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Role of pumping statistics and dynamics of atomic polarization in quantum fluctuations of laser sources

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We solve the problem of a laser with variable pumping statistics for any relative magnitude of the atomic- and cavity-decay constants, and obtain a different regime of sub-Poissonian light generation. We show that, even for Poissonian pumping, the noise in the amplitude quadrature outside the cavity can be reduced up to 50% below the shot-noise level when the polarization but not the populations can be adiabatically eliminated. Maximum noise reduction in this case is obtained when the lower level decays much faster than the upper one and occurs at a frequency given by the geometrical mean of the decay rates of the field and the lower-level population. Furthermore, the full consideration of atomic memory effects leads to a generalization of previous results on regularly pumped lasers. We find that, for regular pumping, maximum amplitude-noise reduction (up to complete quieting) still occurs at zero frequency in all cases. However, a long-living polarization leads to an increase in amplitude noise and may even eliminate the dip at zero frequency, at the same time leading to a significant quenching of the laser linewidth in the bad-cavity limit.

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I. INTRODUCTION

Quenching of quantum noise in lasers has been an active field of theoretical and experimental research [1–6], motivated by the intrinsic interest of these devices, from the point of view of fundamental physics, and also by multiple possible applications. As compared to noise quenching in passive devices [7–11], it has the potential advantage of leading to intense sources of squeezed light.

Both squeezing of photon number [1–3] and phase fluctuations [4–6] have been considered in the literature, leading to sub-Poissonian and subnatural linewidth lasers, respectively.

Most of the theoretical studies of the nonlinear dynamic behavior of single-mode homogeneously broadened lasers deal with simplified models which are characterized by three dynamic variables, namely, field amplitude, atomic polarization, and population inversion [12–14]. The dynamic evolution of such models is governed by three relaxation rates: $\gamma_\parallel$ for the population inversion, $\gamma_\perp$ for the polarization, and κ for the field intensity in the cavity (frequently one has two different population decay rates, for the upper and the lower lasing levels). Corresponding to the different possible relations between these parameters, single-mode lasers are grouped into four main classes with distinguishable dynamic characteristics [12]:

(1) $\gamma_\perp, \gamma_\parallel \gg \kappa$—dye lasers, for example.

(2) $\gamma_\perp \gg \kappa \sim \gamma_\parallel$—helium-neon (0.6 and 1.15 μm), argon-ion.

(3) $\gamma_\perp \gg \kappa \gg \gamma_\parallel$—ruby, Nd: YAG (yttrium aluminum garnet), carbon dioxide, and semiconductor.

(4) $\kappa \gg \gamma_\perp, \gamma_\parallel$—near-infrared noble-gas lasers and many far-infrared gas lasers.

Of course, it would be highly desirable to have a theory of the quantum fluctuations of the radiation field which would accommodate these four classes of lasers. However, in spite of the fact that earlier work on laser theory has fully taken into account the role of polarization dynamics [17, 18], most of the recent work on noise quenching assumes that the transverse decay time is by far the largest one, so that the polarization just follows the other dynamic variables adiabatically. One knows, on the other hand, that in some systems a careful consideration of the polarization dynamics is essential when considering the fluctuation spectrum of the produced light. Thus, in multiwave mixing in two-level atoms, the Stark splitting of the atom enhances the nonlinear coupling between the fields at the Rabi frequencies [8]. In absorptive optical bistability, it was shown by Carmichael [9], in a treatment which does not eliminate adiabatically the polarization, that a low-Q cavity may produce better squeezing than the high-Q cavity. The general case, also without adiabatic elimination of the polarization, was studied by several authors [10]. They showed that the high-Q cavity was less favorable for inducing squeezing. The importance of taking into account the polarization dynamics [11] is also stressed by the experimental demonstration of squeezing enhancement in the regime of vacuum-field Rabi splitting in dispersive optical bistability. This effect occurs at low intensities and for comparable atomic- and cavity-decay rates [9].

In view of these remarks, we consider, in the following, a quite general approach, which fully includes the effects of polarization *and* population dynamics in the theory of laser fluctuations. It differs from earlier work on laser theory [17, 18] in that we allow for the possibility of variable pumping statistics [1, 2], and make a detailed investigation of the spectrum of the field outside the cavity. The method of solution is also quite different, allowing us to get to the final results without any approximation, in a completely analytical way, thus making possible an easy physical interpretation of the results. We resort to numerical methods only at the very end, in order to get the curves describing the spectrum of fluctuations of the outgoing laser field. Although limited in this paper to the on-resonance homogeneously broadened case, our method can be easily generalized to include dispersive effects and inhomogeneous broadening, as it will be shown elsewhere.

The main results with respect to previous treatments are (i) a reduction of noise—up to 50%—in the spectrum of amplitude fluctuations for the field outside the cavity, *for Poissonian pumping,* and lasers of the third class, around a frequency given by the geometrical mean of the decay rates of the field and the lower-level population, when the decay of this population is much faster than that of the upper level; (ii) a generalization, with the inclusion of population and polarization dynamics, of previous work on regularly pumped lasers [1, 2]: we find that in all cases maximum amplitude noise reduction (up to complete quieting) still occurs at zero frequency, but a long-lived polarization may result in a pronounced *increase* of amplitude noise for the outgoing field, for lasers of the fourth class (bad-cavity lasers), around the frequency of relaxation oscillations of the active atoms in the cavity—this noise increase may even eliminate the zero-frequency dip resulting from the regularization of the pumping. We show that, in the same regime in which this increase in amplitude noise takes place, phase diffusion gets slowed down, due to atomic memory effects, resulting in a significant *reduction* of the laser linewidth. The complementarity between phase and amplitude noise quenching becomes thus quite apparent.

We start our investigation by writing down, in the following section, the Langevin equations of motion for the laser, taking into account the pumping statistics. Then, in Sec. III, we derive the corresponding classical equations of motion, using a normal-ordering representation for field and atomic operators. These classical equations will allow us to obtain the steady-state solutions, as well as the spectra of the field-quadrature fluctuations. In Sec. IV, we discuss the non-Markovian behavior of the phase, generalizing the results obtained in Refs. [5, 6] (which applied only to the good-cavity case), and calculate the laser line shape, which suffers an important narrowing in the bad cavity limit. In Sec. V, we calculate the spectrum of fluctuations of the field outside the cavity, and analyze the effects of population and polarization dynamics. In particular, we show that even a Poissonian-pumped laser can emit sub-Poissonian light. In Sec. VI, we summarize our conclusions.

II. QUANTUM LANGEVIN EQUATIONS

We consider a system of homogeneously broadened two-level atoms with transition frequency ω_{ab}, assuming that the lower level is not the ground state. The atoms fill a resonant ring cavity of length L and volume V with intensity transmission coefficient of the coupling mirror T. We assume the finesse of the cavity to be sufficiently high so as to justify a mean field approximation (spatial variations neglected). The atoms interact with the radiation field of a single excited mode of the cavity, which we consider approximately as a plane traveling wave with frequency ω_c.

To describe the influence of the pumping statistics of the active atoms on the quantum fluctuations of the laser radiation field, we follow the model described in Ref. [2], in which the random excitation of the atoms in the laser cavity is mimicked by a random injection of excited atoms in the same cavity. We restrict ourselves here to a time-independent probability distribution of the injection (or excitation) times, corresponding to a time-independent average pumping rate. As shown in [2], in this case the influence of the pumping statistics can be characterized by a single statistical parameter $p \leq 1$. The two extreme cases of Poissonian and regular statistics correspond respectively to $p = 0$ and $p = 1$.

We assume that the jth atom gets excited from some lower nonresonant states at the instant t_j, and starts then to interact with the one-mode radiation field in the cavity. This interaction is described by the following Langevin equations:

$$\dot{a}(t) = -(i\omega_c + \kappa/2)a(t) - ig\sum_j \Theta(t-t_j)\sigma^j(t) + F_\gamma(t), \tag{2.1}$$

$$\begin{aligned}\dot{\sigma}^j(t) = &-(i\omega_{ab} + \gamma_{ab})\sigma^j(t)\\ &+ig\Theta(t-t_j)[\sigma_a^j(t) - \sigma_b^j(t)]a(t) + f_\sigma^j(t),\end{aligned} \tag{2.2}$$

$$\begin{aligned}\dot{\sigma}_a^j(t) = &-(\gamma_a + \gamma_a')\sigma_a^j(t)\\ &+ig\Theta(t-t_j)[a^\dagger(t)\sigma^j(t) - \sigma^{\dagger j}(t)a(t)] + f_a^j(t),\end{aligned} \tag{2.3}$$

$$\begin{aligned}\dot{\sigma}_b^j(t) = &-\gamma_b\sigma_b^j(t) + \gamma_a'\sigma_a^j(t)\\ &-ig\Theta(t-t_j)[a^\dagger(t)\sigma^j(t) - \sigma^{\dagger j}(t)a(t)] + f_b^j(t),\end{aligned} \tag{2.4}$$

where $\sigma_a^j(t) = (|\, a\rangle\langle a\,|)^j$ and $\sigma_b^j(t) = (|\, b\rangle\langle b\,|)^j$ are the projection operators for the upper and lower states of the jth atom, $\sigma^j(t) = (|\, b\rangle\langle a\,|)^j$ is the spin-flip operator, representing the complex polarization of the jth atom, $\Theta(t)$ is the step function, κ is the cavity damping constant, given by (in the limit of negligible internal losses)

$$\kappa = cT/L, \tag{2.5}$$

γ_a and γ_b are the decay rates of the populations of the upper and lower levels to the other atomic levels; γ_a' is the spontaneous decay rate between the lasing levels, and γ_{ab} is the decay rate of the atomic polarization, which obeys the inequality $2\gamma_{ab} \geq \gamma_a + \gamma_a' + \gamma_b$. The coupling constant g is given by

$$g = \sqrt{\frac{1}{2\hbar\epsilon_0\omega_c V}}\omega_{ab}\mu, \tag{2.6}$$

where μ is the magnitude of the atomic dipole moment. The photon annihilation operator $a(t)$ of the laser field is normalized so that the mean value $\langle a^\dagger(t)a(t)\rangle$ gives the mean photon number inside the laser cavity.

The Langevin noise operators are fully defined by their first- and second-order moments. The calculation of these moments has been done by many authors [18, 16]. The normally ordered correlation functions of the field Langevin forces $F_\gamma(t)$ stemming from the interaction with a heat bath read as

$$\langle F_\gamma(t)\rangle = 0, \tag{2.7}$$

$$\langle F_\gamma^\dagger(t)F_\gamma(t')\rangle = \kappa\,\overline{n}_T\delta(t-t'), \tag{2.8}$$

$$\langle F_\gamma(t)F_\gamma(t')\rangle = 0, \tag{2.9}$$

where $\overline{n}_T$ is the average number of thermal photons in the laser cavity. In this paper we assume for simplicity that the heat reservoir is at zero temperature, $T = 0$, and the average number of thermal photons in the cavity is zero. The generalization of our results to nonzero temperatures is straightforward.

The correlation functions of the atomic noise operators in (2.2)–(2.4) can be found, for instance, in [15,16]. In case of radiative decay from atomic levels $\mid a\rangle$ and $\mid b\rangle$ to some ground state, the nonvanishing correlation functions of the atomic Langevin noise operators are given by

$$\langle f_a^j(t)f_a^j(t')\rangle = (\gamma_a+\gamma_a')\langle\sigma_a^j(t)\rangle\delta(t-t'), \tag{2.10}$$

$$\langle f_b^j(t)f_b^j(t')\rangle = [\gamma_b\langle\sigma_b^j(t)\rangle + \gamma_a'\langle\sigma_a^j(t)\rangle]\delta(t-t'), \tag{2.11}$$

$$\langle f_a^j(t)f_b^j(t')\rangle = -\gamma_a'\langle\sigma_a^j(t)\rangle\delta(t-t'), \tag{2.12}$$

$$\langle f_\sigma^{\dagger j}(t)f_\sigma^j(t')\rangle = (2\gamma_{ab}-\gamma_a-\gamma_a')\langle\sigma_a^j(t)\rangle\delta(t-t'), \tag{2.13}$$

$$\langle f_\sigma^{\dagger j}(t)f_b^j(t')\rangle = \gamma_b\langle\sigma^{\dagger j}(t)\rangle\delta(t-t'), \tag{2.14}$$

$$\langle f_\sigma^j(t)f_a^j(t')\rangle = (\gamma_a+\gamma_a')\langle\sigma^j(t)\rangle\delta(t-t'), \tag{2.15}$$

$$\langle f_\sigma^j(t)f_b^j(t')\rangle = -\gamma_a'\langle\sigma^j(t)\rangle\delta(t-t'), \tag{2.16}$$

$$\langle f_\sigma^j(t)f_\sigma^{\dagger j}(t')\rangle = [(2\gamma_{ab}-\gamma_b)\langle\sigma_b^j(t)\rangle + \gamma_a'\langle\sigma_a^j(t)\rangle]\delta(t-t'). \tag{2.17}$$

All the other correlation functions of atomic Langevin noise operators are zero.

It is convenient to define the slowly varying field and spin-flip operators in the frame rotating at frequency ω_c:

$$\tilde{a}(t) = e^{i\omega_c t}a(t),\ \tilde{\sigma}^j(t) = e^{i\omega_c t}\sigma^j(t). \tag{2.18}$$

For simplicity we consider an exact resonance between the cavity mode frequency and the atomic transition frequency, i.e., $\omega_{ab} = \omega_c$. In this case the Langevin equations for the slowly varying operators $\tilde{a}(t)$ and $\tilde{\sigma}^j(t)$ are the same as those for $a(t)$ and $\sigma^j(t)$ with the only difference that the terms proportional to ω_c and ω_{ab} disappear. In the following we drop the tilde on the operators, keeping in mind that now all the operators are defined in the rotating frame.

Now, following [2], we define macroscopic atomic operators, by adding up the individual atomic operators, taking into account the corresponding injection times into the cavity. We have, then,

$$M(t) = -i\sum_j \Theta(t-t_j)\sigma^j(t), \tag{2.19}$$

$$N_a(t) = \sum_j \Theta(t-t_j)\sigma_a^j(t), \tag{2.20}$$

$$N_b(t) = \sum_j \Theta(t-t_j)\sigma_b^j(t), \tag{2.21}$$

The additional factor $(-i)$ in (2.19) is introduced for mathematical convenience. The operator $M(t)$ represents the macroscopic atomic polarization. The operators $N_a(t)$ and $N_b(t)$ represent the macroscopic population of the upper and lower levels, respectively. It is worth noting that when calculating any average values or correlation functions with the macroscopic operators (2.19)–(2.21), one has to perform not only the quantum mechanical average but also the classical average over the pumping statistics, i.e., over the statistics of the arrival times t_j of the atoms into the cavity.

With the definitions (2.19)–(2.21) of the atomic macroscopic operators Eq. (2.1) for the electromagnetic field simplifies to

$$\dot{a}(t) = -\kappa/2\ a(t) + g\ M(t) + F_\gamma(t). \tag{2.22}$$

The Langevin equations for the macroscopic atomic operators can be found by differentiating Eqs. (2.19)–(2.21) and substituting Eqs. (2.2)–(2.4) for the individual atomic operators, respectively. For example, for the operator $N_a(t)$ corresponding to the macroscopic population of the upper level we obtain

$$\begin{aligned}\dot{N}_a(t) &= \sum_j \Big[\delta(t-t_j)\sigma_a^j(t) + \Theta(t-t_j)\dot{\sigma}_a^j(t)\Big] \\ &= \sum_j \delta(t-t_j)\sigma_a^j(t_j) - (\gamma_a+\gamma_a')N_a(t) \\ &\quad -g\Big[a^\dagger(t)M(t) + M^\dagger(t)a(t)\Big] \\ &\quad +\sum_j \Theta(t-t_j)f_a^j(t).\end{aligned} \tag{2.23}$$

The first term on the right-hand side of Eq. (2.23) corresponds to the pumping of the atoms into the upper lasing level. Indeed the expectation value of this term is given by

$$\left\langle \sum_j \delta(t-t_j)\sigma_a^j(t_j) \right\rangle = \left\langle \sum_j \delta(t-t_j)\langle\sigma_a^j(t_j)\rangle \right\rangle_S = \left\langle \sum_j \delta(t-t_j) \right\rangle_S . \quad (2.24)$$

As was mentioned above, we have to perform a double average. When doing it, we use the fact that the atoms are initially prepared in the upper state so that $\langle\sigma_a^j(t_j)\rangle = 1$. But we still have to perform the classical average over the injection times, i.e., over the pump statistics, which is indicated by the index S on the angle brackets in Eq. (2.24). The sum in Eq. (2.24) yields then the mean pumping rate of the upper lasing level:

$$\left\langle \sum_j \delta(t-t_j) \right\rangle_S = R\int_{-\infty}^{+\infty} dt_j \delta(t-t_j) = R. \quad (2.25)$$

Note that Eq. (2.25) can be regarded as the definition of the mean pumping rate R.

In order to separate the drift terms from the noise terms in Eq. (2.23) we add and subtract the expectation value of the first term and obtain

$$\dot{N}_a(t) = R - (\gamma_a + \gamma_a')N_a(t) - g\left[a^\dagger(t)M(t) + M^\dagger(t)a(t)\right] + F_a(t), \quad (2.26)$$

with

$$F_a(t) = \sum_j \Theta(t-t_j)f_a^j(t) + \sum_j \delta(t-t_j)\sigma_a^j(t_j) - R. \quad (2.27)$$

The new Langevin operator $F_a(t)$ is the total noise operator for the macroscopic atomic population $N_a(t)$. It incorporates the fluctuations of the population of the upper level due to the radiative decay and also due to the pump fluctuations.

In a similar way we can derive the equations for the macroscopic population of the lower level and for the macroscopic atomic polarization

$$\dot{N}_b(t) = -\gamma_b N_b(t) + \gamma_a' N_a(t) + g\left[a^\dagger(t)M(t) + M^\dagger(t)a(t)\right] + F_b(t), \quad (2.28)$$

$$\dot{M}(t) = -\gamma_{ab}M(t) + g\left[N_a(t) - N_b(t)\right]a(t) + F_M(t), \quad (2.29)$$

with

$$F_b(t) = \sum_j \Theta(t-t_j)f_b^j(t) + \sum_j \delta(t-t_j)\sigma_b^j(t_j), \quad (2.30)$$

$$F_M(t) = -i\left[\sum_j \Theta(t-t_j)f_\sigma^j(t) + \sum_j \delta(t-t_j)\sigma^j(t_j)\right]. \quad (2.31)$$

Note that there are no pumping terms in Eqs. (2.30) and (2.31), due to the assumption that the atoms are prepared initially at the upper level.

The evaluation of the correlation functions of the macroscopic Langevin forces defined by Eqs. (2.27), (2.30), and (2.31) can be done in the same fashion as in [2]. As shown in that reference,

$$\left\langle \sum_{j,k} \delta(t-t_j)\delta(t'-t_k) \right\rangle_S - R^2 = R(1-p)\delta(t-t'), \quad (2.32)$$

where p is a parameter which characterizes the pumping statistics: a Poissonian excitation statistics corresponds to $p = 0$, and for a regular statistics we have $p = 1$. The intermediate cases between these two extremes are described by values of p between one and zero.

Using Eq. (2.32), we find for the correlation functions of the macroscopic forces the following results:

$$\langle F_a(t)F_a(t')\rangle = \left[(\gamma_a + \gamma_a')\langle N_a(t)\rangle + R(1-p)\right]\delta(t-t'), \quad (2.33)$$

$$\langle F_M^\dagger(t)F_M(t')\rangle = \left[(2\gamma_{ab} - \gamma_a - \gamma_a')\langle N_a(t)\rangle + R\right]\delta(t-t'), \quad (2.34)$$

$$\langle F_b(t)F_b(t')\rangle = \left[\gamma_b\langle N_b(t)\rangle + \gamma_a'\langle N_a(t)\rangle\right]\delta(t-t'), \quad (2.35)$$

$$\langle F_b(t)F_M(t')\rangle = \gamma_b\langle M(t)\rangle\delta(t-t'), \quad (2.36)$$

$$\langle F_a(t)F_b(t')\rangle = -\gamma_a'\langle N_a(t)\rangle\delta(t-t'), \quad (2.37)$$

$$\langle F_M(t)F_a(t')\rangle = (\gamma_a + \gamma_a')\langle M(t)\rangle\delta(t-t'), \quad (2.38)$$

$$\langle F_M(t)F_b(t')\rangle = -\gamma_a'\langle M(t)\rangle\delta(t-t'), \quad (2.39)$$

$$\langle F_M(t)F_M^\dagger(t')\rangle = \left[(2\gamma_{ab} - \gamma_b)\langle N_b(t)\rangle + \gamma_a'\langle N_a(t)\rangle\right] \times\delta(t-t'). \quad (2.40)$$

It is worth noting that there is always a noise contribution from the pumping process in the normally ordered correlation function of the macroscopic atomic polarization, Eq. (2.34), which does not depend on the pumping statistics, being present even for regular pumping ($p = 1$).

Equations (2.22), (2.26), (2.28), and (2.29) with the correlation functions (2.7)–(2.9) and (2.33)–(2.40) describe completely the laser dynamics as well as the dynamics of the quantum fluctuations for arbitrary pumping statistics.

III. SPECTRA OF THE FLUCTUATIONS OF THE FIELD QUADRATURE COMPONENTS

A. Equivalent c-number stochastic Langevin equations for a normally ordered product of operators

Now we derive the stochastic c-number Langevin equations which are equivalent to the operator Langevin equations (2.22), (2.26), (2.28), and (2.29). For this we have to choose some particular ordering for products of atomic and field operators. This is necessary because the c-number variables commute with each other while the operators do not. Therefore, we obtain a unique relationship between operator and c-number equations only if we define the correspondence between a product of operators and a product of corresponding c-number variables. We choose here, as in [2], the normal ordering of atomic and field operators, i.e., $a^\dagger(t), M^\dagger(t), N_a(t), N_b(t), M(t), a(t)$. The stochastic c-number variables corresponding to the operators $a(t), M(t), N_a(t)$, and $N_b(t)$ are denoted by $\mathcal{A}(t), \mathcal{M}(t), \mathcal{N}_a(t)$, and $\mathcal{N}_b(t)$, respectively. We will derive the equations for these stochastic variables from the requirement that the equations for the first- and second-order moments of operators and c-number variables are identical. Equations (2.22), (2.26), (2.28), and (2.29) are already written in normal order, so that it is easy to obtain the equation for the corresponding c-number variables

$$\dot{\mathcal{A}}(t) = -\kappa/2\ \mathcal{A}(t) + g\mathcal{M}(t) + \mathcal{F}_\gamma(t), \tag{3.1}$$

$$\dot{\mathcal{M}}(t) = -\gamma_{ab}\mathcal{M}(t) + g\Big[\mathcal{N}_a(t) - \mathcal{N}_b(t)\Big]\mathcal{A}(t) + \mathcal{F}_{\mathcal{M}}(t), \tag{3.2}$$

$$\dot{\mathcal{N}_a}(t) = R - (\gamma_a + \gamma_a')\mathcal{N}_a(t) - g\Big[\mathcal{A}^*(t)\mathcal{M}(t) + \mathcal{M}^*(t)\mathcal{A}(t)\Big] + \mathcal{F}_a(t), \tag{3.3}$$

$$\dot{\mathcal{N}_b}(t) = -\gamma_b\mathcal{N}_b(t) + \gamma_a'\mathcal{N}_a(t) + g\Big[\mathcal{A}^*(t)\mathcal{M}(t) + \mathcal{M}^*(t)\mathcal{A}(t)\Big] + \mathcal{F}_b(t). \tag{3.4}$$

Here the functions $\mathcal{F}_k(t)$ with $k = \gamma, \mathcal{M}, a$, or b, are the stochastic c-number Langevin forces with the properties

$$\langle \mathcal{F}_k(t)\rangle = 0, \tag{3.5}$$

$$\langle \mathcal{F}_k(t)\mathcal{F}_l(t')\rangle = 2\mathcal{D}_{kl}\delta(t-t'). \tag{3.6}$$

The diffusion coefficients $\mathcal{D}_{kl}$ for the c-number Langevin forces are different, in general, from the corresponding diffusion coefficients D_{kl} for the operator Langevin forces, defined by (2.7)–(2.9) and (2.33)–(2.40). The diffusion coefficients $\mathcal{D}_{kl}$ are determined from the requirement that the c-number equations for the second moments should be identical to the corresponding normally ordered operator equations. It is easy to see that the diffusion coefficients for the c-number Langevin force in the field equation $\mathcal{F}_\gamma(t)$ are the same as for the Langevin noise operators $F_\gamma(t)$, so that

$$\mathcal{D}_{\gamma^*\gamma} = 0,\ \mathcal{D}_{\gamma\gamma} = 0. \tag{3.7}$$

However, some of the atomic diffusion coefficients $\mathcal{D}_{kl}$ for the c-number Langevin forces are different from the corresponding diffusion coefficients D_{kl}. As an example, let us calculate the diffusion coefficient $\mathcal{D}_{aa}$. From the operator equation (2.26), we have

$$\frac{d}{dt}\Big[N_a(t)N_a(t)\Big] = 2RN_a(t) - 2(\gamma_a + \gamma_a')N_a(t)N_a(t) - gN_a(t)\Big[a^\dagger(t)M(t) + M^\dagger(t)a(t)\Big] - g\Big[M^\dagger(t)a(t) + a^\dagger(t)M(t)\Big]N_a(t) + N_a(t)F_a(t) + F_a(t)N_a(t). \tag{3.8}$$

We note here that the third term on the right-hand side of the equation is not in the chosen order, because the operator $N_a(t)$ is to the left of $M^\dagger(t)$. To bring it into the chosen order we have to use the commutator $[N_a(t), M(t)] = -M(t)$, so that we get

$$\frac{d}{dt}\langle N_a(t)N_a(t)\rangle = 2R\langle N_a(t)\rangle - 2(\gamma_a + \gamma_a')\langle N_a(t)N_a(t)\rangle - 2g\Big[\langle a^\dagger(t)N_a(t)M(t) + M^\dagger(t)N_a(t)a(t)\rangle\Big] - g\Big[\langle M^\dagger(t)a(t) + a^\dagger(t)M(t)\rangle\Big] + 2D_{aa}, \tag{3.9}$$

where we have made use of the fact that $\langle N_a(t)F_a(t)\rangle = \langle F_a(t)N_a(t)\rangle = D_{aa}$ (see, for instance, [16]). On the other hand, using the corresponding c-number equation (3.3), we get

$$\frac{d}{dt}\langle \mathcal{N}_a(t)\mathcal{N}_a(t)\rangle = 2R\langle \mathcal{N}_a(t)\rangle - 2(\gamma_a + \gamma_a')\langle \mathcal{N}_a(t)\mathcal{N}_a(t)\rangle - 2g\langle \mathcal{A}^*(t)\mathcal{N}_a(t)\mathcal{M}(t) + \mathcal{M}^*(t)\mathcal{N}_a(t)\mathcal{A}(t)\rangle + 2\mathcal{D}_{aa}. \tag{3.10}$$

If we require the left-hand sides of Eqs. (3.9) and (3.10) to be equal we can see that the diffusion coefficient $\mathcal{D}_{aa}$ is connected with the diffusion coefficient D_{aa} as

$$2\mathcal{D}_{aa} = 2D_{aa} - g\Big[\langle \mathcal{M}^*(t)\mathcal{A}(t) + \mathcal{A}^*(t)\mathcal{M}(t)\rangle\Big]. \tag{3.11}$$

In the same fashion we can get the relationship between all the other atomic diffusion coefficients for c-number Langevin forces and operator Langevin forces. All the nonvanishing diffusion coefficients for the c-number Langevin forces read as

$$2\mathcal{D}_{aa} = (\gamma_a + \gamma_a')\langle \mathcal{N}_a(t)\rangle + R(1-p) - g\Big[\langle \mathcal{M}^*(t)\mathcal{A}(t) + \mathcal{A}^*(t)\mathcal{M}(t)\rangle\Big], \tag{3.12}$$

$$2\mathcal{D}_{bb} = \gamma_b\langle \mathcal{N}_b(t)\rangle + \gamma_a'\langle \mathcal{N}_a(t)\rangle - g\Big[\langle \mathcal{M}^*(t)\mathcal{A}(t) + \mathcal{A}^*(t)\mathcal{M}(t)\rangle\Big], \tag{3.13}$$

$$2\mathcal{D}_{ab} = -\gamma_a'\langle \mathcal{N}_a(t)\rangle + g\Big[\langle \mathcal{M}^*(t)\mathcal{A}(t) + \mathcal{A}^*(t)\mathcal{M}(t)\rangle\Big], \tag{3.14}$$

$$2\mathcal{D}_{\mathcal{M}\mathcal{M}} = 2g\langle \mathcal{M}(t)\mathcal{A}(t)\rangle, \tag{3.15}$$

$$2\mathcal{D}_{\mathcal{M}^*\mathcal{M}} = (2\gamma_{ab} - \gamma_a - \gamma_a')\langle \mathcal{N}_a(t)\rangle + R, \tag{3.16}$$

$$2\mathcal{D}_{b\mathcal{M}} = \gamma_b\langle \mathcal{M}(t)\rangle. \tag{3.17}$$

We are now in a position to solve the c-number Langevin equations and to calculate the spectra of the fluctuations of the field quadrature components. We start by calculating the steady-state solutions for the field and atomic variables.

B. Steady-state solution for above-threshold operation

The steady-state solutions for the mean values of the field and atomic variables for laser operation above threshold are obtained by dropping the noise terms in Eqs. (3.1)–(3.4) and setting the time derivatives equal to zero. These solutions are denoted by the subscript zero. For the mean intensity of the laser field above threshold, $I_0 \equiv \mathcal{A}_0^2$, we get the well-known expression

$$I_0 = I_s(R/R_{\rm th} - 1), \tag{3.18}$$

where I_s is the saturation intensity,

$$I_s = \frac{\gamma_{ab}\gamma_b}{2g^2}\frac{\gamma_a + \gamma_a'}{\gamma_a + \gamma_b}, \tag{3.19}$$

and $R_{\rm th}$ it the threshold pumping rate,

$$R_{\rm th} = \frac{\kappa\gamma_{ab}\gamma_b}{2g^2}\frac{\gamma_a + \gamma_a'}{\gamma_b - \gamma_a'}. \tag{3.20}$$

From (3.20) we note that the necessary condition for laser oscillation is $\gamma_b > \gamma_a'$. The steady-state populations of the upper and lower levels are given by

$$\mathcal{N}_{a_0} = \frac{R - \kappa I_0}{\gamma_a + \gamma_a'}, \quad \mathcal{N}_{b_0} = \frac{\gamma_a' R + \kappa\gamma_a I_0}{\gamma_b(\gamma_a + \gamma_a')}. \tag{3.21}$$

Using Eq. (3.18), we can express these steady-state populations in terms of the mean intensity I_0 and the saturation intensity I_s:

$$\mathcal{N}_{a_0} = \frac{\kappa}{\gamma_b - \gamma_a'}\left(I_0 + \frac{\gamma_a + \gamma_b}{\gamma_a + \gamma_a'}I_s\right), \tag{3.22}$$

$$\mathcal{N}_{b_0} = \frac{\kappa}{\gamma_b - \gamma_a'}\left(I_0 + \frac{\gamma_a'}{\gamma_b}\frac{\gamma_a + \gamma_b}{\gamma_a + \gamma_a'}I_s\right). \tag{3.23}$$

From Eqs. (3.22) and (3.23) we can see that far above threshold, i.e., when $R \gg R_{\rm th}$ and $I_0 \gg I_s$, the populations of the upper and lower lasing level approach each other and the inversion goes to zero, corresponding to saturation as expected.

The steady-state value of the atomic polarization can be expressed in terms of the mean value of the field as

$$\mathcal{M}_0 = \frac{\kappa}{2g}\mathcal{A}_0. \tag{3.24}$$

The mean optical phase of the laser field does not appear in the steady-state equations. This is related to the fact that in the steady state the optical phase is randomly distributed between 0 and 2π. We can, therefore, choose the arbitrary mean value of the phase to be equal to zero, which is quite convenient since then both the field $\mathcal{A}_0$ and the polarization $\mathcal{M}_0$ become real.

The evolution of the quantum fluctuations are now obtained by linearizing Eqs. (3.1)–(3.4) around the steady-state solution. We assume that the working point around which the fluctuations are calculated is in the stable region, as analyzed in Ref. [12].

C. Quantum fluctuations of the laser field around steady state

To investigate the small fluctuations of the field and atomic variables around steady state we consider all these variables, as usual, as the sum of the steady-state solution and a small fluctuating term. For example, for $\mathcal{N}_a(t)$ we set $\mathcal{N}_a(t) = \mathcal{N}_{a_0} + \delta\mathcal{N}_a(t)$ and in the same way for the other variables. Thus, we get the following equations for the fluctuations

$$\delta\dot{\mathcal{N}}_a(t) = -(\gamma_a + \gamma_a')\delta\mathcal{N}_a(t) - g\mathcal{A}_0\Big[\delta\mathcal{M}(t) + \delta\mathcal{M}^*(t)\Big] - g\mathcal{M}_0\Big[\delta\mathcal{A}(t) + \delta\mathcal{A}^*(t)\Big] + \mathcal{F}_a(t), \tag{3.25}$$

$$\delta\dot{\mathcal{N}}_b(t) = -\gamma_b\delta\mathcal{N}_b(t) + \gamma_a'\delta\mathcal{N}_a(t) + g\mathcal{A}_0\Big[\delta\mathcal{M}(t) + \delta\mathcal{M}^*(t)\Big] + g\mathcal{M}_0\Big[\delta\mathcal{A}(t) + \delta\mathcal{A}^*(t)\Big] + \mathcal{F}_b(t), \tag{3.26}$$

$$\delta\dot{\mathcal{M}}(t) = -\gamma_{ab}\delta\mathcal{M}(t) + g\left(\mathcal{N}_{a_0} - \mathcal{N}_{b_0}\right)\delta\mathcal{A}(t) + g\mathcal{A}_0\left[\delta\mathcal{N}_a(t) - \delta\mathcal{N}_b(t)\right] + \mathcal{F}_{\mathcal{M}}(t), \quad (3.27)$$

$$\delta\dot{\mathcal{A}}(t) = -\kappa/2\ \delta\mathcal{A}(t) + g\delta\mathcal{M}(t) + \mathcal{F}_\gamma(t), \quad (3.28)$$

where we have made use of the fact that $\mathcal{A}_0$ and $\mathcal{M}_0$ are real.

At this stage, the adiabatic elimination of atomic variables is frequently used, under the assumption that the polarization decay rate γ_{ab} and the population decay rates $\gamma_a(+\gamma_a')$ and γ_b are much larger than the cavity decay rate κ. Here we do not want to restrict ourselves to some particular class of lasers and, therefore, we do not adiabatically eliminate any of the variables in Eqs. (3.25)–(3.28). Instead, we take the Fourier transform of all the variables and convert the differential equations into algebraic equations.

Therefore, we take the Fourier transform of $\delta\mathcal{N}_a(t)$

$$\delta\mathcal{N}_a(\Omega) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} dt\ e^{i\Omega t}\delta\mathcal{N}_a(t), \quad (3.29)$$

as well as of all the other variables. In order not to overcharge the notation, we adopt the same symbol for both members of a Fourier-transform pair, which will therefore get distinguished through the time or frequency argument.

In the above equations, we may set $\mathcal{F}_\gamma(t) = 0$ in Eq. (3.28), since the mean value and the correlation functions of this force with all the other variables, as well as the autocorrelation function, are zero (this is a consequence of the normal ordering of the operators). In this way we get the following equations for the Fourier amplitudes:

$$-i\Omega\delta\mathcal{N}_a(\Omega) = -(\gamma_a + \gamma_a')\delta\mathcal{N}_a(\Omega) - g\mathcal{A}_0\left[\delta\mathcal{M}(\Omega) + \delta\mathcal{M}^*(-\Omega)\right] - g\mathcal{M}_0\left[\delta\mathcal{A}(\Omega) + \delta\mathcal{A}^*(-\Omega)\right] + \mathcal{F}_a(\Omega), \quad (3.30)$$

$$-i\Omega\delta\mathcal{N}_b(\Omega) = -\gamma_b\delta\mathcal{N}_b(\Omega) + \gamma_a'\delta\mathcal{N}_a(\Omega) + g\mathcal{A}_0\left[\delta\mathcal{M}(\Omega) + \delta\mathcal{M}^*(-\Omega)\right] + g\mathcal{M}_0\left[\delta\mathcal{A}(\Omega) + \delta\mathcal{A}^*(-\Omega)\right] + \mathcal{F}_b(\Omega), \quad (3.31)$$

$$-i\Omega\delta\mathcal{M}(\Omega) = -\gamma_{ab}\delta\mathcal{M}(\Omega) + g(\mathcal{N}_{a_0} - \mathcal{N}_{b_0})\delta\mathcal{A}(\Omega) + g\mathcal{A}_0\left[\delta\mathcal{N}_a(\Omega) - \delta\mathcal{N}_b(\Omega)\right] + \mathcal{F}_{\mathcal{M}}(\Omega), \quad (3.32)$$

$$-i\Omega\delta\mathcal{A}(\Omega) = -\kappa/2\ \delta\mathcal{A}(\Omega) + g\delta\mathcal{M}(\Omega), \quad (3.33)$$

where the Fourier-transformed fluctuation forces satisfy now the equations [which follow immediately from the definition (3.29) and from (3.6)]:

$$\langle\mathcal{F}_k(\Omega)\mathcal{F}_l(\Omega')\rangle = 2\mathcal{D}_{kl}\delta(\Omega + \Omega'). \quad (3.34)$$

Note that, since $\mathcal{F}_{\mathcal{M}^*}(t) = \mathcal{F}^*_{\mathcal{M}}(t)$, it follows that $\mathcal{F}_{\mathcal{M}^*}(\Omega) = \mathcal{F}^*_{\mathcal{M}}(-\Omega)$. Also, since $\mathcal{F}_a(t)$ and $\mathcal{F}_b(t)$ are real, we must have $\mathcal{F}_i(\Omega) = \mathcal{F}_i^*(-\Omega), i = a, b$.

The solution of the above linear system is straightforward. Because finally we are interested in the spectra of fluctuations of the field quadrature components, we express the Fourier amplitudes of the field fluctuations $\delta\mathcal{A}(\Omega)$ and $\delta\mathcal{A}^*(-\Omega)$ in terms of the Fourier amplitudes of the Langevin forces. The final expression for $\delta\mathcal{A}(\Omega)$ reads as

$$\delta\mathcal{A}(\Omega) = \frac{ig}{\Omega(\kappa/2 + \gamma_{ab} - i\Omega)(\gamma_b - i\Omega)[C(\Omega) + 2\mathcal{A}_0^2]} \times\Big\{[C(\Omega) + \mathcal{A}_0^2](\gamma_b - i\Omega)\mathcal{F}_{\mathcal{M}}(\Omega) + gC(\Omega)\mathcal{A}_0\frac{\gamma_b - \gamma_a' - i\Omega}{\gamma_a + \gamma_a' - i\Omega}\mathcal{F}_a(\Omega) - gC(\Omega)\mathcal{A}_0\mathcal{F}_b(\Omega) - \mathcal{A}_0^2(\gamma_b - i\Omega)\mathcal{F}^*_{\mathcal{M}}(-\Omega)\Big\}, \quad (3.35)$$

with the following shorthand

$$C(\Omega) = \frac{-i\Omega(\kappa/2 + \gamma_{ab} - i\Omega)(\gamma_b - i\Omega)(\gamma_a + \gamma_a' - i\Omega)}{g^2(\gamma_a + \gamma_b - 2i\Omega)(\kappa - i\Omega)}. \quad (3.36)$$

The expression for $\delta\mathcal{A}^*(-\Omega)$ is obtained by performing a complex conjugation and substitution of $-\Omega$ instead of Ω in Eq. (3.35).

We define now the amplitude and phase quadrature components of the field fluctuations inside the cavity as

$$\delta X(\Omega) = \frac{1}{2}\left[\delta\mathcal{A}(\Omega) + \delta\mathcal{A}^*(-\Omega)\right], \tag{3.37}$$

$$\delta Y(\Omega) = \frac{1}{2i}\left[\delta\mathcal{A}(\Omega) - \delta\mathcal{A}^*(-\Omega)\right]. \tag{3.38}$$

These definitions correspond to our choice of phase for the steady-state solution (real field). Note that $\delta X^*(\Omega) = \delta X(-\Omega)$, $\delta Y^*(\Omega) = \delta Y(-\Omega)$, so that $\delta X(t)$ and $\delta Y(t)$ are real, as expected.

Using Eq. (3.35) for $\delta\mathcal{A}(\Omega)$ and the corresponding expression for $\delta\mathcal{A}^*(-\Omega)$ we get the following expressions for these quadrature components:

$$\delta X(\Omega) = \frac{\gamma_a + \gamma_a' - i\Omega}{g(\gamma_a+\gamma_b-2i\Omega)(\kappa - i\Omega)[C(\Omega)+2\mathcal{A}_0^2]} \times\left\{\frac{1}{2}\left[\mathcal{F}_\mathcal{M}(\Omega)+\mathcal{F}_\mathcal{M}^*(-\Omega)\right](\gamma_b - i\Omega) + g\mathcal{A}_0\frac{\gamma_b-\gamma_a'-i\Omega}{\gamma_a+\gamma_a'-i\Omega}\mathcal{F}_a(\Omega) - g\mathcal{A}_0\mathcal{F}_b(\Omega)\right\}, \tag{3.39}$$

$$\delta Y(\Omega) = \frac{g}{2\Omega(\kappa/2+\gamma_{ab}-i\Omega)}\left[\mathcal{F}_\mathcal{M}(\Omega) - \mathcal{F}_\mathcal{M}^*(-\Omega)\right]. \tag{3.40}$$

At this point two remarks should be made. First, from (3.39) and (3.40) one can see that the phase quadrature component $\delta Y(\Omega)$ diverges when $\Omega \to 0$, while the amplitude quadrature component $\delta X(\Omega)$ remains finite. The divergence of $\delta Y(\Omega)$ is related to the phase diffusion process which leads to the infinite growth of phase fluctuations when $t\to\infty$, that is $\langle\delta\varphi^2(t)\rangle_{t\to\infty}\to\infty$. Second, as seen from (3.40), the phase quadrature component of the field fluctuations depends only on the phase quadrature component of the Langevin force $\mathcal{F}_\mathcal{M}(t)$, which is associated with fluctuations of the atomic polarization. This is again a consequence of choosing the normal ordering of operators. For other orderings (for instance, the symmetrical ordering could have been used), we could get a contribution from the Langevin force $\mathcal{F}_\gamma(t)$, associated with the vacuum fluctuations.

We note also that, in view of (3.34), the auto- and cross-correlation functions of the amplitude and phase quadratures are δ-function correlated, that is

$$\langle\delta X(\Omega)\delta X(\Omega')\rangle = (\delta X^2)_\Omega\delta(\Omega+\Omega'), \tag{3.41}$$

$$\langle\delta Y(\Omega)\delta Y(\Omega')\rangle = (\delta Y^2)_\Omega\delta(\Omega+\Omega'), \tag{3.42}$$

$$\langle\delta X(\Omega)\delta Y(\Omega')\rangle = (\delta X\delta Y)_\Omega\delta(\Omega+\Omega'). \tag{3.43}$$

Now we evaluate the spectra of the amplitude and phase quadrature components of the field fluctuations, defined as the following Fourier transforms:

$$\mathcal{W}_X(\Omega) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} dt e^{i\Omega t}\langle\delta X(0)\delta X(t)\rangle, \tag{3.44}$$

with the analogous definition for $\mathcal{W}_Y(\Omega)$. For a calculation of these spectra we use the Wiener-Khinchine theorem [18, 19], which states that for a real stationary random process $X(t)$, the spectrum $\mathcal{W}_X(\Omega)$ is equal to $(\delta X^2)_\Omega$, defined by (3.41) as the coefficient of the δ function in the correlation function $\langle\delta X(\Omega)\delta X(\Omega')\rangle$.

Using also Eq. (3.34), we get the following expressions for the spectra of quadrature components of the field fluctuations inside the cavity:

$$(\delta X^2)_\Omega = \frac{(\gamma_a+\gamma_a')^2+\Omega^2}{g^2((\gamma_a+\gamma_b)^2+4\Omega^2)(\kappa^2+\Omega^2)\mid C(\Omega)+2\mathcal{A}_0^2\mid^2} \times\left\{(\gamma_b^2+\Omega^2)\left[(\gamma_{ab}-\gamma_a-\gamma_a')\mathcal{N}_{a_0}+R\right]+g^2\mathcal{A}_0^2\frac{(\gamma_b-\gamma_a')^2+\Omega^2}{(\gamma_a+\gamma_a')^2+\Omega^2}\left[2(\gamma_a+\gamma_a')\mathcal{N}_{a_0}-pR\right]+2g^2\mathcal{A}_0^2\gamma_a'\mathcal{N}_{a_0} -2g^2\mathcal{A}_0^2\frac{(\gamma_a+\gamma_a')(\gamma_b-\gamma_a')+\Omega^2}{(\gamma_a+\gamma_a')^2+\Omega^2}\left(\gamma_b\mathcal{N}_{b_0}-2\gamma_a'\mathcal{N}_{a_0}\right)-\kappa\mathcal{A}_0^2\gamma_b^2\right\}, \tag{3.45}$$

$$(\delta Y^2)_\Omega = \frac{g^2\gamma_{ab}\mathcal{N}_{a_0}}{\Omega^2[(\kappa/2+\gamma_{ab})^2+\Omega^2]}. \tag{3.46}$$

We turn now, in the next section, to a physical discussion of these results.

IV. PHASE DIFFUSION COEFFICIENT AND LINEWIDTH

Let us consider first the phase quadrature, Eq. (3.46), which yields the phase-diffusion coefficient and the laser line shape. Even though the influence of polarization dynamics on the diffusion coefficient has been considered before [17, 18], we include this discussion here in order to stress the complementarity between phase and amplitude fluctuations (to be discussed in Sec. V), and also in order to show how the line shape can be calculated within the present formalism.

For a small fluctuation of the phase the spectrum of the phase fluctuations is simply related to the spectrum of the phase quadrature component of the field fluctuations, namely,

$$(\delta\varphi^2)_\Omega = \frac{1}{I_0}(\delta Y^2)_\Omega. \tag{4.1}$$

From Eq. (3.46) as well as Eqs. (3.18) and (3.21) for the steady-state field intensity and the population of the upper level, it then follows that

$$(\delta\varphi^2)_\Omega = \frac{D_{\rm ST}}{\Omega^2}\,\frac{(\kappa/2+\gamma_{ab})^2}{(\kappa/2+\gamma_{ab})^2+\Omega^2}\left(\frac{\gamma_{ab}}{\kappa/2+\gamma_{ab}}\right)^2, \tag{4.2}$$

where $D_{\rm ST}$ is the Schawlow-Townes diffusion coefficient [16]

$$D_{\rm ST} = \frac{g^2\mathcal{N}_{a_0}}{I_0\gamma_{ab}}. \tag{4.3}$$

We can see from Eq. (4.2) that our spectrum of the phase fluctuations is different from the usual result given by $(\delta\varphi^2)_\Omega = D_{\rm ST}/\Omega^2$ (see, for example, [20]), since there are two additional factors in Eq. (4.2).

The first factor (frequency dependent) plays an important role in the high-frequency range, $\Omega \gg \kappa/2+\gamma_{ab}$. Since the linewidth of the laser radiation can be estimated as $\kappa/2I_0$, this factor is of no importance for the spectral width and modifies the line shape only in the line wings. Nevertheless, as shown in Refs. [5, 6], this factor can improve the sensitivity of the short-time measurements of the phase if the duration of such measurement t_m is much shorter than $(\kappa/2+\gamma_{ab})^{-1}$.

This factor has its origin in the atomic memory effect associated with the transient behavior of the polarization [5, 6]. For short observation (or measurement) times one cannot consider the spontaneous-emission events as δ-function-like uncorrelated impulses (a model which leads to the Schawlow-Townes formula). There is a certain characteristic time $\tau \sim (\kappa/2+\gamma_{ab})^{-1}$ which can be considered as a "memory time" or a time during which a spontaneous-emission event takes place. As it was shown in [5, 6], on a time scale much shorter than this characteristic time τ, the spontaneous-emission events are correlated, and this leads to the reduction of the phase noise.

The second factor in (4.2) is the ratio $\gamma_{ab}^2/(\kappa/2+\gamma_{ab})^2$ which modifies the phase diffusion coefficient from the Schawlow-Townes value $D_{\rm ST}$ to

$$D = D_{\rm ST}\left(\frac{\gamma_{ab}}{\kappa/2+\gamma_{ab}}\right)^2. \tag{4.4}$$

This expression coincides with the one given in Refs. [17, 18]. If the atomic polarization decay rate is much faster than the cavity decay rate, i.e., $\gamma_{ab} \gg \kappa/2$, this additional factor is close to unity and we get the usual Schawlow-Townes diffusion coefficient. But in the opposite case, $\gamma_{ab} \ll \kappa/2$—which corresponds to the fourth class of lasers described in the Introduction—this factor can be very small and we have $D \ll D_{\rm ST}$.

Thus, when the lifetime of the atomic polarization is much larger than the lifetime of a photon in the cavity (bad-cavity case), which is just the opposite limit with respect to the one considered in Refs. [5, 6], we may obtain a significant slowing down of the phase diffusion process. As will be shown next, this may imply a large quenching of the spectral width of the laser radiation.

Using the phase fluctuation spectrum (4.2) we can evaluate the time correlation function of the phase fluctuations and the time-dependent behavior of the phase mean-square uncertainty, which can then be used for calculating the field spectrum. We consider first the correlation function of the time derivative of the phase fluctuation, namely

$$\begin{aligned}\langle\delta\dot\varphi(t)\delta\dot\varphi(t')\rangle &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\Omega e^{-i\Omega(t-t')}\Omega^2(\delta\varphi^2)_\Omega\\ &= \frac{D}{2\pi}\int_{-\infty}^{+\infty} d\Omega e^{-i\Omega(t-t')}\frac{(\kappa/2+\gamma_{ab})^2}{(\kappa/2+\gamma_{ab})^2+\Omega^2}\\ &= (D/2)\ (\kappa/2+\gamma_{ab})\ \exp\{-(\kappa/2+\gamma_{ab})\mid t-t'\mid\}.\end{aligned} \tag{4.5}$$

Let us compare this result with the usual expression for this correlation function

$$\langle\delta\dot\varphi(t)\delta\dot\varphi(t')\rangle = D_{\rm ST}\delta(t-t'), \tag{4.6}$$

which is valid when the time behavior of the phase is described by a Markovian diffusion process, $\dot\varphi(t) = F(t)$, with a δ-function-correlated random force $F(t)$. It is seen from Eq. (4.5), that due to the finite relaxation time of the atomic polarization we do not have a Markovian behavior of the phase: the random force $F(t)$ has a memory, with a characteristic time $\tau = (\kappa/2+\gamma_{ab})^{-1}$. If the atomic polarization relaxes very fast, the memory time goes to zero and we get the correlation function (4.6).

We calculate now the time-dependent behavior of the mean-square uncertainty of the phase integrating Eq. (4.5) twice over time:

$$\langle\delta\varphi^2(t)\rangle = \int_0^t dt' \int_0^t dt'' \langle\delta\dot{\varphi}(t')\delta\dot{\varphi}(t'')\rangle$$
$$= D\left[\,|\,t\,| + \frac{e^{-(\kappa/2+\gamma_{ab})|t|} - 1}{\kappa/2 + \gamma_{ab}}\right]. \tag{4.7}$$

If the time $t(>0)$ of measurement is short compared to the memory time τ, we can expand the exponential and obtain a quadratic dependence on time for the phase fluctuation [5, 6]:

$$\langle\delta\varphi^2(t)\rangle \approx Dt\left(\frac{1}{2}\frac{t}{\tau}\right) \ll Dt. \tag{4.8}$$

Thus, for a short-time measurement the quantum noise of the phase fluctuations can be reduced by a factor $t/2\tau \ll 1$ with respect to the usual expression. For times long compared to the memory time we get a linear growth with time of the phase fluctuations, Dt. These results generalize those of Refs. [5, 6], which apply only to the good-cavity case, that is for $\gamma_{ab} \gg \kappa/2$. The main differences between our results and those of Refs. [5, 6] can be summarized in the following way: (i) In the good-cavity case considered in [5, 6], one has $D \approx D_{\rm ST}$ and therefore there is no appreciable change in the spectral line shape, while in the opposite bad-cavity limit, $\gamma_{ab} \ll \kappa/2$, which is obtained as a particular case of our general solution, one can obtain an important quenching of the linewidth, as we are going to show; (ii) our memory time, $\tau = (\gamma_{ab} + \kappa/2)^{-1}$, differs from the one obtained in [5,6], $\tau = \gamma_{ab}^{-1}$, reducing to it, however, in the good-cavity limit.

We calculate now explicitly the power spectrum of the laser field inside the laser cavity, using the above results for the phase diffusion. We assume a Gaussian statistics for the phase fluctuations, so that the spectrum of the field inside the cavity is given by

$$(\mathcal{A}^2)_\Omega = \int_{-\infty}^{+\infty} dt\; e^{i\Omega t}\langle\mathcal{A}^*(t)\mathcal{A}(0)\rangle$$
$$\approx I_0 \int_{-\infty}^{+\infty} dt\; e^{i(\Omega-\omega_c)t} e^{-\frac{1}{2}\langle\delta\varphi^2(t)\rangle}, \tag{4.9}$$

where we have neglected the amplitude fluctuations (far-above threshold case). We substitute Eq. (4.7) for the phase mean-square uncertainty into Eq. (4.9) and use, as in [6], the series expansion of the exponential in (4.9), calculating the Fourier integrals of the different terms of the expansion. We get then the following infinite sum of Lorentzians:

$$(\mathcal{A}^2)_\Omega = 2I_0 \exp\left[\frac{D}{2(\kappa/2+\gamma_{ab})}\right]\sum_{m=0}^{\infty}\frac{1}{m!}\left[-\frac{D}{2(\kappa/2+\gamma_{ab})}\right]^m \frac{D/2 + (\kappa/2+\gamma_{ab})m}{(\omega_c-\Omega)^2 + [D/2 + (\kappa/2+\gamma_{ab})m]^2}\,. \tag{4.10}$$

The Schawlow-Townes expression is recovered when $\gamma_{ab} \to \infty$. In this case only the first term in the sum, corresponding to $m = 0$, survives, and we get the usual formula

$$(\mathcal{A}^2)_\Omega = 2I_0 \frac{D_{\rm ST}/2}{(\omega_c-\Omega)^2 + (D_{\rm ST}/2)^2}\,. \tag{4.11}$$

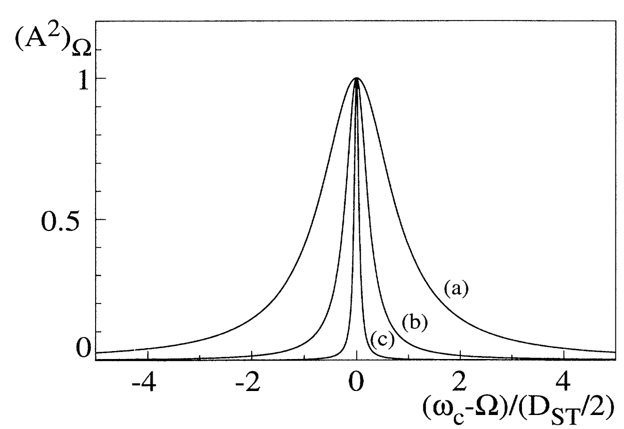

FIG. 1. Normalized power spectrum of the laser with atomic-memory effect. For all three curves $D_{\rm ST}/\kappa = 10^{-10}$. (a) $2\gamma_{ab}/\kappa = 10$, i.e., Schawlow-Townes limit, (b) $2\gamma_{ab}/\kappa = 1$, and (c) $2\gamma_{ab}/\kappa = 0.3$ (long-lived atomic polarization).

The spectrum of the field for the general case is shown in Fig. 1, where it is plotted as a function of the dimensionless frequency $(\omega_c - \Omega)/(D_{\rm ST}/2)$. It depends on two dimensionless parameters, namely, $D_{\rm ST}/\kappa$ and $\gamma_{ab}/(\kappa/2)$. Because the first parameter is inversely proportional to the mean number of photons inside the cavity and is very small, the sum (4.10) converges very rapidly and can be approximated in the general case by the first Lorentzian with $m = 0$ and diffusion coefficient D instead of $D_{\rm ST}$. Figure 1 displays the quenching of the spectral width of the laser as the lifetime of the atomic polarization increases.

We turn now to the main results of this paper, those pertaining to the spectrum of fluctuations of the outgoing light.

V. SPECTRUM OF FLUCTUATIONS OF THE OUTPUT FIELD

In this section we investigate the spectrum of fluctuations for the field transmitted through the cavity port, expressing it, by means of well-known methods

[10, 21, 22], in terms of correlation functions of the fluctuations obtained from the linearized semiclassical equations. The normalized spectrum of fluctuations corresponding to a quadrature

$$X_\theta = a_{\rm out}(t)e^{-i\theta} + a^\dagger_{\rm out}(t)e^{i\theta} \tag{5.1}$$

is defined as

$$V(\theta,\Omega) = \int_{-\infty}^{\infty} e^{i\Omega\tau}\langle X_\theta(t+\tau), X_\theta(t)\rangle\, d\tau, \tag{5.2}$$

where $\langle X,Y\rangle = \langle XY\rangle - \langle X\rangle\langle Y\rangle$. For a stationary field, this quantity is independent of time.

The spectrum defined in this way actually corresponds to the normalized photocurrent obtained in a homodyne measurement of the field quadrature component, defined by the angle θ, transmitted by the cavity port. One can express this spectrum in terms of the amplitude and phase quadrature of the field inside the cavity. For $\theta = 0$, we get the spectrum of the amplitude fluctuations, while for $\theta = \pi/2$, Eq. (5.2) gets reduced to the spectrum of phase fluctuations. For each frequency Ω, the minimum noise is obtained by minimizing $V(\theta,\Omega)$ with respect to θ.

Here we examine the spectrum of amplitude fluctuations. It is worth noting that for an exact resonance between the atoms and the laser field, amplitude and phase fluctuations are decoupled. Moreover, due to the phase diffusion process there is an excess noise in the phase quadrature (inside the spectral bandwidth, corresponding to the linewidth of the laser), so that adding the phase quadrature to the amplitude quadrature will only increase the resultant noise power spectrum.

The spectrum in (5.2) can be related to the c-number averages of the field fluctuations defined in the normally ordered representation [10, 21, 22], so that the spectrum of amplitude fluctuations ($\theta = 0$) can be rewritten as

$$V_A(\Omega) = V(0,\Omega) = 1 + 4\kappa(\delta X^2)_\Omega, \tag{5.3}$$

where $(\delta X^2)_\Omega$ is given by Eq. (3.45). The first term on the right-hand side of this equation comes from the commutator of the outgoing boson operators, and corresponds to the shot-noise contribution.

We turn now to a detailed analysis of this spectrum, expressing it in a dimensionless form convenient for graphical representation. We introduce the dimensionless parameters

$$\tilde\Omega \equiv \Omega/\kappa, \quad a \equiv \gamma_a/\kappa, \quad b \equiv \gamma_b/\kappa, \quad c \equiv \gamma_{ab}/\kappa, \text{ and } a' \equiv \gamma'_a/\kappa. \tag{5.4}$$

In terms of these spectral parameters and also the statistical parameter p and the dimensionless pump parameter $r \equiv R/R_{\rm th}$ the spectrum of the amplitude fluctuations reads as

$$\begin{aligned} V_A(\Omega) &= 1 + \frac{2bc(a+a')}{b-a'}\,\frac{1}{D(\tilde\Omega)}\Bigg((b^2+\tilde\Omega^2)\left[(a+a')^2+\tilde\Omega^2\right]\left[r + n\left(\frac{c}{a+a'}-1\right)\right] \\ &\quad + 2w^2\Bigg\{\left[(b-a')^2+\tilde\Omega^2\right]\left[n-\tfrac{1}{2}pr\right] - \left[(b-a')(a+a')+\tilde\Omega^2\right]\left(r - \frac{a+2a'}{a+a'}n\right) \\ &\quad + \left[(a+a')^2+\tilde\Omega^2\right]\left(\frac{a'}{a+a'}n - \frac{b}{c}\,\frac{b-a'}{a+a'}\right)\Bigg\}\Bigg)\,. \end{aligned} \tag{5.5}$$

The following shorthands have been introduced:

$$D(\tilde\Omega) = \left| -i\tilde\Omega(\tfrac{1}{2}+c-i\tilde\Omega)(b-i\tilde\Omega)(a+a'-i\tilde\Omega) + 2w^2(a+b-2i\tilde\Omega)(1-i\tilde\Omega)\right|^2, \tag{5.6}$$

$$n = \frac{ra+b+a'(r-1)}{a+b}, \quad w^2 = \frac{(a+a')bc}{2(a+b)}\,(r-1), \tag{5.7}$$

with w being the dimensionless Rabi frequency for the laser medium inside the cavity.

We investigate now the behavior of the spectrum given by Eq. (5.5) for the four different classes of lasers described in the Introduction. We start with the simplest case (1) when all the atomic relaxation rates are much faster than the field relaxation rate. This is the situation for which the atomic variables can be adiabatically eliminated.

(i) First class of lasers ($a, b, c, a' \gg 1$). In this case, for dimensionless frequencies of order of unity, $\tilde\Omega \sim 1$, we can simplify the general result (5.5) for the spectrum to

$$V_A(\Omega) = 1 - \frac{r}{(r-1)^2 + r^2\tilde\Omega^2}\left[\frac{b-a'}{b+a}\,p(r-1) - 2\frac{a+a'}{b+a}\,\frac{a'}{b-a'}\,(r-1) - 2\frac{b}{b-a'}\right]. \tag{5.8}$$

One can see from (5.8) that the parameter c does not appear in the fluctuation spectrum of photocurrent even when being close to the laser threshold. Far above threshold, $r \gg 1$, this result looks even more simple

$$V_A(\Omega) = 1 - \frac{1}{1+\tilde{\Omega}^2}\left[\frac{b-a'}{b+a}\,p - 2\,\frac{a+a'}{b+a}\,\frac{a'}{b-a'}\right]. \tag{5.9}$$

For regular pumping, $p = 1$, and in the absence of spontaneous emission in the laser transition, $a' = 0$, this formula coincides with the results of previous work [1] and shows the possibility of perfect noise reduction in the region of low frequencies. The necessary condition for this is a low relaxation rate of the upper lasing level in comparison with the relaxation rate of the lower level, $a \ll b$. Formula (5.9) shows also the influence on the photon statistics of the spontaneous emission between the lasing levels. One can see that spontaneous emission destroys the squeezing at low frequencies. This is in agreement with the conclusion obtained by Kennedy and Walls in [1] for the particular case $a = 0$; our formula (5.9) coincides in this case with the corresponding result in their paper. Figure 2 displays the spectrum of the amplitude fluctuations for several pump parameters r. Noise reduction at low frequencies increases with the pumping rate, largest values being obtained when the laser operates far above threshold.

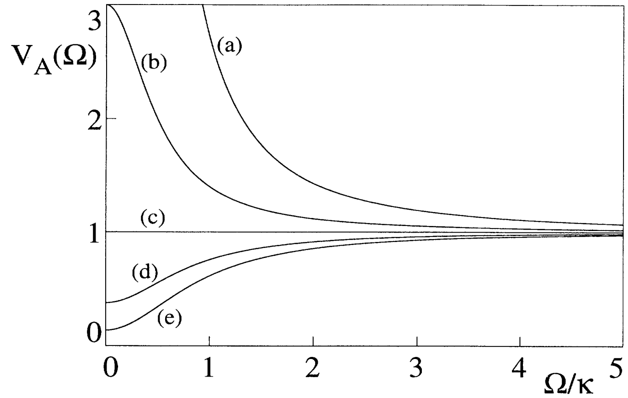

FIG. 2. Normalized spectrum of amplitude fluctuations for a laser of the first class for several pump parameters r. (a) $r = 1.1$, (b) $r = 2$, (c) $r = 3$, (d) $r = 5$, and (e) $r = 10$. For all curves $a' = a = 0$, $p = 1$.

(ii) Second class ($c \gg 1 \sim a, b, a'$) and third class ($c \gg 1 \gg a, b, a'$) of lasers. We consider now the second and third classes of lasers, as defined in the Introduction. We set $c \to \infty$ in the general expression (5.5) but keep the parameters a and b arbitrary. For simplicity we set $a' = 0$. After some algebra we get for the spectrum of amplitude fluctuations

$$V_A(\Omega) = 1 + \frac{2a}{Q^2(\Omega_1^2-\tilde{\Omega}^2)^2+\tilde{\Omega}^2(\Omega_2^2-\tilde{\Omega}^2)^2}\left\{(b^2+\tilde{\Omega}^2)(a^2+\tilde{\Omega}^2)n/a + \varepsilon\left[(b^2+\tilde{\Omega}^2)(n-\tfrac{1}{2}pr)-(ab+\tilde{\Omega}^2)(r-n)\right]\right\}, \tag{5.10}$$

with

$$\varepsilon \equiv \frac{ab}{a+b}\,(r-1),\quad Q \equiv a+b+2\varepsilon,$$
$$\Omega_1^2 \equiv \frac{ab(a+b)(r-1)}{a^2+b^2+2abr},\ \text{and}\ \Omega_2^2 \equiv \frac{2ab}{a+b}(r-1)+abr. \tag{5.11}$$

From (5.10) one can see that, because $r-n = (r-1)b/(a+b)$, the last term on the right-hand side is always negative. Note also that this term does not depend on the statistical parameter p and is present even for Poissonian pumping. Thus, the question arises as to whether this could result in shot noise reduction for some region of the parameters a, b, and r. From (5.10) one can also see that the first term in square brackets is proportional to r, while the second term (negative contribution included) is proportional to r^2. This suggests that one should consider the situation $r \gg 1$, corresponding to the laser operating far above threshold. We also set now $p = 0$ to keep only the negative contribution in question, thus isolating it from the effects associated with the regularization of the pumping process. We arrive to

$$V_A(\Omega) = 1 + \frac{2(a-b)\tilde{\Omega}^2}{4b(\Omega_1^2-\tilde{\Omega}^2)^2+b(2+a+b)^2\tilde{\Omega}^2} \tag{5.12}$$

with $\Omega_1^2 \approx (a+b)/2$. From (5.12) we can see that maximum shot noise reduction is obtained for $a \ll b$. In this case the spectrum (5.12) becomes

$$V_A(\Omega) = 1 - \frac{1}{2}\frac{\tilde{\Omega}^2}{(b/2-\tilde{\Omega}^2)^2+\tilde{\Omega}^2(1+b/2)^2} \tag{5.13}$$

which exhibits a minimum noise level equal to

$$V_A(\Omega)_{\min} = 1 - \frac{1}{2}\,\frac{1}{(1+b/2)^2} \tag{5.14}$$

at the frequency $\tilde{\Omega}_0 = (b/2)^{1/2}$. For $b/2 \ll 1$, which is typical of the third class of lasers mentioned in the Introduction, this dip is 50% below the shot-noise level. Going back to the original frequencies and decay constants [cf. Eq. (5.4)], we see that the frequency at which the minimum noise occurs is given by

$$\Omega_0 = \sqrt{\kappa\gamma_b/2}, \tag{5.15}$$

that is, just the geometrical mean between the decay constant of the lower level γ_b and the field decay constant $\kappa/2$.

Therefore, we can summarize the present analysis in the following way: when a, $b \ll 1 \ll c$ (third class of lasers), amplitude noise can be reduced to a level *up to 50% below shot noise,* at the frequency given by (5.15), *even for Poissonian pumping statistics.*

This result should be contrasted with the dynamic pump-noise suppression effect recently discussed by Ritsch *et al.*, Hart and Kennedy, and Ralph and Savage [1, 3]. In these papers, a closed system of atomic levels is considered, and the reduction of quantum noise below the shot level is due to regularization of the pumping statistics by the internal dynamics of the system. Noise quenching is obtained in the good-cavity limit: the spectrum of amplitude fluctuations has the shape of a Lorentzian function with a dip below the shot-noise level around zero frequency.

On the other hand, our model involves an *open* system of states, which decay to other levels. Therefore, the depletion of the ground state does not play a role, and the origin of the subshot noise behavior is quite different. We note in particular that the minimum noise level in our case is not at zero frequency, and the shape of the amplitude fluctuation spectrum is not Lorentzian.

Figure 3 shows the dependence of the spectrum (5.10) on the statistical parameter p, for operation far above threshold. One can see that in addition to the dip at nonzero frequency which shows up when the pumping is Poissonian, there is a narrow dip (with a width given by the atomic decay rate) around zero frequency, due to the regularization of the pumping. For regular pumping, $p = 1$, one can get complete noise quieting at zero frequency, as in the case of lasers of the first class.

In Fig. 4 the spectrum of amplitude fluctuations is displayed for different pump parameters r, when the pumping is Poissonian. The noise reduction at nonzero frequencies takes place only when operation is very far above threshold. As the parameter r decreases, we get excess noise for a wide range of frequencies, and the dip below the shot-noise level disappears.

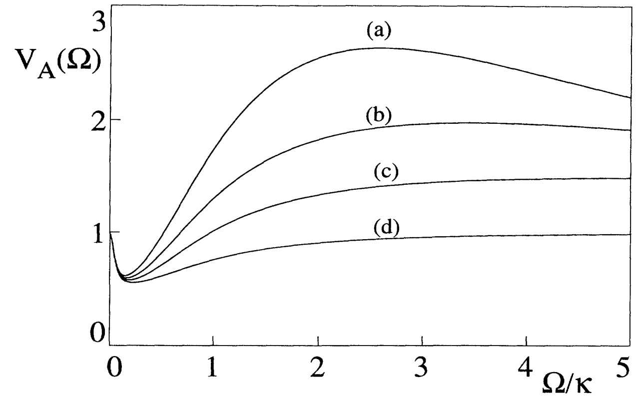

FIG. 4. Normalized spectrum of amplitude fluctuations for a laser of the third class for Poissonian pumping, $p = 0$, and different pump parameters r. (a) $r = 3 \times 10^3$, (b) $r = 5 \times 10^3$, (c) $r = 10^4$, and (d) $r = 10^6$. For all curves $a' = 0$, $a = 0.001$, $b = 0.1$.

A discussion related to ours, and applicable also to lasers of the second and third classes, has been presented in Ref. [23], in which the authors have used a simple classical approach with rate equations for the populations of the lasing levels and for the photon number inside the cavity for calculation of the photocurrent spectrum of the laser radiation. They have considered two particular cases, $a = b$ and $a = 0, b \to \infty$. Our general formula (5.10) reproduces their results in these two cases. This can be considered as a justification of the intuitive model used in Ref. [23]. One should remark, however, that our new result on the 50% noise reduction in the Poissonian pumping case cannot be obtained in those two limits. Even though $b \gg a$ in our case, we must still keep b fi-

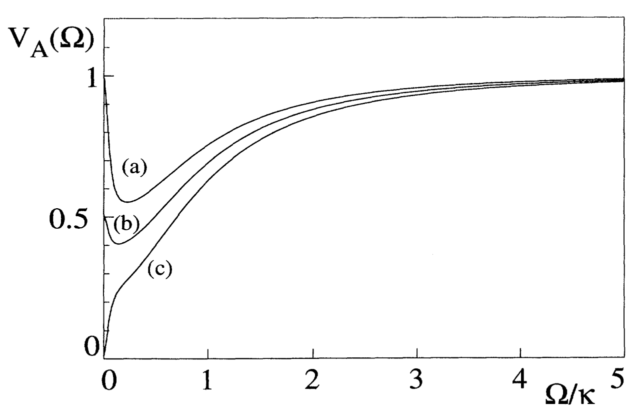

FIG. 3. Normalized spectrum of amplitude fluctuations for a laser of the third class for different statistical parameters p. (a) $p = 0$ (Poissonian pumping), (b) $p = 0.5$ (intermediate case), (c) $p = 1$ (regular pumping). High-above-threshold operation, $r = 10^6$. For all curves $a' = 0$, $a = 0.001$, $b = 0.1$.

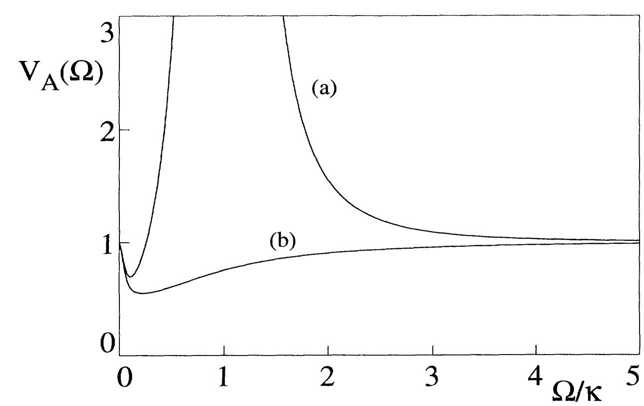

FIG. 5. Normalized spectrum of amplitude fluctuations for a laser of the fourth class for Poissonian pumping, $p = 0$, and different pump parameters r. (a) $r = 10^3$, (b) $r = 10^6$. For both curves $a' = 0$, $a = 0.001$, $b = 0.1$, $c = 0.5$.

nite, so that the minimum of the spectrum, given by Eq. (5.15), is not displaced to infinity.

(iii) Fourth class of lasers $(a,b,c \leq 1)$. Now we analyze the last family of lasers, for which $a,b,c \leq 1$ (we consider again the case when $a' = 0$). It is interesting to compare this class with the second and third classes (for which $c \to \infty$) and to understand the influence of the atomic polarization dynamics on the noise spectrum of the laser radiation.

We consider the laser operation very far above threshold, $r \gg 1$. Then, we can simplify the general result (5.6), which becomes

$$D(\tilde{\Omega}) = \left\{2w^2(a+b-2\tilde{\Omega}^2) - \tilde{\Omega}^2\left[ab - \tilde{\Omega}^2 + (\tfrac{1}{2}+c)(a+b)\right]\right\}^2 + \tilde{\Omega}^2\left[(\tfrac{1}{2}+c)(ab-\tilde{\Omega}^2) + 2w^2(2+a+b) - (a+b)\tilde{\Omega}^2\right]^2$$
$$\approx (2w^2)^2\left[(a+b-2\tilde{\Omega}^2) + \tilde{\Omega}^2(2+a+b)^2\right] \tag{5.16}$$

in the range of frequencies $\tilde{\Omega} \sim 1$. For the spectrum of amplitude fluctuations we get an expression which coincides precisely with (5.12) (for $p = 0$). Thus, when very far above threshold, the dynamics of the atomic polarization plays no role in the determination of the spectrum of the quantum fluctuations of the laser radiation.

On the other hand, as seen from Fig. 4, when the pump parameter r decreases, the noise grows and the dynamics of the polarization starts affecting the results. A long-lived polarization contributes, however, to *increasing* of the noise of the amplitude quadrature. This is illustrated in Fig. 5. The new feature, associated with the transient of the atomic polarization, is the sharp resonance peak around the frequency of the relaxation oscillations [24]. The frequency of the relaxation oscillations decreases as r decreases. Because of this large excess noise which shifts towards lower frequency values as r decreases, the dip below the shot-noise level disappears for laser operation close to threshold. This behavior of the amplitude noise spectrum should be contrasted with the narrowing of the line shape discussed in Sec. IV, which occurred for the same region of parameters: a long-living polarization in a laser is thus seen to decrease the phase noise and correspondingly increase the amplitude noise.

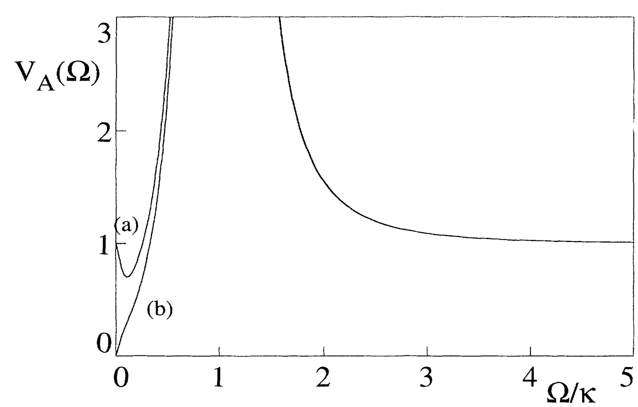

FIG. 6. Normalized spectrum of amplitude fluctuations for a laser of the fourth class for Poissonian pumping, $p = 0$ [curve (a)] and regular pumping, $p = 1$ [curve (b)]. For both curves $a' = 0$, $a = 0.001$, $b = 0.1, c = 0.5$; $r = 10^3$.

Figure 6 represents amplitude noise spectra for the fourth class of lasers, operating far above threshold, for two different pumping statistics: Poissonian (a), and regular (b). One can see that also in this case there is a 100% dip below the shot-noise level around zero frequency, due to the regularization of the pumping. The width of this dip is of the order of the atomic relaxation rates, i.e., much less than unity for this class of lasers.

VI. CONCLUSION

Although of great theoretical and practical interest, the problem of quantum noise quenching in lasers has been treated, up to now, only in some special limiting cases, due to the inherent complexity of the system. Proposals for noise quenching stemming from these treatments have included the regularization of the pumping [1, 2], correlated-emission schemes [5], the reduction of spontaneous-emission noise for short measurement times due to atomic memory effects [6], or dynamical pump-noise suppression, through the recycling of the active laser electron from the lower to the upper laser level via a multistep process, in a closed system of states [3].

We have developed in this paper a quite general analytical approach to this problem, which does not rely on the adiabatic elimination of the atomic variables, and at the same time takes into account the possibility of non-Poissonian pumping statistics. Our theory can therefore be applied to a variety of laser systems, covering a wide spectrum of relations between atomic and field decay constants. As simplifying assumptions, we have considered the active medium to be homogeneously broadened, and have only dealt with a zero-temperature bath. Of course, inhomogeneous broadening and temperature effects will tend to increase noise, and should be considered in a more complete treatment. We have decided, however, to avoid these extra complexities in this first approach, postponing the consideration of these effects to a later publication.

Our theory generalizes previous treatments of regularly pumped lasers, allowing a detailed investigation of the influence of polarization and population dynamics on noise quenching. We have shown that a long-lived polarization increases the noise in the amplitude quadrature, at the same time that it quenches the phase fluctuations. Furthermore, we have shown that careful consideration of atomic memory effects leads to a new possibility for generating sub-Poissonian light, *even for Poissonian pumping and for an open two-level system:* for fast polarization damping but slow population decay, as compared to the field decay rate, we obtain up to 50% noise quenching below the shot-noise level, at a frequency given by the geometrical mean of the cavity and lower-level decay rates. The width of the noise-reduced region increases with the pumping rate. This result stands out particularly if we consider the difficulties associated with the production of a regularized pumping, and the realization of a closed system of states.

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