \text{for } x\to\infty \quad \text{and } \phi_1\to\infty \quad \text{for } x\to-\infty , \tag{2.11}$$

$$\phi_1\to\infty \quad \text{for } x\to\infty \text{ and } x\to-\infty , \tag{2.12}$$

$$\phi_1\to 0 \quad \text{for } x\to\infty \text{ and } x\to-\infty . \tag{2.13}$$

We are not interested in the overall sign of ϕ_1 nor in any particular choice of the direction of the x axis. Therefore the above three cases cover essentially all possibilities.

Let us note some relevant features of the three cases. In the case of Eq. (2.11), neither ϕ_1 nor ϕ_2 $(=1/\phi_1)$ is normalizable. For the S defined by Eq. (2.10) with this ϕ_1, H_D has no zero mode. The H_1 and H_2 share the same energy spectrum. In the case of Eq. (2.12), ϕ_1 is not normalizable but ϕ_2 is. The H_1 and H_2 share the same energy spectrum except that H_2 has one extra, the lowest state of $\varepsilon=-m/2$. The H_D has a zero mode. The case of Eq. (2.13) is the same as that of Eq. (2.12) except that ϕ_1 and ϕ_2 are interchanged. It is sufficient to consider Eqs. (2.11) and (2.12). The spectra for H_1, H_2, and H_D of the two cases are schematically shown in Fig. 1. The asymptotic behavior of ϕ_1 is relevant to the asymptotic behavior of S. For the starting potential U_1 we assume either $U_1(\infty)=U_1(-\infty)=0$ or $U_1(\infty)/U_1(-\infty)=1$. Then we will see that

$$\frac{m+S(\infty)}{m+S(-\infty)}=\begin{cases}+1 , & \text{if } \phi_1 \text{ is like Eq. (2.11)} ,\\ -1 , & \text{if } \phi_1 \text{ is like Eq. (2.12)} .\end{cases} \tag{2.14}$$

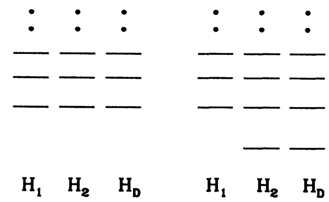

FIG. 1. Schematic energy spectra of H_1, H_2, and H_D are shown for the two situations which correspond to Eqs. (2.11) and (2.12), respectively, regarding the asymptotic behavior of function ϕ_1.

We refer to the type of S related to Eq. (2.11) as the nontopological type, and the one related to Eq. (2.12) the topological type.

If U_1 and U_2 are both localized, i.e., if they vanish as $x\to\pm\infty$, we can think of the transmission problem; a wave is incident, say, from the left and it is partially transmitted to the right and partially reflected to the left. We will be particularly interested in transparent potentials. A transparent potential is such that the transmission probability is unity for a wave of any shape. If the incient wave is a plane wave with a definite energy, the transparency has to hold for any value of the energy. It is known that the SUSY pair H_1 and H_2 are phase equivalent; that is, the resulting transmission coefficient (and also the reflection coefficient) is the same between H_1 and H_2 [7]. Therefore, if U_1 is a transparent potential, so is U_2. Then the associated S is a transparent potential for the Dirac equation.

III. ILLUSTRATIONS

We examine several examples and illustrate various features of the SUSY Dirac equation. We start with a solvable U_1 and determine S through Eq. (2.10) and then U_2 through Eq. (2.5). For U_1 we consider the Pöschl-Teller potential [8], the Kay-Moses potentials [9], and the harmonic-oscillator potential.

Example A. The Pöschl-Teller potential is defined by

$$U_1(x)=-\frac{\nu(\nu+1)\kappa^2}{2m}\text{sech}^2(\kappa x) , \tag{3.1}$$

where ν and κ are positive constants. The Schrödinger equation with this U_1 has bound states with eigenvalues

$$\varepsilon^{(n)}=-\frac{(\nu-n)^2\kappa^2}{2m} , \quad n(=0,1,2,\ldots)\le\nu . \tag{3.2}$$

If ν is an integer, U_1 is a transparent potential [8,9]. In this case $\varepsilon^{(n)}=0$ for $n=\nu$; this is a "half-bound" state [10,11].

In this example we consider the case of $\nu=1$. There is one bound state with

$$\psi_1^{(1)}(x)=(\kappa/2)^{1/2}\text{sech}(\kappa x) , \quad \varepsilon^{(1)}=-\frac{\kappa^2}{2m} . \tag{3.3}$$

Also U_1 is a transparent potential. Define the "scattering solution" $\chi(k,x)$ by

$$\chi(k,x)=\frac{ik-\kappa\tanh(\kappa x)}{ik+\kappa}e^{ikx}\,. \tag{3.4}$$

This $\chi(k,x)$ satisfies Eq. (2.4) with $\varepsilon=k^2/2m$. It represents a transmission process in which no reflection takes place. Apart from a constant factor, $\psi_1^{(1)}$ is given by $\chi(i\kappa,x)$. Similarly, solution ϕ_1 of Eq. (2.9) is given by

$$\phi_1(x)=\chi(im,x)=\frac{m+\kappa\tanh(\kappa x)}{m-\kappa}e^{-mx}\,, \tag{3.5}$$

which behaves like Eq. (2.11). This ϕ_1 and Eq. (2.10) lead to

$$S(x)=\frac{-2\kappa^2}{m+E_\kappa\cosh[2(\kappa x+\lambda)]}\,, \tag{3.6}$$

where $E_\kappa=(m^2-\kappa^2)^{1/2}$ and

$$e^{2\lambda}=\left[\frac{m+\kappa}{m-\kappa}\right]^{1/2}\,. \tag{3.7}$$

The U_2 turns out to be

$$U_2(x)=-\frac{\kappa^2}{m}\mathrm{sech}^2(\kappa x+2\lambda)\,, \tag{3.8}$$

which is the same as U_1 of Eq. (3.1) with $\nu=1$ except that U_2 is displaced by $2\lambda/\kappa$. Obviously U_1 and U_2 have the same energy spectrum and they are both transparent potentials. The S of Eq. (3.6) is a transparent potential for the Dirac equation [11,12]. Figure 2 shows U_1, U_2, and S; $\kappa=0.5m$. If we shift the origin such that $\kappa x\to\kappa x-\lambda$, S of Eq. (3.6) becomes an even function of x, and $U_1(x)=U_2(-x)$.

We can choose ϕ_1 of the type of Eq. (2.12),

$$\phi_1=\chi(im,x)+\gamma\chi(im,-x)\,, \tag{3.9}$$

where γ is an arbitrary positive constant. If we take $\gamma=1$, the above ϕ_1 is even with respect to x and $m+S$ is odd. The S of this case is given by

$$S(x)=-\frac{2E_\kappa\{E_\kappa+m\cosh[2(\kappa x-\lambda)]\}e^{mx}+2\kappa^2e^{-mx}}{\{m+E_\kappa\cosh[2(\kappa x-\lambda)]\}e^{mx}+\{m+E_\kappa\cosh[2(\kappa x+\lambda)]\}e^{-mx}}\,. \tag{3.10}$$

This S is topological; note that $m+S\to m$ as $x\to-\infty$ while $m+S\to-m$ as $x\to\infty$. Equation (2.5) with this S gives the partner potential U_2, which is even with respect to x. For $\gamma\neq1$, however, neither $m+S$ nor U_2 exhibits such symmetry.

Figure 3 compares U_1, U_2, and S of this case of $\gamma=1$ with $\kappa=0.5m$. The U_1 of Fig. 3 is the same as that of Fig. 2. Each of U_1 and U_2 has a bound state with $\varepsilon=-\kappa^2/2m$. In addition U_2 has a bound state with $\varepsilon=-m/2$ and wave function ϕ_2 $(=1/\phi_1)$. The Dirac equation with this S has two bound states which correspond to the two bound states of U_2. One of them is the zero mode ($E=0$, $\varepsilon=-m/2$), of which ϕ consists of ϕ_2 alone [4]. These U_1 and U_2 are both transparent potentials for the Schrödinger equation and the S is a transparent potential for the Dirac equation. In Ref. [12] a complete solution to the problem of constructing a transparent potential of the Lorentz scalar type for the one-dimensional Dirac equation was given. It was understood in Ref. [12], however, that S was a localized potential. Hence S of the topological type was not included in Ref. [12].

Example B. The $\nu=2$ case of Eq. (3.1). There are two eigenstates of H_1 with eigenvalues $\varepsilon^{(1)}=-\kappa^2/2m$ and $\varepsilon^{(2)}=-(2\kappa)^2/2m$, where the superscript refers to n of Eq. (3.2). The corresponding wave functions are

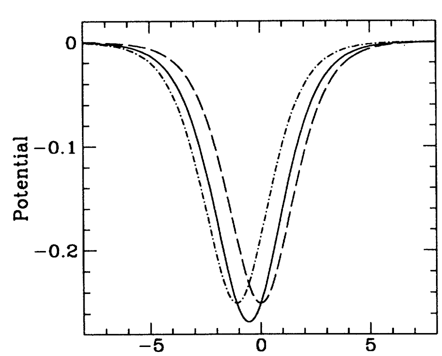

FIG. 2. Example A: Potentials U_1 (dashed curve), U_2 (dash-dotted curve), and S (solid curve) in units of $1/m$. The S is nontopological. $\kappa=0.5m$.

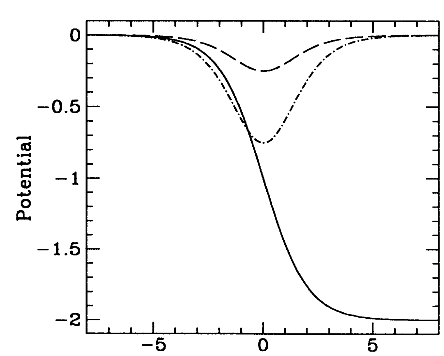

FIG. 3. Example A: Potentials U_1 (dashed curve), U_2 (dash-dotted curve), and S (solid curve) in units of $1/m$. The S is topological. $\kappa=0.5m$ and $\gamma=1$.

$$\psi_1^{(1)} = (3\kappa/2)^{1/2}\tanh(\kappa x)\operatorname{sech}(\kappa x) \ , \tag{3.11}$$

$$\psi_1^{(2)} = \frac{\sqrt{3\kappa}}{2}\operatorname{sech}^2(\kappa x) \ . \tag{3.12}$$

For the scattering solution we take

$$\chi(k,x) = \frac{k^2+\kappa^2+3ik\kappa\tanh(\kappa x)-3\kappa^2\tanh^2(\kappa x)}{(ik+\kappa)(ik+2\kappa)} e^{ikx} \ . \tag{3.13}$$

Apart from a constant factor this $\chi(k,x)$ is related to the wave functions of the bound states by $\chi(in\kappa,x)=\psi_1^{(n)}$. If we define $\phi_1=\chi(im,x)$, ϕ_1 conforms to Eq. (2.11). The rest goes in the same way as in example A. The U_1, U_2 and S of this case are displayed in Fig. 4; $\kappa=0.25m$. The U_1 is symmetric, i.e., an even function of x. But, close scrutiny reveals that neither U_2 nor S has symmetry (with respect to any value of x). Solution ϕ_1 of the type of Eq. (2.12) can be constructed in the same way as Eq. (3.9). The resulting S is topological, but we do not show the results of this case because they are qualitatively similar to those of Fig. 3.

Example C. The case of nonintegral ν of Eq. (3.1). The scattering solution can be chosen as [8]

$$\chi(k,x) = \operatorname{sech}^{-ik/\kappa}(\kappa x)\,{}_2F_1(a,b,c;\tfrac{1}{2}[1+\tanh(\kappa x)]) \ , \tag{3.14}$$

where

$$a=\nu+c \ , \quad b=-\nu-1+c \ , \quad c=1-\frac{ik}{\kappa} \ . \tag{3.15}$$

The wave function for the bound state with $\varepsilon=-\kappa'^2/2m$ is given by $\psi_1(\kappa',x)=\chi(i\kappa',x)$. The rest goes again in the same way as in Example A. The U_1 and U_2 for nonintegral ν are not transparent.

Example D. The Kay-Moses potentials. For the Schrödinger equation, Kay and Moses gave a complete solution for the problem of constructing a transparent potential [9]. There is an infinite family of the Kay-Moses potentials including examples A and B. A Kay-Moses potential can be constructed as follows. Choose an integer N (any of 1,2, . . .), and then $2N$ positive constants κ_i and A_i $(i=1,2,\ldots,N)$. The Kay-Moses potential for this parameter set is

$$V(x) = -2\frac{d^2}{dx^2}[\ln\{\det[I+\hat{A}(x)]\}] \ , \tag{3.16}$$

where I and $\hat{A}(x)$ are $N\times N$ matrices with matrix elements

$$I_{ij}=\delta_{ij} \ , \quad \hat{A}_{ij}(x)=(A_iA_j)^{1/2}\frac{e^{(\kappa_i+\kappa_j)x}}{\kappa_i+\kappa_j} \ . \tag{3.17}$$

The V has N bound states with energies $\varepsilon^{(i)}=-\kappa_i^2/2m$, $i=1,2,\ldots$. For an arbitrary choice of the A_i's, V has no symmetry. If we require that V be symmetric, V is uniquely determined (for specified values of κ_i's). The A_i's in this case are such that [13]

$$\frac{A_i}{2\kappa_i} = \prod_{j\,(\neq i)}\frac{\kappa_i+\kappa_j}{|\kappa_i-\kappa_j|} \ . \tag{3.18}$$

The U_1 assumed in examples A and B can be identified with V of Eq. (3.16) with the following parameters: $\kappa_1=\kappa$, $A_1/(2\kappa)=1$ for example A, and $\kappa_1=\kappa$, $\kappa_2=2\kappa$, $A_1/2\kappa_1=A_2/2\kappa_2=3$ for example B. For the U_2 of example A, U_2 of Eq. (3.8) corresponds to the parameter set at $\kappa_1=\kappa$ and $A_1/2\kappa=e^{2\lambda}$. On the other hand, the U_2 associated with a topological S of Eq. (3.10) bears four parameters: $\kappa_1=\kappa$, $\kappa_2=m$, A_1, and A_2. The A_1 and A_2 depend on γ of Eq. (3.9). If $\gamma=1$, the U_2 is symmetric and Eq. (3.18) applies: $A_1/(2\kappa)=A_2/(2m)=(m+\kappa)/(m-\kappa)$.

For the scattering solution we can take [14]

$$\chi(k,x) = \left[1+\sum_{i=1}^{N}\frac{(A_i)^{1/2}g_i(x)e^{\kappa_i x}}{ik+\kappa_i}\right]e^{ikx} \ , \tag{3.19}$$

where $g_i(x)$'s are defined by the N linear algebraic equations

$$\sum_{j=1}^{N}[\delta_{ij}+\hat{A}_{ij}(x)]g_j(x)=0 \ . \tag{3.20}$$

The $\chi(im,x)$ derived from the $\chi(k,x)$ given above con-

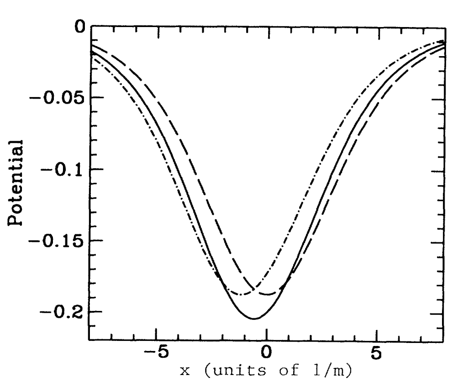

FIG. 4. Example B: Potentials U_1 (dashed curve), U_2 (dash-dotted curve), and S (solid curve) in units of $1/m$. The S is nontopological. $\kappa=0.25m$.

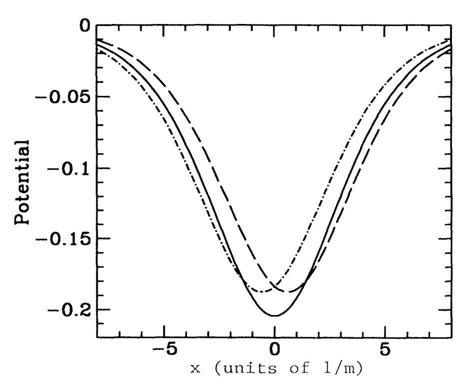

FIG. 5. Example D: The symmetrized version of Fig. 4.

forms to Eq. (2.11). With this $\chi(k,x)$ we can proceed in the same way as in examples A and B.

If we start with a symmetric U_1, the ensuing nontopological S and U_2 have no symmetry in general as we saw in example B. It would be interesting to find a U_1 which leads to a symmetric S. In that case U_2 will be related to U_1 by $U_1(x)=U_2(-x)$. This can be done as follows. Choose the A_i's according to [12,15]

$$\frac{A_i}{2\kappa_i}=\left[\frac{m-\kappa_i}{m+\kappa_i}\right]^{1/2}\prod_{j\,(\neq i)}\frac{\kappa_i+\kappa_j}{|\kappa_i-\kappa_j|}\ . \tag{3.21}$$

Then, with $\phi_1=\chi(im,\chi)$, where χ is that of Eq. (3.19), we can proceed in the same way as before. Figure 5 shows the symmetrized version of example B. If we opt for a topological S, the choice of $\gamma=1$ leads to an odd $m+S$ and an even U_2.

Example E. The harmonic oscillator. We start with

$$U_1=\tfrac{1}{2}m\omega^2x^2\ . \tag{3.22}$$

The eigenvalues are $\varepsilon_n=(n+\frac{1}{2})\omega$. In place of $\chi(k,x)$ we now use the parabolic cylindrical function

$$D_\nu(z)=2^{\nu/2}e^{-z^2/4}\left[\frac{\Gamma(\frac{1}{2})}{\Gamma[(1-\nu)/2]}\,{}_1F_1\left[-\frac{\nu}{2};\frac{1}{2};\frac{z^2}{2}\right]+\frac{z}{\sqrt{2}}\frac{\Gamma(-\frac{1}{2})}{\Gamma(-\nu/2)}\,{}_1F_1\left[\frac{1-\nu}{2};\frac{3}{2};\frac{z^2}{2}\right]\right]\ , \tag{3.23}$$

where $z^2=2m\omega x^2$ [16]. The $D_\nu(z)$ consists of two terms as seen in Eq. (3.23). Each of them satisfies Eq. (2.4) with U_1 of Eq. (3.22) and with $\varepsilon_\nu=(\nu+\frac{1}{2})\omega$. Each of them, however, diverges as $x\to\pm\infty$. In $D_\nu(z)$ the two terms are combined such that $D_\nu(z)\to 0$ as $x\to\infty$. Unless ν is a nonnegative integer, however, $D_\nu(z)$ diverges as $x\to-\infty$; it behaves like Eq. (2.11). We take ϕ_1 to be

$$\phi_1(x)=D_\nu(z)\ ,\quad \nu=-\frac{1}{2}\left[1+\frac{m}{\omega}\right]\ , \tag{3.24}$$

which leads to a nontopological S. Figure 6 shows U_1, U_2, and S of this example with $\omega=\sqrt{0.1}m$.

The case of topological S can be done by means of

$$\phi_1(x)=D_\nu(z)+\gamma D_\nu(-z)\ , \tag{3.25}$$

where γ is an arbitrary positive constant. If $\gamma=1$, ϕ_1 is even with respect to x, $m+S$ odd, and U_2 even. The results for the choice of $\gamma=1$ are shown in Fig. 7. In Appendix B we examine a modified version of example E for which ϕ_1, etc., become very simple.

The nontopological S obtained above (Fig. 6) is not symmetric. It is possible to construct a symmetric version of the model in a manner similar to example D. In example D we chose the same κ_i's as those of example B. This time we choose κ_i's which correspond to the ε_n, $n=1,2,\ldots$. In practice we can accommodate only a finite number of energy levels, but this number can be increased arbitrarily. Such calculations, under the name of the "inverse problem," have been done in nonrelativistic quantum mechanics [13,17]. There are two points to be remembered in the relativistic case. (i) For A_i's we have to use Eq. (3.21) rather than Eq. (3.18). (ii) Since ε_n is positive, the corresponding relativistic energy E_n is greater than m. In order for the Dirac equation to have a bound state with such an energy, we have to assume a rest mass greater than m. Note that when the potential is the confining type such as the harmonic-oscillator potential, the rest mass loses its meaning in any case.

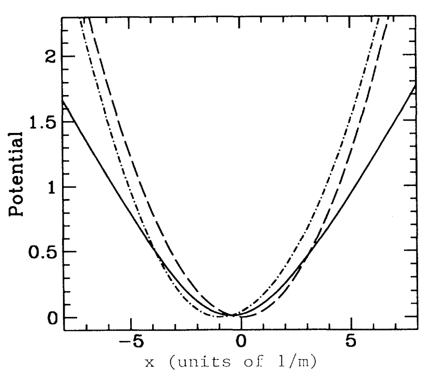

FIG. 6. Example E: Potentials U_1 (dashed curve), U_2 (dash-dotted curve), and S (solid curve) in units of $1/m$. The S is nontopological. $\omega=\sqrt{0.1}m$.

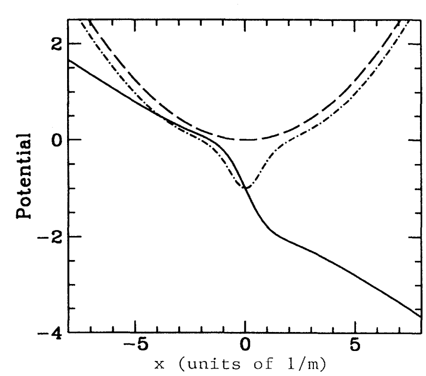

FIG. 7. Example E: Potentials U_1 (dashed curve), U_2 (dash-dotted curve), and S (solid curve) in units of $1/m$. The S is topological. $\omega=\sqrt{0.1}m$ and $\gamma=1$.

IV. DISCUSSION

We have examined various SUSY features of the Dirac equation in one dimension with a Lorentz scalar potential S. We have shown that, starting with a potential U_1 for the Schrödinger equation, an S that leads to the same energy spectrum and transmission coefficient can be constructed. In particular, a transparent potential for the Dirac equation can be constructed by starting with a transparent potential for the Schrödinger equation. Below Eq. (2.10) we mentioned that Eq. (2.10) is a special case of a general relation between S and ϕ_1. The general formula is

$$m+S=-\frac{1}{\phi_1}\frac{d\phi_1}{dx}-\frac{1/\phi_1^2}{c+\int^x dy/\phi_1^2(y)} , \tag{4.1}$$

where c is an arbitrary constant. This generalization yields a further variety of phase equivalent potentials as it was the case for the Schrödinger equation [3,6].

The method of constructing a transparent potential that we have presented in this paper is complementary to the other method that we recently developed [11,12]. The other method is based on the relationship between transparent potentials and solutions of a class of nonlinear Dirac equations [18]. In Refs. [11,12] it was understood that the scalar potential S is a localized one. Hence S of the topological type we discussed in this paper was not included in Refs. [11,12]. One might suspect that a transparent potential of the topological type is also related to solutions of nonlinear Dirac equations in the same manner as observed in Refs. [11,12]. We found that this is not the case.

We have confined ourselves to the one-dimensional case. Can the results be extended to the two- and three-dimensional cases? This does not seem to be possible. When the potential is a central one, the Dirac Hamiltonian in two or three dimensions can be reduced to the form of

$$H_D=\alpha_r p_r+i\alpha_r\beta k/r+\beta(m+S) , \tag{4.2}$$

where k is a constant related to the angular momentum, p_r is the radial part of $\mathbf{p}$, and $\alpha_r=\boldsymbol{\alpha}\cdot\mathbf{r}$ [19]. This H_D is similar to the H_D of Eq. (2.1) in the sense that α_r and β can be chosen as 2×2 Pauli matrices. However, the term with $\alpha_r\beta$ makes it impossible to reduce the Dirac equation to a Schrödinger equation like Eq. (2.4) with an *energy-independent* potential.

ACKNOWLEDGMENT

This work was supported by the Natural Sciences and Engineering Research Council of Canada.

APPENDIX A

Let us show how the Dirac equation can be "derived" from the Schrödinger equation. Start with Eq. (2.4) with subscript $i=1$. Assume that there exists a function S which is related to the given potential U_1 through Eq. (2.5). Then Eq. (2.4) can be written as

$$\psi_1''-(2mS+S^2-S')\psi_1+(E^2-m^2)\psi_1=0 , \tag{A1}$$

where $S'=dS/dx$, etc. By adding and subtracting $(m+S)\psi_1'$ to its left-hand side, Eq. (A1) can be rewritten as

$$[\psi_1'+(m+S)\psi_1]'-(m+S)\psi_1'+[E^2-(m+S)^2]\psi_1=0 . \tag{A2}$$

Then if we *define* ψ_2 by

$$\psi_1'+(m+S)\psi_1=E\psi_2 , \tag{A3}$$

Eq. (A2) becomes

$$-\psi_2'+(m+S)\psi_2=E\psi_1 . \tag{A4}$$

Equations (A3) and (A4) are nothing but Eqs. (2.2) and (2.3). The assumed existence of S can be justified by Eqs. (2.9) and (2.10).

The "derivation" given above may give the impression that the Schrödinger equation and the Dirac equation (with a scalar potential) are completely equivalent. This is not quite right, however. If the Dirac wave function for a bound state is normalized by $\int_{-\infty}^{\infty}\psi^\dagger\psi\,dx=\int_{-\infty}^{\infty}(|\psi_1|^2+|\psi_2|^2)dx=1$, the "Schrödinger wave function" ψ_1 is not normalized by itself. The *combination* of the two Schrödinger Hamiltonians H_1 and H_2 is equivalent to the Dirac Hamiltonian H_D.

APPENDIX B

Let us examine the following somewhat artificial modification of U_1 of Eq. (3.22),

$$U_1=\tfrac{1}{2}m\omega^2x^2-\tfrac{1}{2}(m-\omega) . \tag{B1}$$

The eigenvalues of H_1 are $\varepsilon_n=(n+1)\omega-m/2$; $n=0,1,2,\ldots$. If we write ε as $\varepsilon_\nu=(\nu+1)\omega-m/2$, the factorization energy is $\varepsilon_{-1}=-m/2$. For $\nu=-1$, $D_\nu(z)$ of Eq. (3.23) becomes simple and we find

$$\phi_1(x)=D_{-1}(z)$$
$$=\sqrt{\pi/2}e^{m\omega x^2/2}\mathrm{erfc}(\sqrt{m\omega}x) . \tag{B2}$$

Recall that $\mathrm{erfc}(y)\to2$ for $y\to-\infty$ and $\sqrt{\pi}\mathrm{erfc}(y)\sim e^{-y^2}/y$ for $y\gg1$. This ϕ_1 is of the type of Eq. (2.11). The S is determined by

$$m+S(x)=-\left[\sqrt{m\omega}x-\frac{2}{\sqrt{\pi}}\frac{e^{-m\omega x^2}}{\mathrm{erfc}(\sqrt{m\omega}x)}\right]\sqrt{m\omega} , \tag{B3}$$

which asymptotically behaves like

$$m+S(x)\to\pm m\omega x , \quad \text{for } x\to\pm\infty . \tag{B4}$$

Note that $S(0)=[-1+(2/\sqrt{\pi})]m\simeq0.128m$. The U_2 is determined through Eq. (2.5).

A topological S can be obtained through Eq. (3.25). If we take $\gamma=1$ for simplicity, we find

$$\phi_1(x)=\sqrt{2\pi}e^{m\omega x^2/2} , \tag{B5}$$

which behaves like Eq. (2.12) and leads to the following:

$$m+S(x)=-m\omega x , \tag{B6}$$

$$U_2=\tfrac{1}{2}m\omega^2x^2-\tfrac{1}{2}(m+\omega) . \tag{B7}$$

[1] F. Cooper, A Khare, R. Musto, and A. Wipf, Ann. Phys. (N.Y.) **187**, 1 (1987).

[2] A. A. Andrianov, N. V. Borisov, and M. V. Ioffe, Phys. Lett. **105A**, 19 (1984).

[3] C. V. Sukumar, J. Phys. A **18**, 2917 (1985).

[4] For the zero mode of the Dirac equation, see R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976); and also F. A. B. Coutinho, Y. Nogami, and F. M. Toyama, Am. J. Phys. **56**, 904 (1988). The Dirac matrices in the present paper are the same as those of Jackiw and Rebbi, while Coutinho *et al.* used $\alpha=\sigma_y$ and $\beta=\sigma_z$. If S is a localized potential, i.e., $S\rightarrow 0$ for $x\rightarrow\pm\infty$, no zero mode is allowed.

[5] In a sense the Dirac equation can be "derived" from the Schrödinger equation as shown in Appendix A.

[6] M. M. Nieto, Phys. Lett. **145B**, 208 (1984).

[7] C. V. Sukumar, J. Phys. A **19**, 2297 (1986).

[8] S. Flügge, *Practical Quantum Mechanics* (Springer-Verlag, Berlin, 1974), pp. 94–100; P. G. Drazin and R. S. Johnson, *Solitons: An Introduction* (Cambridge University Press, Cambridge, 1989), pp. 45–48.

[9] I. Kay and H. E. Moses, J. Appl. Phys. **27**, 1503 (1956).

[10] The existence of the half-bound state is characteristic of a transparent potential. P. Senn, Am. J. Phys. **56**, 916 (1988).

[11] Y. Nogami and F. M. Toyama, Phys. Rev. A **45**, 5258 (1992).

[12] F. M. Toyama, Y. Nogami, and Z. Zhao, Phys. Rev. A **47**, 897 (1993). The ψ_+ and ψ_- of this reference and of Ref. [11], respectively, correspond to ψ_2 and ψ_1 of the present paper. Similarly, U_+ and U_- correspond to U_2 and U_1, respectively.

[13] P. Asthana and A. N. Kamal, Z. Phys. C **19**, 37 (1983).

[14] This $\chi(k,x)$ is equal to $u(x,\sqrt{E})$ of Eq. (2.1) of Ref. [9].

[15] The A_i of Eq. (3.21) corresponds to A_{i-} of Ref. [12].

[16] E. Merzbacher, *Quantum Mechanics,* 2nd ed. (Wiley, New York, 1961), Chap. 5.

[17] J. F. Schonfeld, W. Kwong, J. L. Rosener, C. Quigg, and H. B. Thacker, Ann. Phys. (N.Y.) **128**, 1 (1980); W. Kwong and J. L. Rosener, Prog. Theor. Phys. Suppl. **86**, 366 (1986).

[18] For the nonrelativistic counterpart of the method developed in Refs. [11,12], see Y. Nogami and C. S. Warke, Phys. Lett. **59A**, 251 (1976); Phys. Rev. C **17**, 1905 (1978).

[19] See, e.g., L. I. Schiff, *Quantum Mechanics,* 3rd ed. (McGraw-Hill, New York, 1968), pp. 483–484.