Frequency shifts in spontaneous emission from two interacting atoms

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A model radiating system consisting of two atoms in close proximity is analyzed. This system demonstrates the influence of spatial coherence on the spectrum of the radiation field. Explicit expressions for the degree of coherence, the source spectrum, and the spectrum of the radiation field are derived. The results are discussed in terms of Wolf's work [Phys. Rev. Lett. **56**, 1370 (1986)] on this effect, which can be considered in terms of a multiple-atom analog of the effects of radiation reaction on a single atom, i.e., spontaneous decay and the Lamb shift.

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I. INTRODUCTION

In the past few years there has been a good deal of research into the effect of the spatial coherence properties of sources of radiation on the spectra of the field that the sources emit. It was first demonstrated theoretically [1] that unless the spectral degree of coherence of a planar secondary quasihomogeneous source obeys a certain scaling law, the normalized radiation spectrum of the field produced by such a source will differ from the normalized source spectrum [2]. This prediction has been subsequently verified experimentally by several groups [3–7]. Similar effects have been considered for radiation from primary sources [8–10], from illuminated pinholes (Young's interference experiment) [11–14], and for scattering, both from static [15,16] and from dynamic random media [17–19].

To get a better insight into the physics underlying this effect, it is desirable to examine the various processes by which radiation sources can become spatially correlated. In the investigations carried out so far, three possible mechanism have been considered: direct electronic manipulation of the coherence between the signals driving a pair of sources (e.g., acoustic transducers, radio antennae) [20–22]; control of spatial coherence of a secondary source by propagating incoherent light through specially designed optical systems [3–7,12–14]; and scattering of light from spatially correlated random media [15–19]. Primary sources of radiation have not received as much attention as other types of source, because of the inherent difficulty in controlling the spatial coherence of spatial distributions of optical radiators. The notable exception is the experiment of Bocko, Douglass, and Knox, who used acoustical rather than optical waves [21]. In this experiment the correlation of two random acoustical sources (transducers, driven by the filtered outputs of two noise generators) was controlled electronically.

It is generally assumed that when atomic vapor radiates, the fluorescence spectrum is proportional to the spectrum of the fluctuations of the dipole moments of the individual atoms. This assumption underlies all of atomic spectroscopy. However, because of the effect of spatial coherence of a source on the spectrum of the emitted radiation, it is clear that this assumption cannot be true in general. In fact, under certain circumstances the fluorescence spectrum may significantly differ from the spectrum of the atomic fluctuations. It is, therefore, important to identify the mechanisms which can give rise to statistical correlations between the fluctuations of two spatially separated atoms.

One such mechanism has recently been discussed by Varada and Agarwal [23]. When two atoms are in close proximity of each other, they will interact via their radiation fields. As we shall see, this interaction gives rise to three distinct effects: shifting of energy levels, changing of the lifetimes, and *introducing correlations between the fluctuations of the polarizations of the atoms.* Varada and Agarwal considered effects of correlations between two two-level atoms irradiated by a thermal field. They analyzed this system by numerically solving the master equation which describes the fluctuations of the two atoms. However, this model has the drawback that it is inherently anisotropic. The strength of the coupling between the two atoms is strongly dependent on the directional orientation of the fluctuations of the dipole moment. Therefore the vector nature of the dipole moment must be carefully considered.

In this paper we present an analysis of a two-atom system. We introduce an isotropic atomic model, thereby avoiding the problems associated with a two-level model. We perform the calculation in the Heisenberg picture and obtain explicit formulas for the correlation functions and the spectra of the atoms and of the field.

II. AN ISOTROPIC MODEL FOR THE ATOM

We will label the individual atoms of the atomic system under consideration by the subscripts α, β, etc. ($\alpha,\beta=1,2,\ldots,N$); thus $\mathbf{r}_\alpha$ is the position vector of the αth atom. Its dipole moment operator will be denoted by $\hat{\boldsymbol{\mu}}_\alpha$ [24]. Boldface characters denote vectors and boldface characters with overbars denote dyadics. The lowercase italic letters i, j, k, etc. will enumerate the three Cartesian components of vectors and dyadics. Greek letters λ, μ, ν, etc. will label different plane-wave modes of the electromagnetic field. Each of the atoms will be described by

a simple four-state model: three degenerate upper levels (a P state) and a singlet ground state (S state). The effects of electron spin are ignored. This model has the advantage over the more well-known two-level atom model in that there is *no* breaking of spherical symmetry. We will label the four states by their angular momentum quantum numbers l and m (see Fig. 1).

The dipole moment operator of the αth atom is given by the formula

$$\hat{\boldsymbol{\mu}}_\alpha = \sum_{m=-1}^{m=1} \langle l=0,m=0|_\alpha \hat{\boldsymbol{\mu}}_\alpha |l=1,m\rangle_\alpha |l=0,m=0\rangle_\alpha \times \langle l=1,m|_\alpha + \text{H.a.} , \tag{1}$$

where the ket $|l,m\rangle_\alpha$ denotes the state of the αth atom labeled by quantum numbers l and m, and H.a. stands for the Hermitian adjoint of the expression which precedes it. The transition dipole matrix elements $\langle l=0,m=0|_\alpha \hat{\boldsymbol{\mu}}_\alpha |l=1,m\rangle_\alpha$ can be calculated easily using the spherical harmonic representation of the angular momentum eigenstates. They are tabulated in many books (see, for example, Ref. [25]), and are

$$\begin{aligned} &\langle l=0,m=0|_\alpha \hat{\boldsymbol{\mu}}_\alpha |l=1,m=1\rangle_\alpha = -p\frac{1}{\sqrt{2}}(\mathbf{e}_1 + i\mathbf{e}_2) , \\ &\langle l=0,m=0|_\alpha \hat{\boldsymbol{\mu}}_\alpha |l=1,m=0\rangle_\alpha = p\mathbf{e}_3 , \\ &\langle l=0,m=0|_\alpha \hat{\boldsymbol{\mu}}_\alpha |l=1,m=-1\rangle_\alpha = -p\frac{1}{\sqrt{2}}(\mathbf{e}_1 - i\mathbf{e}_2) . \end{aligned} \tag{2}$$

Here $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are unit vectors along the three axes of quantization and p is a real-valued constant, dependent on the integral of the radial parts of the wave functions of the levels in question, e.g.,

$$p = \sqrt{3}e\int_0^\infty R_{n0}^*(r) R_{n'1}(r) r^3 dr , \tag{3}$$

where e is the charge of the electron and $R_{nl}(r)$ is the radial part of the wave function [labeled by the principal quantum numbers n (lower level) or n' (upper level), and the angular momentum quantum numbers $l=0$ and 1].

The notation may be simplified in the following way: Instead of using the eigenstates of the z component of the angular momentum operator $\hat{l}_3$ to describe the degenerate upper level, we use a set of linear combinations of them, chosen so that the transition dipole matrix elements are real and aligned along the three axes of quantization. We will use the following notation:

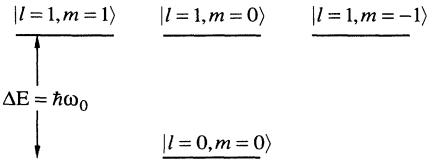

FIG. 1. A schematic diagram of the four-level atomic model introduced in Sec. II.

$$\begin{aligned} |0\rangle_\alpha &\equiv |l=0,m=0\rangle_\alpha , \\ |1\rangle_\alpha &\equiv -\frac{1}{\sqrt{2}}(|l=1,m=1\rangle_\alpha + |l=1,m=-1\rangle_\alpha) , \\ |2\rangle_\alpha &\equiv -\frac{i}{\sqrt{2}}(|l=1,m=1\rangle_\alpha - |l=1,m=-1\rangle_\alpha) , \\ |3\rangle_\alpha &\equiv |l=1,m=0\rangle_\alpha . \end{aligned} \tag{4}$$

We then have the simpler expression for the dipole moment operator:

$$\hat{\boldsymbol{\mu}}_\alpha = p(\mathbf{e}_1|0\rangle_\alpha\langle 1|_\alpha + \mathbf{e}_2|0\rangle_\alpha\langle 2|_\alpha + \mathbf{e}_3|0\rangle_\alpha\langle 3|_\alpha) + \text{H.a.} \tag{5}$$

Let us introduce the non-Hermitian vector transition operator for the αth atom,

$$\hat{\mathbf{b}}_\alpha = \sum_{i=1}^{3} \mathbf{e}_i |0\rangle_\alpha \langle i|_\alpha , \tag{6}$$

with $i=1$, 2, and 3 enumerating the three Cartesian directions x, y, and z, respectively. These operators have the following commutation properties:

$$\begin{aligned} &[\hat{\mathbf{b}}_\alpha, \hat{\mathbf{b}}_\beta] = 0 , \\ &[\hat{\mathbf{b}}_\alpha^\dagger, \hat{\mathbf{b}}_\beta] = 0 \quad (\alpha \neq \beta) \\ &[\hat{\mathbf{b}}_\alpha^\dagger, \hat{\mathbf{b}}_\alpha] = \sum_{i,j=1}^{3} \mathbf{e}_i \mathbf{e}_j |i\rangle_\alpha \langle j|_\alpha - \bar{\mathbf{I}}|0\rangle_\alpha\langle 0|_\alpha . \end{aligned} \tag{7}$$

Here $\bar{\mathbf{I}}$ denotes the unit dyadic.

Using this notation, it is not difficult to show that the Hamiltonian operator for the N-atom system interacting with the electromagnetic field is given by

$$\hat{H} = \hbar\omega_0 \sum_\alpha \hat{\mathbf{b}}_\alpha^\dagger \cdot \hat{\mathbf{b}}_\alpha + \sum_\lambda \hbar\omega_\lambda \hat{a}_\lambda^\dagger \hat{a}_\lambda + i\hbar \sum_\alpha \sum_\lambda \mathbf{g}^\lambda \cdot \{\hat{a}_\lambda^\dagger (\hat{\mathbf{b}}_\alpha + \hat{\mathbf{b}}_\alpha^\dagger) \exp(-i\mathbf{k}_\lambda \cdot \mathbf{r}_\alpha) - \text{H.a.}\} , \tag{8}$$

where ω_0 is the resonance frequency of the atom transition in question, $\hat{a}_\lambda$ is the annihilation operator for the λth mode of the electromagnetic field, with wave vector $\mathbf{k}_\lambda$, frequency ω_λ ($|\mathbf{k}_\lambda| = \omega_\lambda/c$, c being the speed of light), and polarization given by the unit vector $\boldsymbol{\epsilon}^\lambda$ [26]. The dagger denotes the Hermitian adjoint operator. The symbol $\hbar$ is Planck's constant divided by 2π. The vector $\mathbf{g}^\lambda$ is given by the expression

$$\mathbf{g}^\lambda = p\left[\frac{\omega_\lambda}{2\epsilon_0 V\hbar}\right]^{1/2} \boldsymbol{\epsilon}^\lambda , \tag{9}$$

where V is the quantization volume for the electromagnetic field, p is given by Eq. (3), and ϵ_0 is the vacuum permittivity.

It is possible to define a vector operator analogous to $\hat{\mathbf{b}}_\alpha$ for a more general electric dipole transition, from a $(2l+1)$-fold degenerate l state to a $(2l+3)$-fold degenerate $l+1$ state. The Hamiltonian for such an atomic model may be shown to be formally the same as that given by Eq. (8). However, the commutation relations are considerably more complicated than those given in Eq. (7).

III. DERIVATION OF THE EQUATIONS OF MOTION

The Heisenberg equation of motion for the mode annihilation operator of the electromagnetic field is

$$\frac{d\hat{a}_\lambda}{dt}=\frac{1}{i\hbar}[\hat{a}_\lambda,\hat{H}] . \tag{10}$$

Substituting the Hamiltonian given by Eq. (8) and using the well-known commutation relations for the mode operators, viz.

$$\begin{aligned}&[\hat{a}_\lambda,\hat{a}_\mu]=0 ,\\ &[\hat{a}_\lambda,\hat{a}_\mu^\dagger]=\delta_{\lambda\mu} ,\end{aligned} \tag{11}$$

we obtain the following equation of motion for the field mode annihilation operator:

$$\frac{d\hat{a}_\lambda}{dt}=-i\omega_\lambda\hat{a}_\lambda+\sum_\alpha \mathbf{g}^\lambda\cdot(\hat{\mathbf{b}}_\alpha+\hat{\mathbf{b}}_\alpha^\dagger)\exp(-i\mathbf{k}_\lambda\cdot\mathbf{r}_\alpha) . \tag{12}$$

This equation may be solved in the Markov approximation (see Ref. [27], Sec. 7.4), to give

$$\begin{aligned}\hat{a}_\lambda(t)=\hat{a}_\lambda(0)\exp(-i\omega_\lambda t)&\\ -i\sum_\alpha \mathbf{g}^\lambda\cdot\{\hat{\mathbf{b}}_\alpha(t)\zeta^*(\omega_\lambda-\omega_0)+\hat{\mathbf{b}}_\alpha^\dagger(t)\zeta^*(\omega_\lambda+\omega_0)\}&\\ \times\exp(-i\mathbf{k}_\lambda\cdot\mathbf{r}_\alpha) ,&\end{aligned} \tag{13}$$

where ζ is the zeta function of Heitler (Ref. [28], p. 69),

$$\zeta(\omega)\equiv -i\lim_{T\to\infty}\int_0^T\exp(i\omega t)dt=\mathrm{P}\frac{1}{\omega}-i\pi\delta(\omega) , \tag{14}$$

the symbol P denoting the fact that the Cauchy principal part of the integral is taken.

Let us now consider the equation of motion for some general atomic variable described by the operator $\hat{Q}$. Using the Heisenberg equation again, with the same Hamiltonian, and substituting the expression for the mode operator (14) (using normal ordering of the operators), we obtain, after some calculation and dropping rapidly oscillating terms, the equation [29]

$$\begin{aligned}\frac{d\hat{Q}}{dt}=&-i(\omega_0+\Delta)\sum_\alpha[\hat{Q},\hat{\mathbf{b}}_\alpha^\dagger\cdot\hat{\mathbf{b}}_\alpha]-i\sum_{\substack{\alpha\beta\\ \alpha\neq\beta}}[\hat{Q},\hat{\mathbf{b}}_\alpha^\dagger\cdot\overline{\mathbf{\Omega}}(\mathbf{r}_{\alpha\beta})\cdot\hat{\mathbf{b}}_\beta]\\ &+\sum_{\alpha\beta}\tfrac{1}{2}\{\hat{\mathbf{b}}_\alpha^\dagger\cdot\overline{\mathbf{\Gamma}}(\mathbf{r}_{\alpha\beta})\cdot[\hat{Q},\hat{\mathbf{b}}_\beta]+[\hat{\mathbf{b}}_\alpha^\dagger,\hat{Q}]\cdot\overline{\mathbf{\Gamma}}(\mathbf{r}_{\alpha\beta})\cdot\hat{\mathbf{b}}_\beta\}+\frac{ip}{\hbar}\sum_\alpha\sum_i\{\hat{\mathbf{E}}_0^{(-)}(\mathbf{r}_\alpha,t)\cdot[\hat{Q},\hat{\mathbf{b}}_\alpha]+[\hat{Q},\hat{\mathbf{b}}_\alpha^\dagger]\cdot\hat{\mathbf{E}}_0^{(+)}(\mathbf{r}_\alpha,t)\} .\end{aligned} \tag{15}$$

In Eq. (15) $\mathbf{r}_{\alpha\beta}=\mathbf{r}_\alpha-\mathbf{r}_\beta$ and the operators $\hat{\mathbf{E}}_0^{(+)}$ and $\hat{\mathbf{E}}_0^{(-)}$ represent the positive and the negative frequency parts of the vacuum field, respectively [Ref. [30], Eq. (3.40)], i.e.,

$$\begin{aligned}\hat{\mathbf{E}}_0^{(+)}(\mathbf{r}_\alpha,t)=&\sum_\lambda i\left[\frac{\hbar\omega_\lambda}{2\epsilon_0 V}\right]^{1/2}\boldsymbol{\epsilon}^\lambda\hat{a}_\lambda(0)\exp(i\mathbf{k}_\lambda\cdot\mathbf{r}_\alpha-i\omega_\lambda t) ,\\ \hat{\mathbf{E}}_0^{(-)}(\mathbf{r}_\alpha,t)=&-\sum_\lambda i\left[\frac{\hbar\omega_\lambda}{2\epsilon_0 V}\right]^{1/2}\boldsymbol{\epsilon}^\lambda\hat{a}_\lambda^\dagger(0)\\ &\times\exp(-i\mathbf{k}_\lambda\cdot\mathbf{r}_\alpha+i\omega_\lambda t) .\end{aligned} \tag{16}$$

On taking the continuum limit ($V\to\infty$), we obtain, after some calculation, the following expressions for the coefficients in Eq. (15):

$$\begin{aligned}\overline{\mathbf{\Omega}}(\mathbf{r}_{\alpha\beta})=&\frac{3A}{4}\left\{\overline{\mathbf{I}}\left[y_0(k_0r_{\alpha\beta})-\frac{y_1(k_0r_{\alpha\beta})}{k_0r_{\alpha\beta}}\right]\right.\\ &\left.+\frac{\mathbf{r}_{\alpha\beta}\mathbf{r}_{\alpha\beta}}{r_{\alpha\beta}^2}y_2(k_0r_{\alpha\beta})\right\}-2\pi A\overline{\mathbf{I}}\delta^{(3)}(k_0\mathbf{r}_{\alpha\beta}) ,\\ \overline{\mathbf{\Gamma}}(\mathbf{r}_{\alpha\beta})=&\frac{3A}{2}\left\{\overline{\mathbf{I}}\left[j_0(k_0r_{\alpha\beta})-\frac{j_1(k_0r_{\alpha\beta})}{k_0r_{\alpha\beta}}\right]\right.\\ &\left.+\frac{\mathbf{r}_{\alpha\beta}\mathbf{r}_{\alpha\beta}}{r_{\alpha\beta}^2}j_2(k_0r_{\alpha\beta})\right\} ,\\ \Delta=&\frac{A}{k_0^3}\frac{1}{\pi}\mathrm{P}\int_0^\infty\frac{k^3(k-2k_0)}{k^2-k_0^2}dk .\end{aligned} \tag{17}$$

Here j_n and y_n are the nth-order spherical Bessel functions of the first and of the second kind, respectively (Ref. [31], Chap. 10), $k_0=\omega_0/c$ and A is the natural life time of the transition, given by the expression

$$A=\frac{k_0^3}{3\pi\epsilon_0\hbar}p^2 . \tag{18}$$

In Eq. (17) the integral expression for the frequency shift Δ due to the self-interaction (i.e., the Lamb shift) is divergent. This was to be expected since, as is well known, the Lamb shift cannot be calculated in a nonrelativistic model such as the one which we are using here. Therefore we will ignore this divergence, and we will assume from now on that the Lamb-shifted resonance frequency of the transition, $\bar{\omega}_0=\omega_0+\Delta$, is a finite quantity [32].

Equation (16) as it stands can be used to obtain the equations of motion governing the behavior of the atomic operators $\hat{\mathbf{b}}_\alpha$ by employing the commutation relations (7). The result is a rather complicated nonlinear equation which cannot be solved exactly. A great simplification can be made by introducing the *harmonic-oscillator approximation*. The resulting model is then a quantum analog of Lorentz's model for the atom. In essence this approximation implies that, in place of the complicated commutation relation for the atomic operators $\hat{\mathbf{b}}_\alpha$ given by Eq. (7), we use the approximation

$$[\hat{\mathbf{b}}_\alpha^\dagger,\hat{\mathbf{b}}_\alpha]\approx-\overline{\mathbf{I}} . \tag{19}$$

This approximation is valid in the case when the probability of occupation of the upper-level states of the atoms is always much smaller than the probability of occupation of the lower-level states. In the case of thermal equilibrium that we will be considering, this requirement places the following condition on the temperature T:

$$\exp\left[-\frac{\hbar\bar{\omega}_0}{K_B T}\right] \ll 1 , \tag{20}$$

K_B being Boltzmann's constant. For optical frequencies this condition is satisfied if $T \ll 3000$ K. By making this approximation we are also ignoring a temperature-dependent change of the widths the resonances which appear in the spectra. This effect is analogous to power broadening in laser-atom interactions. At the temperatures for which the inequality (20) is satisfied this effect is small.

Using the equation of motion, Eq. (15), the simplified commutation relation (19) and the fact that, for each atom, $\bar{\Gamma}(\mathbf{r}_{\alpha\alpha}) = A\bar{\mathbf{I}}$, the equation of motion for the vector atomic operator $\hat{\mathbf{b}}_\alpha$ is

$$\frac{d\hat{\mathbf{b}}_\alpha}{dt} + (i\bar{\omega}_0 - \tfrac{1}{2}A)\hat{\mathbf{b}}_\alpha + \sum_{\substack{\beta \\ \beta\neq\alpha}} [i\bar{\Omega}(\mathbf{r}_{\alpha\beta}) - \tfrac{1}{2}\bar{\Gamma}(\mathbf{r}_{\alpha\beta})]\cdot\hat{\mathbf{b}}_\beta = \frac{ip}{\hbar}\hat{\mathbf{E}}_0^{(+)}(\mathbf{r}_\alpha, t) . \tag{21}$$

IV. THE RADIATED FIELD

The far-zone field generated by a collection of atoms in a direction along some unit vector $\mathbf{n}$ may be shown to be given by the expression [33]

$$\hat{\mathbf{E}}_r^{(+)}(r\mathbf{n}, t) \approx \frac{p\omega_0^2}{4\pi\epsilon_0 c^2} \sum_\alpha \frac{\mathbf{n}\times[\mathbf{n}\times\hat{\mathbf{b}}_\alpha(t - R_\alpha/c)]}{R_\alpha} , \tag{22}$$

where $R_\alpha = |r\mathbf{n} - \mathbf{r}_\alpha|$ (see Fig. 2). We will make the following approximation, appropriate to the far zone:

$$R_\alpha = r - \mathbf{n}\cdot\mathbf{r}_\alpha . \tag{23}$$

The autocorrelation function of the far-zone field is then given by

$$\Gamma^{(1,1)}(r\mathbf{n}, r\mathbf{n}, \tau) \equiv \langle \hat{\mathbf{E}}_r^{(-)}(r\mathbf{n}, t)\cdot\hat{\mathbf{E}}_r^{(+)}(r\mathbf{n}, t+\tau)\rangle = \frac{I_0}{r^2}\sum_{\alpha,\beta} \mathrm{Tr}\left\{(\bar{\mathbf{I}} - \mathbf{nn})\cdot\bar{\mathbf{g}}^{\alpha\beta}\left[\tau - \frac{\mathbf{n}\cdot\mathbf{r}_{\alpha\beta}}{c}\right]\right\} , \tag{24}$$

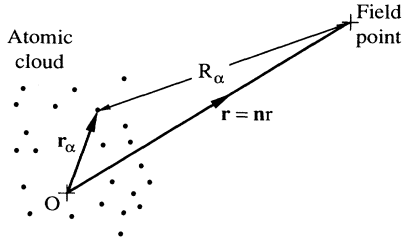

FIG. 2. Illustrating the notation used in Eq. (22).

where $I_0 = 3A\hbar\omega_0/16\pi\epsilon_0 c$, Tr stands for the trace operator of a dyadic, and we have assumed that the field and polarizations of the atoms are all stationary random processes. The atomic correlation dyadic which appears in Eq. (24) is given by

$$\bar{\mathbf{g}}^{\alpha\beta}(\tau) = \langle \hat{\mathbf{b}}_\alpha^\dagger(t)\hat{\mathbf{b}}_\beta(t+\tau)\rangle , \tag{25}$$

where the atomic polarizations have also been assumed to be statistically stationary.

As is well known, the power spectrum of the field is related to the autocorrelation function by the Wiener-Khintchine theorem (Ref. [34], p. 133)

$$S(r\mathbf{n}, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma^{(1,1)}(r\mathbf{n}, r\mathbf{n}, \tau)\exp(i\omega\tau)d\tau . \tag{26}$$

On substituting from Eq. (24) into Eq. (26) we see that

$$S(r\mathbf{n}, \omega) = \frac{I_0}{r^2}\sum_{\alpha,\beta} \mathrm{Tr}\left\{(\bar{\mathbf{I}} - \mathbf{nn})\cdot\bar{\mathbf{W}}^{\alpha\beta}(\omega)\exp\left[i\frac{\omega}{c}\mathbf{n}\cdot\mathbf{r}_{\alpha\beta}\right]\right\} , \tag{27}$$

where the atomic cross-spectral tensor $\bar{\mathbf{W}}^{\alpha\beta}(\omega)$ is defined in terms of the atomic correlation tensor as

$$\bar{\mathbf{W}}^{\alpha\beta}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\mathbf{g}}^{\alpha\beta}(\tau)\exp(i\omega\tau)d\tau . \tag{28}$$

The power spectrum of the fluctuations of the polarization of the αth atom may be defined in terms of the atomic correlation tensor by the formula

$$S_\alpha^{(\mathrm{at})}(\omega) \equiv \mathrm{Tr}\{\bar{\mathbf{W}}^{\alpha\alpha}(\omega)\} , \tag{29}$$

where trace operation is being performed over the dyadic and there is no summation over atoms. In the incoherent limit, the cross-spectral dyadic of the fluctuations of the atomic polarizations is given by the formula

$$\bar{\mathbf{W}}^{\alpha\beta}(\omega) = \tfrac{1}{3}\delta_{\alpha\beta}\bar{\mathbf{I}}S_\alpha^{(\mathrm{at})}(\omega) . \tag{30}$$

In this case the power spectrum of the field in the far zone may be found using Eqs. (27) and (30) to be

$$S(r\mathbf{n}, \omega) = \frac{2}{3}\frac{I_0 N}{r^2}S_\alpha^{(\mathrm{at})}(\omega) , \tag{31}$$

where N is the total number of atoms and we have assumed that the spectrum $S_\alpha^{(\mathrm{at})}(\omega)$ is the same for every atom. Thus in the incoherent limit the field spectrum in the far zone depends only on the power spectrum of the polarization fluctuations of the individual atoms. However, in general, there will exist a correlation between different atoms, i.e., $\bar{\mathbf{W}}^{\alpha\beta}(\omega) \neq 0$ for $\alpha \neq \beta$, and it is clear, on comparing Eqs. (27) and (29), that *there is no simple relationship between the field spectrum and the spectrum of the fluctuations of the polarization of the various atoms.*

We can also determine the power spectrum of the far-zone field, averaged over all directions of propagation. It is evidently given by the formula

$$S^{(\mathrm{av})}(\omega) = \frac{1}{4\pi}\int_{(4\pi)} S(r\mathbf{n}, \omega)d\Omega_{\mathbf{n}} , \tag{32}$$

where $d\Omega_{\mathbf{n}}$ is the element of solid angle about the direc-

tion $\mathbf{n}$. Using Eqs. (27) and (17) it is not difficult to show that

$$S^{(\mathrm{av})}(\omega)=\frac{2I_0}{3A}\sum_{\alpha,\beta}\mathrm{Tr}\{\overline{\mathbf{W}}^{\alpha\beta}(\omega)\cdot\overline{\mathbf{\Gamma}}(\mathbf{r}_{\alpha\beta})\}\ . \tag{33}$$

V. SOLUTION OF THE EQUATIONS OF MOTION FOR A PAIR OF ATOMS IN A THERMAL FIELD

In this section we will solve the equations of motion for the case of a system consisting of two atoms. Two atoms interacting via the quantized radiation field is a system which has been a subject of a great deal of study (see, for example, Refs. [35–38]). Usually some initial condition is imposed; for example, one or both of the atoms are assumed to be initially completely excited, and the subsequent evolution of the system is studied. Here we take a different approach. We assume that the atoms have been illuminated by a thermal field for a period of time that is sufficiently long for any effect due to the initial conditions to have completely decayed. The nature of the atomic fluctuations is then related to the coherence properties of the thermal light.

We assume that the atoms are located on the z axis of a Cartesian coordinate system, separated by a distance d, with the origin of the coordinates midway between the two (see Fig. 3). The radiation field emitted by the atoms is observed in the far zone in some direction specified by the unit vector $\mathbf{n}$, whose spherical polar angles are (θ,ϕ).

Starting with the equation of motion (21), we obtain the following six coupled equations of motion:

$$\frac{d\hat{b}_{1x}}{dt}+\left[i\bar{\omega}_0+\frac{A}{2}\right]\hat{b}_{1x}+\left[i\Omega_\perp+\frac{\Gamma_\perp}{2}\right]\hat{b}_{2x}=\frac{ip}{\hbar}\hat{E}_x^{(+)}(\mathbf{r}_1,t)\ ,$$

$$\frac{d\hat{b}_{2x}}{dt}+\left[i\bar{\omega}_0+\frac{A}{2}\right]\hat{b}_{2x}+\left[i\Omega_\perp+\frac{\Gamma_\perp}{2}\right]\hat{b}_{1x}=\frac{ip}{\hbar}\hat{E}_x^{(+)}(\mathbf{r}_2,t)\ ,$$

$$\frac{d\hat{b}_{1y}}{dt}+\left[i\bar{\omega}_0+\frac{A}{2}\right]\hat{b}_{1y}+\left[i\Omega_\perp+\frac{\Gamma_\perp}{2}\right]\hat{b}_{2y}=\frac{ip}{\hbar}\hat{E}_y^{(+)}(\mathbf{r}_1,t)\ ,$$

$$\frac{d\hat{b}_{2y}}{dt}+\left[i\bar{\omega}_0+\frac{A}{2}\right]\hat{b}_{2y}+\left[i\Omega_\perp+\frac{\Gamma_\perp}{2}\right]\hat{b}_{1y}=\frac{ip}{\hbar}\hat{E}_y^{(+)}(\mathbf{r}_2,t)\ , \tag{34}$$

$$\frac{d\hat{b}_{1z}}{dt}+\left[i\bar{\omega}_0+\frac{A}{2}\right]\hat{b}_{1z}+\left[i\Omega_\parallel+\frac{\Gamma_\parallel}{2}\right]\hat{b}_{2z}=\frac{ip}{\hbar}\hat{E}_z^{(+)}(\mathbf{r}_1,t)\ ,$$

$$\frac{d\hat{b}_{2z}}{dt}+\left[i\bar{\omega}_0+\frac{A}{2}\right]\hat{b}_{2z}+\left[i\Omega_\parallel+\frac{\Gamma_\parallel}{2}\right]\hat{b}_{1z}=\frac{ip}{\hbar}\hat{E}_z^{(+)}(\mathbf{r}_2,t)\ .$$

The coefficients $\Omega_\perp$, $\Omega_\parallel$, and $\Gamma_\perp$, and $\Gamma_\parallel$ which appear in these equations are functions of the atomic separation distance d. Their functional form can be found by applying the general expressions (17) given above. The explicit expressions for these coefficients then are

$$\begin{aligned}
\Omega_\perp&=\frac{3A}{4}\left[y_0(k_0d)-\frac{y_1(k_0d)}{k_0d}\right]\ ,\\
\Omega_\parallel&=\frac{3A}{4}\left[y_0(k_0d)-\frac{y_1(k_0d)}{k_0d}+y_2(k_0d)\right]\ ,\\
\Gamma_\perp&=\frac{3A}{2}\left[j_0(k_0d)-\frac{j_1(k_0d)}{k_0d}\right]\ ,\\
\Gamma_\parallel&=\frac{3A}{2}\left[j_0(k_0d)-\frac{j_1(k_0d)}{k_0d}+j_2(k_0d)\right]\ .
\end{aligned} \tag{35}$$

The variations of these coefficients with the atomic separation distance are shown in Figs. 4 and 5.

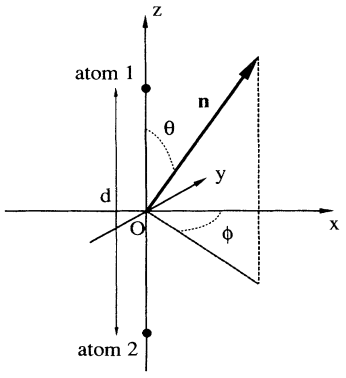

FIG. 3. The geometrical arrangement for the two-atom problem.

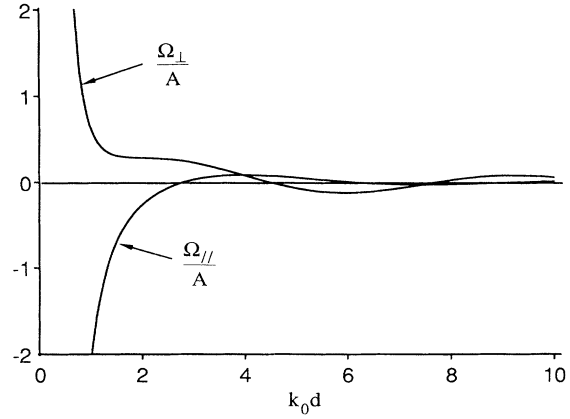

FIG. 4. The variation of the coefficients $\Omega_\perp$ and $\Omega_\parallel$ with the atomic separation d.

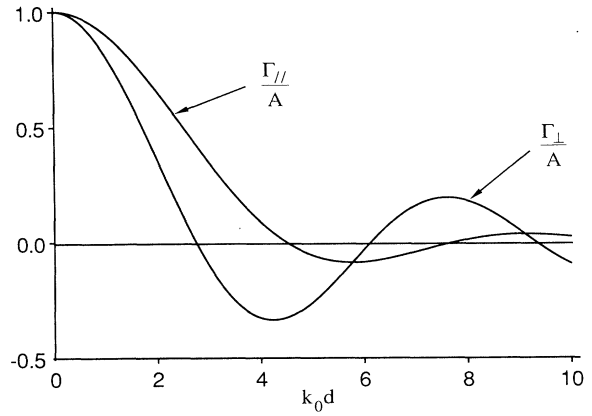

FIG. 5. The variation of the coefficients $\Gamma_\perp$ and $\Gamma_\parallel$ with the atomic separation d.

Methods for solving coupled equations such as Eqs. (34) are well known. We introduce the *normal modes* of oscillations, defined by the formulas

$$\hat{\mathbf{u}}(t)=\frac{1}{\sqrt{2}}[\hat{\mathbf{b}}_1(t)+\hat{\mathbf{b}}_2(t)] ,$$
$$\hat{\mathbf{v}}(t)=\frac{1}{\sqrt{2}}[-\hat{\mathbf{b}}_1(t)+\hat{\mathbf{b}}_2(t)] . \tag{36}$$

On substituting for $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$ from Eqs. (36) into Eqs. (34), it can shown that the normal modes $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are governed by the six uncoupled equations

$$\frac{d\hat{u}_x}{dt}+[i(\bar{\omega}_0+\Omega_\perp)+\tfrac{1}{2}(A+\Gamma_\perp)]\hat{u}_x=\hat{F}_x ,$$
$$\frac{d\hat{u}_y}{dt}+[i(\bar{\omega}_0+\Omega_\perp)+\tfrac{1}{2}(A+\Gamma_\perp)]\hat{u}_y=\hat{F}_y ,$$
$$\frac{d\hat{u}_z}{dt}+[i(\bar{\omega}_0+\Omega_\parallel)+\tfrac{1}{2}(A+\Gamma_\parallel)]\hat{u}_z=\hat{F}_z ,$$
$$\frac{d\hat{v}_x}{dt}+[i(\bar{\omega}_0-\Omega_\perp)+\tfrac{1}{2}(A-\Gamma_\perp)]\hat{v}_x=\hat{G}_x , \tag{37}$$
$$\frac{d\hat{v}_y}{dt}+[i(\bar{\omega}_0-\Omega_\perp)+\tfrac{1}{2}(A-\Gamma_\perp)]\hat{v}_y=\hat{G}_y ,$$
$$\frac{d\hat{v}_z}{dt}+[i(\bar{\omega}_0-\Omega_\parallel)+\tfrac{1}{2}(A-\Gamma_\parallel)]\hat{v}_z=\hat{G}_z .$$

The driving terms in these equations are given by a linear combination of the incident blackbody field at the locations of the two atoms, specifically

$$\hat{\mathbf{F}}(t)=\frac{ip}{\sqrt{2}\hbar}[\hat{\mathbf{E}}_0^{(+)}(\mathbf{r}_1,t)+\hat{\mathbf{E}}_0^{(+)}(\mathbf{r}_2,t)] ,$$
$$\hat{\mathbf{G}}(t)=\frac{ip}{\sqrt{2}\hbar}[-\hat{\mathbf{E}}_0^{(+)}(\mathbf{r}_1,t)+\hat{\mathbf{E}}_0^{(+)}(\mathbf{r}_2,t)] . \tag{38}$$

The solution of a set of first-order linear differential equations such as those appearing in Eq. (37) is not difficult to obtain (see the Appendix). For our purposes, the results are best expressed in terms of the cross-spectral dyadics, defined by the formulas

$$\overline{\mathbf{W}}^{(u)}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\langle\hat{\mathbf{u}}^\dagger(t)\hat{\mathbf{u}}(t+\tau)\rangle\exp(i\omega\tau)d\tau ,$$
$$\overline{\mathbf{W}}^{(v)}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\langle\hat{\mathbf{v}}^\dagger(t)\hat{\mathbf{v}}(t+\tau)\rangle\exp(i\omega\tau)d\tau , \tag{39}$$
$$\overline{\mathbf{W}}^{(uv)}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\langle\hat{\mathbf{u}}^\dagger(t)\hat{\mathbf{v}}(t+\tau)\rangle\exp(i\omega\tau)d\tau .$$

To determine these cross-spectral dyadics we need to find the cross-spectral dyadics for the inhomogeneous terms $\hat{\mathbf{F}}$ and $\hat{\mathbf{G}}$. They are given by the formulas

$$\overline{\mathbf{W}}^{(F)}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\langle\hat{\mathbf{F}}^\dagger(t)\hat{\mathbf{F}}(t+\tau)\rangle\exp(i\omega\tau)d\tau ,$$
$$\overline{\mathbf{W}}^{(G)}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\langle\hat{\mathbf{G}}^\dagger(t)\hat{\mathbf{G}}(t+\tau)\rangle\exp(i\omega\tau)d\tau , \tag{40}$$
$$\overline{\mathbf{W}}^{(FG)}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\langle\hat{\mathbf{F}}^\dagger(t)\hat{\mathbf{G}}(t+\tau)\rangle\exp(i\omega\tau)d\tau .$$

Using Eq. (38) these quantities can be expressed in terms of the cross-spectral tensor of the incident radiation field $\hat{\mathbf{E}}_0(\mathbf{r},t)$. For thermal radiation this tensor is known [39]. It is given by the formula

$$\begin{aligned}\overline{\mathbf{W}}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\omega)&\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}\langle\hat{\mathbf{E}}_0^{(-)}(\mathbf{r}_1,t)\hat{\mathbf{E}}_0^{(+)}(\mathbf{r}_2,t+\tau)\rangle\\&\quad\times\exp(i\omega\tau)d\tau\\&=\frac{S_0(\omega)}{2}\left\{\bar{\mathbf{I}}\left[j_0(kr_{12})-\frac{j_1(kr_{12})}{kr_{12}}\right]\right.\\&\quad\left.+\frac{\mathbf{r}_{12}\mathbf{r}_{12}}{r_{12}^2}j_2(kr_{12})\right\} ,\end{aligned} \tag{41}$$

where $k=\omega/c$, $\mathbf{r}_{12}=\mathbf{r}_1-\mathbf{r}_2$, and $r_{12}=|\mathbf{r}_{12}|$. The spectrum of the thermal field is given by the expression

$$\begin{aligned}S_0(\omega)&\equiv\mathrm{Tr}\{\overline{\mathbf{W}}^{(e)}(\mathbf{r},\mathbf{r},\omega)\}\\&=\frac{\hbar\omega^3}{4\pi^2\epsilon_0 c^3}\frac{1}{\exp(\hbar\omega/K_BT)-1} .\end{aligned} \tag{42}$$

Since $\overline{\mathbf{W}}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ is a slowly varying function of frequency ω, and since the resonances of the normal modes are very sharp, we can, to a good approximation, replace the frequency argument of $\overline{\mathbf{W}}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ by the resonance frequency ω_0. Using Eqs. (35), (38), and (41) we then obtain the following expressions for the components of the cross-spectral dyadics of $\hat{\mathbf{F}}$ and $\hat{\mathbf{G}}$:

$$W_{xx}^{(F)}(\omega)=W_{yy}^{(F)}(\omega)=\frac{\bar{n}}{2\pi}(A+\Gamma_\perp) ,$$
$$W_{zz}^{(F)}(\omega)=\frac{\bar{n}}{2\pi}(A+\Gamma_\parallel) ,$$
$$W_{xx}^{(G)}(\omega)=W_{yy}^{(G)}(\omega)=\frac{\bar{n}}{2\pi}(A-\Gamma_\perp) ,$$
$$W_{zz}^{(G)}(\omega)=\frac{\bar{n}}{2\pi}(A-\Gamma_\parallel) , \tag{43}$$
$$W_{ij}^{(F)}(\omega)=W_{ij}^{(G)}(\omega)=0\quad(i\neq j) ,$$
$$W_{ij}^{(FG)}(\omega)=0\quad(\text{for all } i,j) .$$

In these equations the symbol $\bar{n}$ stands for the photon occupation number at resonance, viz.

$$\bar{n}=\frac{1}{\exp(\hbar\bar{\omega}_0/K_BT)-1}\ . \tag{44}$$

From Eq. (43) we see that the different driving terms appearing in different equations in the set of equations (37) are uncorrelated. Because we are dealing with a situation where there is no dependence on the initial conditions, this implies that the different components of the normal models $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are all uncorrelated. Hence we can solve all the equations (37) to obtain the components of the cross-spectral dyadics of $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$. This may be done by a method outlined in the Appendix, and the result is

$$\begin{aligned}
&W_{xx}^{(u)}(\omega)=W_{yy}^{(u)}(\omega)=\bar{n}S_3(\omega)\ ,\\
&W_{zz}^{(u)}(\omega)=\bar{n}S_1(\omega)\ ,\\
&W_{xx}^{(v)}(\omega)=W_{yy}^{(v)}(\omega)=\bar{n}S_2(\omega)\ ,\\
&W_{zz}^{(v)}(\omega)=\bar{n}S_4(\omega)\ ,\\
&W_{ij}^{(u)}(\omega)=W_{ij}^{(v)}(\omega)=0\quad(i\neq j)\ ,\\
&W_{ij}^{(uv)}(\omega)=0\quad(\text{for all } i,j)\ ,
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
S_1(\omega)&=\frac{(A+\Gamma_\parallel)/2\pi}{(\omega-\bar{\omega}_0-\Omega_\parallel)^2+(A+\Gamma_\parallel)^2/4}\ ,\\
S_2(\omega)&=\frac{(A-\Gamma_\perp)/2\pi}{(\omega-\bar{\omega}_0+\Omega_\perp)^2+(A-\Gamma_\perp)^2/4}\ ,\\
S_3(\omega)&=\frac{(A+\Gamma_\perp)/2\pi}{(\omega-\bar{\omega}_0-\Omega_\perp)^2+(A+\Gamma_\perp)^2/4}\ ,\\
S_4(\omega)&=\frac{(A-\Gamma_\parallel)/2\pi}{(\omega-\bar{\omega}_0+\Omega_\parallel)^2+(A-\Gamma_\parallel)^2/4}\ .
\end{aligned} \tag{46}$$

Equation (45) shows that the six components of the normal modes are all mutually uncorrelated. This suggests that there may be some relation between the normal modes and the coherent mode representation introduced by Wolf in connection with the spectral representation of statistically stationary random sources [40].

It is now possible to calculate the components of the cross-spectral dyadic of the atomic fluctuations. The atomic operators are related to the normal modes by the equations

$$\begin{aligned}
\hat{\mathbf{b}}_1(t)&=\frac{1}{\sqrt{2}}[\hat{\mathbf{u}}(t)-\hat{\mathbf{v}}(t)]\ ,\\
\hat{\mathbf{b}}_2(t)&=\frac{1}{\sqrt{2}}[\hat{\mathbf{u}}(t)+\hat{\mathbf{v}}(t)]\ .
\end{aligned} \tag{47}$$

Equations (46) and (47) imply that the components of the atomic cross-spectral tensor are given by the formulas

$$\begin{aligned}
&W_{xx}^{11}(\omega)=W_{yy}^{11}(\omega)=W_{xx}^{22}(\omega)=W_{yy}^{22}(\omega)\\
&\qquad=\frac{\bar{n}}{2}[S_2(\omega)+S_3(\omega)]\ ,\\
&W_{zz}^{11}(\omega)=W_{zz}^{22}(\omega)=\frac{\bar{n}}{2}[S_1(\omega)+S_4(\omega)]\ ,\\
&W_{xx}^{12}(\omega)=W_{yy}^{12}(\omega)=W_{xx}^{21}(\omega)=W_{yy}^{21}(\omega)\\
&\qquad=\frac{\bar{n}}{2}[S_3(\omega)-S_2(\omega)]\ ,\\
&W_{zz}^{12}(\omega)=W_{zz}^{21}(\omega)=\frac{\bar{n}}{2}[S_1(\omega)-S_4(\omega)]\ .
\end{aligned} \tag{48}$$

All other components of these tensors are zero. It is important to remember that all the components are functions of the separation d between the two atoms because the coefficients $\Omega_\perp$, $\Omega_\parallel$, $\Gamma_\perp$, and $\Gamma_\parallel$, which appear in the definitions of $S_1(\omega)$, $S_2(\omega)$, $S_3(\omega)$, and $S_4(\omega)$, are all functions of the separation.

We can also express the various quantities derived in Sec. IV in terms of the spectra $S_1(\omega)$, $S_2(\omega)$, $S_3(\omega)$, and $S_4(\omega)$. In particular, the power spectrum of the atomic fluctuations, given by Eq. (29), may be expressed as

$$\begin{aligned}
S_1^{(\mathrm{at})}(\omega)&=S_2^{(\mathrm{at})}(\omega)\\
&=\bar{n}\left[\frac{S_1(\omega)}{2}+S_2(\omega)+S_3(\omega)+\frac{S_4(\omega)}{2}\right]\ .
\end{aligned} \tag{49}$$

The change of the atomic spectrum as the two atoms are brought close together is illustrated in Fig. 6.

The spectral degree of coherence between the two atoms is defined by the formula

$$\mu_{ij}(\omega)=\frac{W_{ij}^{12}(\omega)}{\sqrt{W_{ii}^{11}(\omega)}\sqrt{W_{jj}^{22}(\omega)}}\ . \tag{50}$$

Using Eq. (48) we see that the nonzero components of this dyadic are given by the formulas

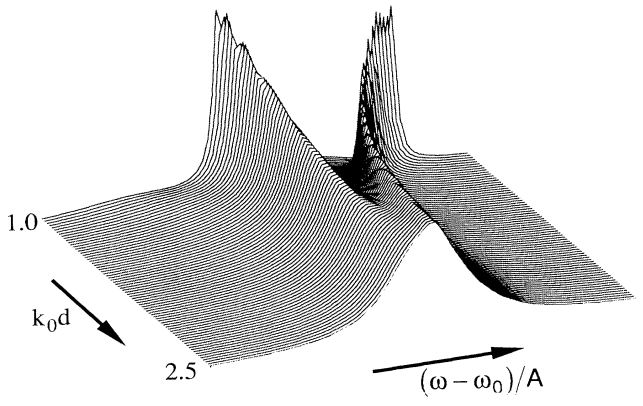

FIG. 6. The variation of the atomic spectrum $S_1^{(\mathrm{at})}(\omega)$ with the atomic separation d.

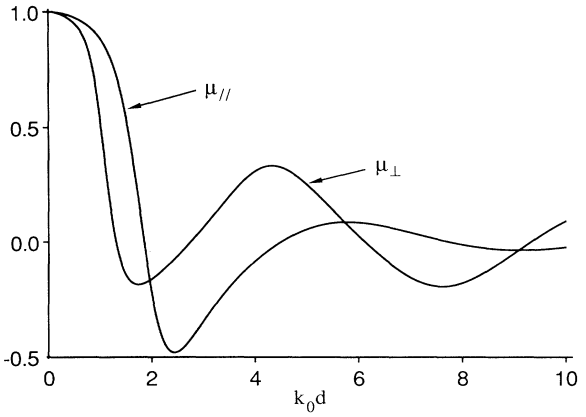

FIG. 7. The variation of the longitudinal and transverse components of the degree of spectral coherence dyadic with atomic separation d at frequency $\omega=\bar{\omega}_0$.

$$\mu_{xx}(\omega)=\mu_{yy}(\omega)=\frac{S_3(\omega)-S_2(\omega)}{S_3(\omega)+S_2(\omega)}\ ,$$
$$\mu_{zz}(\omega)=\frac{S_1(\omega)-S_4(\omega)}{S_1(\omega)+S_4(\omega)}\ . \tag{51}$$

The dependence of these components, at the frequency ω_0, on the atomic separation is illustrated in Fig. 7.

The power spectrum of the radiation field is given by Eq. (27). Using the expression (48) for the cross-spectral dyadic of the atoms, we obtain after some calculations, the following expression for the far-zone radiation spectrum:

$$S(r\mathbf{n},\omega)=\frac{2I_0\bar{n}}{r^2}\left\{[S_1(\omega)(2-\sin^2\theta)+S_2(\omega)\sin^2\theta]\cos^2\left[\frac{\omega d}{c}\cos\theta\right]+[S_3(\omega)(2-\sin^2\theta)+S_4(\omega)\sin^2\theta]\times\sin^2\left[\frac{\omega d}{c}\cos\theta\right]\right\}. \tag{52}$$

Here (θ,ϕ) denote, as before, the spherical polar angles of the vector $\mathbf{n}$ (see Fig. 3). Because of axial symmetry this result is independent of the angle ϕ.

In a similar manner we may determine the power spectrum of the far-zone field averaged over all directions of propagation [Eq. (33)]. One finds that

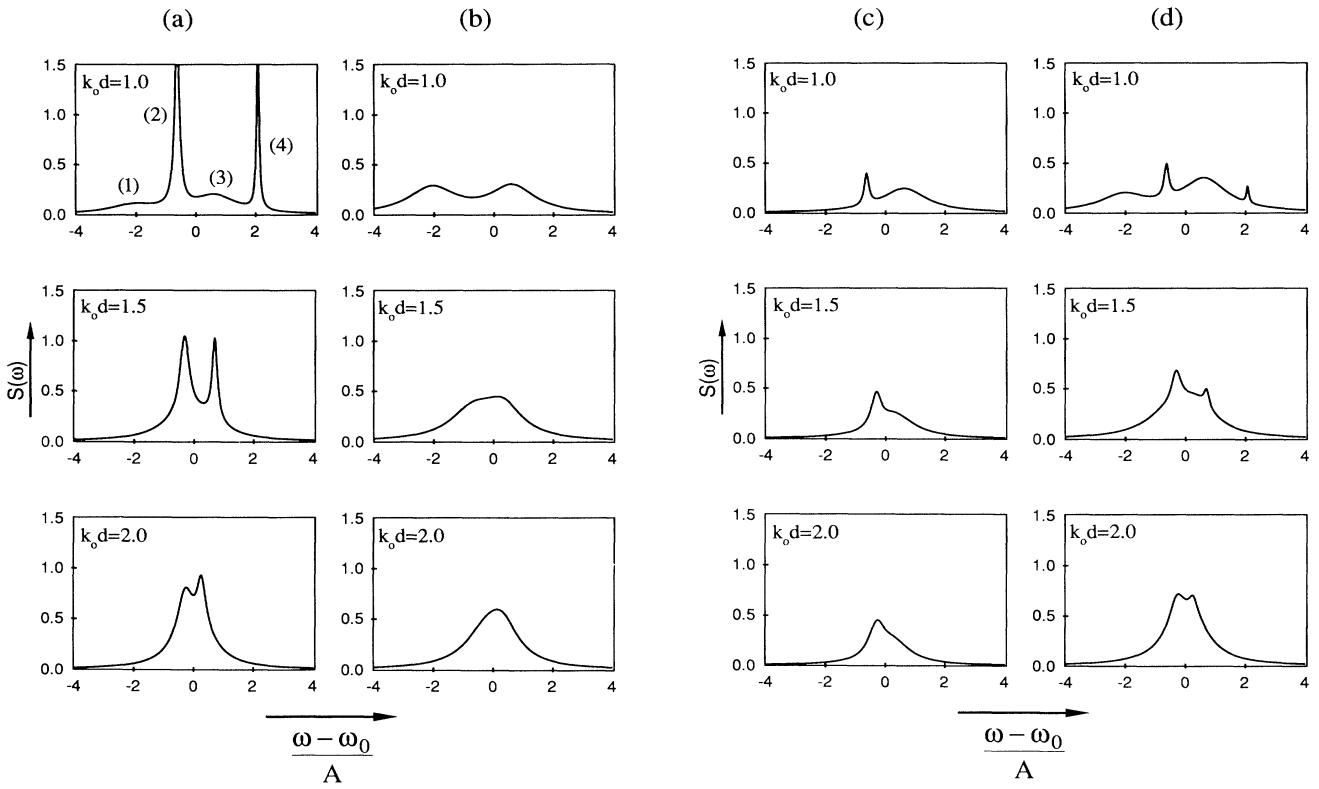

FIG. 8. (a) The spectrum of the atomic fluctuations. (b) The spectrum of the radiation field at an angle $\theta=90°$ (i.e., in the azimuthal plane). (c) The spectrum of the radiation field at an angle $\theta=0°$ (i.e., in the polar direction). (d) The spectrum of the field averaged over all directions of propagation. The spectra are shown for the following values of the atomic separation: $d=1.0/k_0$, $1.5/k_0$, and $2.0/k_0$.

$$S^{(\mathrm{av})}(\omega)=\frac{2I_0\bar{n}}{3r^2}\left[2S_1(\omega)\left[1+\frac{\Gamma_\perp}{A}\right]+S_2(\omega)\left[1+\frac{\Gamma_\parallel}{A}\right]+2S_3(\omega)\left[1-\frac{\Gamma_\perp}{A}\right]+S_4(\omega)\left[1-\frac{\Gamma_\parallel}{A}\right]\right] . \quad (53)$$

In Fig. 8 the following power spectra are shown: the spectrum of the atomic fluctuations; the spectrum of radiation field at an angle $\theta=90°$ (i.e., in the azimuthal plane); the spectrum of the radiation field at an angle $\theta=0°$ (i.e., in the polar direction); and the spectrum of the field averaged over all directions of propagation. The spectra are shown for three different values of the atomic separation, namely $d=1.0/k_0$, $1.5/k_0$, and $2.0/k_0$. These curves are normalized so that, for values of the atomic separation much larger than the wavelength, the atomic spectrum and the field spectrum averaged over all directions are both Lorentzian curves of unit height.

VI. DISCUSSION AND CONCLUSION

We have shown that the fluctuations of a two-atom system can be considered in terms of six normal modes, two for each of three mutually orthogonal directions. Because of the rotational symmetry of our model in the x,y plane, the four normal modes corresponding to directions of oscillation in this plane form two pairs of modes, each pair having a different resonance frequency and a different decay parameter. Thus there are four identifiable features which appear in the spectra that we have calculated; they are the four Lorentzian spectral resonances $S_1(\omega)$, $S_2(\omega)$, $S_3(\omega)$, and $S_4(\omega)$. The peaks corresponding to these four terms are indicated by (1), (2), (3), and (4) in the top left-hand graph of Fig. 8. The terms $S_1(\omega)$ and $S_4(\omega)$ are associated with the fluctuations in the z direction, while $S_2(\omega)$ and $S_3(\omega)$ are associated with the fluctuations in the x,y plane. These four features also appear in the spectra of the radiated field, with various strengths, depending on the direction of observation.

The separation distance d between the atoms affects the central frequencies and the widths of these four resonances. As can be seen from the figures, half of the resonances become broader and half become narrower. The broadening, which is equivalent to the reduction of the decay time of the appropriate normal mode, is associated with cooperative emission by the pair of atoms. This effect is analogous to superradiance in systems of containing many atoms. The narrower resonance corresponds to a longer lifetime mode. These modes can be considered as a photon being swapped back and forth repeatedly between the two atoms, resulting in a quasistable state of the system.

Instead of considering the system in terms of the uncorrelated fluctuations of a set of normal modes, one can also consider it as a pair of atoms whose polarization fluctuations are partially correlated. The atomic fluctuation spectrum, which is the same for both atoms, is given by Eq. (49) and is illustrated in Figs. 6 and 8. The spectrum of the radiation field, given by Eqs. (52) and (53) and illustrated in Fig. 8, is significantly different from the atom fluctuation spectrum. This difference illustrates the effect of spatial coherence of sources on the spectra of radiation fields, as discussed in the Introduction. In this case, the spatial coherence of the source is induced between the two atoms due to their interaction. The corresponding spectral degree of coherence can be calculated [see Eq. (51) and Fig. 7]. It is found that it does not obey Wolf's scaling law [41]. Of particular significance is the difference between the atomic spectrum and the radiation spectrum averaged over all angles. This difference shows that partial coherence does more than just redistribute light of different frequencies in different directions.

Our analysis has elucidated the effect of the radiation interaction between two atoms. As mentioned above, this interaction gives rise to a change in the lifetimes and in the resonance frequencies of the atomic transitions involved. Further, the interaction gives rise to a *correlation between the fluctuations of the atomic polarizations,* which causes the spectra of the emitted radiation to change, as predicted by Wolf [1]. Moreover, the effects of the radiation reaction of an atom upon itself are well known. The atom self-interaction via the radiation field gives rise to the natural decay lifetime and to the Lamb shift in the resonance frequency. Thus the spectral effects we have discussed in this paper could be considered as the analog in multiple-atom systems to the spectral effects seen in one atom systems, as both arise from the interaction with the electromagnetic field generated by the atomic system.

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APPENDIX: SOLUTION OF A FIRST-ORDER LINEAR DIFFERENTIAL EQUATION FOR A RANDOM VARIABLE

Let us consider an operator $\hat{x}(t)$ which is related to some stationary random process $\hat{F}(t)$ by the differential equation

$$\frac{d\hat{x}(t)}{dt}+\left[i\omega_0+\frac{\gamma}{2}\right]\hat{x}(t)=\hat{F}(t) . \quad (A1)$$

The formal solution of this equation is

$$\hat{x}(t)=\int_{-\infty}^{\infty}dt'\hat{F}(t')\Theta(t-t')\exp\left[-\left[i\omega_0+\frac{\gamma}{2}\right](t-t')\right] , \quad (A2)$$

where the effect of the initial value $\hat{x}(-\infty)$ has completely decayed. In Eq. (A2), $\Theta(t)$ is the Heaviside step function defined as

$$\Theta(t)=\begin{cases}1 & \text{if } t>0\\ 0 & \text{if } t<0\end{cases}. \tag{A3}$$

We assume that the random process $\hat{F}(t)$ is statistically stationary. Hence its autocorrelation function depends on t_1 and t_2 only through the difference t_2-t_1:

$$\langle \hat{F}^{\dagger}(t_1)\hat{F}(t_2)\rangle=\Gamma^{(F)}(t_2-t_1)\ . \tag{A4}$$

Using Eq. (A2) we can determine the autocorrelation function of $\hat{x}(t)$. One finds that

$$\langle \hat{x}^{\dagger}(T-\tau/2)\hat{x}(T+\tau/2)\rangle=\int_{-\infty}^{\infty}d\tau'\int_{-\infty}^{\infty}dT'\Theta(T-\tau/2-T'+\tau'/2)\Theta(T+\tau/2-T'-\tau'/2)\times\Gamma^{(F)}(\tau')\exp[-i\omega_0(\tau-\tau')+\gamma(T'-T)]\ . \tag{A5}$$

This expression can be simplified if we recall that a product of Heaviside step functions can be expressed as

$$\Theta(T_1-T')\Theta(T_2-T')=\Theta(\min\{T_1,T_2\}-T')\ , \tag{A6}$$

where $\min\{a,b\}$ is equal to the numerically smaller of the quantities a or b. Applying this result to Eq. (A5), and using the fact that

$$\min\{(T-\tau/2+\tau'/2),(T+\tau/2-\tau'/2)\}=T-|\tau-\tau'|/2\ , \tag{A7}$$

we obtain the following expression for the autocorrelation function of $\hat{x}(t)$:

$$\langle \hat{x}^{\dagger}(T-\tau/2)\hat{x}(T+\tau/2)\rangle=\int_{-\infty}^{\infty}d\tau'\int_{-\infty}^{\infty}dT'\Theta(T-|\tau-\tau'|/2-T')\Gamma^{(F)}(\tau')\times\exp[-i\omega_0(\tau-\tau')+\gamma(T'-T)]\ . \tag{A8}$$

The integration with respect to T' can now be performed and yields an expression for the autocorrelation function of $\hat{x}(t)$,

$$\Gamma^{(x)}(\tau)\equiv\langle \hat{x}^{\dagger}(T-\tau/2)\hat{x}(T+\tau/2)\rangle=\frac{1}{\gamma}\int_{-\infty}^{\infty}d\tau'\Gamma^{(F)}(\tau')\times\exp[-i\omega(\tau-\tau')-\gamma|\tau-\tau'|/2]\ . \tag{A9}$$

We see that $\Gamma^{(x)}(\tau)$ is independent of the variable T, thereby implying that $\hat{x}(t)$ is stationary, at least in the wide sense.

The power spectrum $S^{(x)}(\omega)$ of $\hat{x}(t)$ is given by the Fourier transform of the autocorrelation function (A9), i.e.,

$$S^{(x)}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\tau\,\Gamma^{(x)}(\tau)\exp(i\omega\tau)\ . \tag{A10}$$

On substituting the expression for $\Gamma^{(x)}(\tau)$ given by Eq. (A9), and applying the convolution theorem, we readily find that the spectrum is given by the expression

$$S^{(x)}(\omega)=\frac{S^{(F)}(\omega)}{(\omega-\omega_0)^2+\gamma^2/4}\ . \tag{A11}$$

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