Leading role of optical phase instabilities in the formation of certain laser transverse patterns

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(Received 18 March 1991; revised manuscript received 22 July 1992)

The global dynamical behavior of a model describing the formation of one-dimensional transverse patterns in a laser has been investigated. Solutions of the equation for the complex electric field are compared to those of a reduced phase equation, the laser Kuramoto-Sivashinsky equation, which should be a good approximation for small detuning. The numerical results show strong evidence of the leading role of the dynamics of the phase of the field in the formation of transverse patterns in a laser with small detuning.

PACS number(s): 42.55.−f, 42.50.Lc, 42.60.Mi, 42.65.Pc

I. INTRODUCTION

The investigation of the dynamical behavior of nonlinear systems is one of the central tasks in modern science and has spread to many fields, including physics, chemistry, biology, hydrodynamics, engineering, and economics. Among these, nonlinear optics is one of the most active fields in which instability phenomena have been studied, particularly since the appearance of the laser and its many applications in the past 30 years. Although developments in the studies of instabilities in nonlinear optical systems have parallelled developments in other fields, there have been fewer studies (until quite recently [1]) focused on the spatial patterns or spatiotemporal phenomena rather than on the temporal behavior of optical systems, for example, the onset of spontaneous oscillations and periodic, quasiperiodic, and chaotic pulsations in the output intensity (see, for example, [1–3]). This interest in the temporal aspects has been due to the successful use of a plane-wave approximation to describe many laser phenomena which clearly use beams of finite transverse extent, many of which are described simply by Gaussian, Hermite-Gaussian, or Laguerre-Gaussian functions. The plane-wave approximation drastically simplifies the mathematical complexity of the models, but it eliminates all the spatial effects in the transverse directions.

Only in recent years has extensive attention been drawn to dynamical origins of transverse spatial phenomena in nonlinear optical systems (see, for example, [1,4–22]). An important subject is the spontaneous formation of spatial patterns in an initially homogeneous optical system [15–17,21,22]. The growth of this new research topic in the area of nonlinear optical systems has led to the discovery of interesting analogies and comparisons with other fields, especially chemistry and hydrodynamics, in which the emergence of spatiotemporal structures (or dissipative structures) is also a major topic.

From a mathematical viewpoint, the inclusion of the spatial variable leads to partial differential equations governing the dynamical behavior. The formation of spatial patterns relies on different physical mechanisms for the spatial partial derivatives: diffusion in chemical reactions, convection in hydrodynamics, and diffraction in optical systems. A common feature of these different systems is that their main dynamical behavior under certain conditions may be described by a reduced phase diffusion equation of the Kuramoto-Sivashinsky (KS) type, which is deduced by adiabatically eliminating the modulus of a complex amplitude variable [23–30]. In these cases, the spatial dependence of the phase of the complex variable contains the main information which dominates the formation of spatiotemporal structures.

A laser can be modeled by a single equation for the complex field amplitude when the dynamics of the material variables can be adiabatically eliminated. The result is a complex Ginzburg-Landau equation for the field amplitude [18]. The method of reducing the original laser field equation to a Kuramoto-Sivashinsky type of equation was presented in Refs. [31] and [32], and the numerical results reported there showed approximate agreement with the claim that phase dynamics governs the laser field.

As an extension of that analysis, in the present work we consider the situation of small detuning. In this case, the spatially homogeneous mode dominates the transverse pattern, so the conditions for truncating the perturbation expansion in the derivation of the laser KS equation are more accurately satisfied. Nevertheless, this condition does not prevent emergence of complicated spatiotemporal dynamical behavior. Indeed we find a variety of the phenomena that have been reported previously [15–17,21,33], such as inhomogeneous stationary patterns, spontaneous breaking and restoring of the parity symmetry, and time-dependent solutions, including periodic, period-doubled, and chaotic spatiotemporal oscillations. The numerical results show global qualitative agreement and almost perfect quantitative agreement between the original laser field equation and the reduced KS equation for the phase of the laser field.

II. THE EQUATION FOR THE LASER FIELD WITH CUBIC NONLINEARITY

We consider a simple laser model discussed in Refs. [15,31,32]. The cavity configuration is rectangular, so Cartesian coordinates are used. The longitudinal direction of the laser is the z axis. In the approximations of a slowly varying envelope and a single longitudinal mode, the z coordinate is eliminated. The laser cavity has, in addition to the usual mirrors orthogonal to the propagation direction z, two lateral mirrors, orthogonal to the transverse direction x, which work as a planar conducting waveguide. On the other hand, the cavity is left open in the y direction. Such a cavity configuration is appropriate to capture the essential elements of the one-dimensional patterns that result from slab or stripe semiconductor laser structures [34,35] and has also been explicitly used in waveguided far infrared lasers and microwave oscillators. One-dimensional transverse patterns have recently been observed and studied with the renewed interest in broad-area semiconductor lasers for pumping solid-state lasers and because of the similarity of lasers with one transverse dimension to one-dimensional laser arrays [36].

The TM cavity modes are determined by the waveguide configuration as discussed in Refs. [11,12,37] and are independent of the variable y. We assume that instabilities, which might give rise to a dependence of the electric field on y, do not occur; a sufficient condition to ensure this is that the size of the cavity in the y direction is small enough. As a consequence of these assumptions, the complex electric-field envelope E' depends only on time and the transverse spatial variable x. Thus, in the cubic approximation of the nonlinearity, the laser field equation is

$$\frac{\partial E'(x,t')}{\partial t'} = -E'(1-i\Delta)(|E'|^2-r)+ia\frac{\partial^2 E'(x,t')}{\partial x^2}, \tag{1}$$

where t' and x denote the time and transverse coordinate normalized by the inverse of the cavity linewidth κ and the transverse length of medium b, respectively. Δ is the detuning parameter and r is the pump parameter. The quantity a is defined as

$$a=\frac{\lambda L}{2\pi b^2 T}, \tag{2}$$

where T is the transmissivity coefficient of the mirrors, λ is the wavelength, and L is the longitudinal length of cavity. Physically, however, the parameter a defines the frequency spacing between transverse modes [15]. The nth mode deviates from the fundamental mode $n=0$ by $\omega_n-\omega_0=\kappa a n^2\pi^2$.

One notes that Eq. (1) has the form of a complex Ginzburg-Landau equation, with the peculiar feature that the coefficient of the second derivative is purely imaginary. An accurate adiabatic elimination of the rapidly relaxing variables, based on the center manifold theorem, reveals, however, that one obtains also a real contribution to the coefficient of the second derivative. This result was obtained by Coullet, Gil, and Rocca [18] for laser operation near threshold and generalized by Oppo *et al.* [38] to an equation appropriate for the full phase space. The correction (i.e., the real part in the coefficient of $\partial^2/\partial x^2$) is on the order of the ratio of the cavity linewidth to the atomic linewidth, which is assumed small in the adiabatic limit; it is, however, essential to ensure the stabilization of vortices in the two-dimensional case [18].

We neglect this correction throughout this paper. We are not able to justify this step on the basis of comparisons between results obtained with and without this term. However, we believe that its inclusion does not change the results in an important way for the following reasons.

(a) In our investigations, which are performed in the one-dimensional case neglecting this contribution, we do not meet any sort of singular pathology such as appears in the two-dimensional case.

(b) The inclusion of the real part in the coefficient of $\partial^2/\partial x^2$ amounts to replacing 1, which appears in the first term of the right-hand side of Eq. (7a), by $1+\varepsilon k^2$, where ε is a small number which tends to zero in the adiabatic elimination limit. If, as our results indicate, it is true that in our case the dynamics is governed by quite a limited number of modal amplitudes f_k, we assume that $\varepsilon k^2 \ll 1$ for all relevant modes, so that the correction should be unimportant.

A complete study of this point is left, however, for future investigation, because one cannot exclude *a priori* that modes with arbitrarily large values of k enter into play when the correction is included. As shown in Ref. [15], above the threshold for laser action ($r>0$), the number of independent parameters can be reduced, yielding a simpler equation

$$\frac{\partial E(x,t)}{\partial t} = -E(1-i\Delta)(|E|^2-1)+ia'\frac{\partial^2 E(x,t)}{\partial x^2}, \tag{3}$$

using scaled quantities defined by

$$t=rt', \tag{4a}$$

$$E=E'/r^{1/2}, \tag{4b}$$

and

$$a'=a/r. \tag{4c}$$

We assume that the field envelope E obeys the boundary condition

$$\frac{\partial E}{\partial x}=0 \tag{5}$$

at the lateral mirrors located at $x=0$ and 1. With this Neumann boundary condition, $E(x,t)$ can be expanded in transverse modes

$$E(x,t)=\sum_{n=0}^{\infty} f_n(t)F_n(x), \tag{6}$$

where $\{F_n(x)\}$ is the set of functions $F_0(x)=1$ and $F_n(x)=2^{1/2}\cos(n\pi x)$ for $n\neq 0$. We call the complex functions of time $f_n(t)$ the amplitudes of the field modes. A set of equations for the amplitudes of the field modes can be obtained by substituting Eq. (6) into Eq. (3),

$$\dot{f}_k=[1-i(\Delta+a'\pi^2k^2)]f_k-(1-i\Delta)\sum_{l,m,n}\Gamma_{klmn}f_lf_mf_n^* \quad (k=0,1,2,\ldots)\ , \tag{7a}$$

where the coupling coefficients are given by

$$\Gamma_{klmn}=\int_0^1 F_lF_mF_n^*F_k^*dx\ . \tag{7b}$$

There are both homogeneous and inhomogeneous stationary solutions to Eqs. (7) in which $|f_k|^2$ is time independent for all k. Stability analysis of the homogeneous stationary solution ($|f_0|=1, f_k=0$ for all $k>0$) yields the instability condition [15]

$$a'\pi^2n^2\equiv a'(n)<2\Delta\ . \tag{8}$$

For the inhomogeneous stationary solutions (which involve more than one nonzero modal amplitude), the transverse modes cooperatively select a common operating frequency, which is an average of the transverse mode frequencies $a'(n)$, weighted by the mode intensities $|f_n|^2$,

$$\delta=\frac{\sum_{n=1}^{\infty}a'\pi^2n^2|f_n|^2}{\sum_{n=0}^{\infty}|f_n|^2}\ . \tag{9}$$

III. LASER KURAMOTO-SIVASHINSKY EQUATION

The laser KS equation is a phase diffusion equation, corresponding to the laser field equation. The complex field envelope $E(x,t)$ can be written as

$$E(x,t)=R(x,t)\exp[i\Theta(x,t)]\ . \tag{10}$$

Substituting Eq. (10) into Eq. (3), we obtain a set of coupled equations for the amplitude R and phase Θ

$$\frac{\partial R}{\partial t}=R(1-R^2)-2a'\frac{\partial R}{\partial x}\frac{\partial\Theta}{\partial x}-a'R\frac{\partial^2\Theta}{\partial x^2}\ , \tag{11a}$$

$$\frac{\partial\Theta}{\partial t}=-\Delta(1-R^2)+\frac{a'}{R}\frac{\partial^2R}{\partial x^2}-a'\left[\frac{\partial\Theta}{\partial x}\right]^2\ . \tag{11b}$$

The conventional procedure to eliminate the amplitude starts from writing R in an appropriate perturbative expansion and then eliminating R adiabatically [31,32]. The laser KS equation is obtained as a fourth-order partial differential equation

$$\frac{\partial\Theta}{\partial t}=-a'\Delta\frac{\partial^2\Theta}{\partial x^2}-a'\left[\frac{\partial\Theta}{\partial x}\right]^2+a'^2\Delta\frac{\partial^3\Theta}{\partial x^3}\frac{\partial\Theta}{\partial x}-\frac{1}{2}a'^2\frac{\partial^4\Theta}{\partial x^4}\ . \tag{12}$$

The analysis of Refs. [31] and [32] shows that the condition of validity for the adiabatic elimination of the amplitude R is $\Delta\ll 1$.

From the stationary solutions $\Theta(x)$ of Eq. (12), in turn, we can obtain the corresponding stationary amplitude profile [31,32]

$$R(x)=1-\frac{1}{2}a'\frac{\partial^2\Theta}{\partial x^2}+\frac{1}{2}a'^2\left[\frac{\partial^3\Theta}{\partial x^3}\frac{\partial\Theta}{\partial x}-\frac{1}{4}\left[\frac{\partial^2\Theta}{\partial x^2}\right]^2\right]\ . \tag{13}$$

In this way, the complex field envelope can be recovered and compared with the exact solution of Eq. (3). If Eqs. (12) and (13) reproduce well the results of Eq. (3), then the phase Θ of field contains all the relevant information for the evolution of the whole complex field envelope.

From the boundary conditions given by Eq. (5), we find the same Neumann boundary conditions for the phase Θ,

$$\frac{\partial\Theta}{\partial x}=0\quad(\text{at } x=0,1)\ . \tag{14}$$

Thus the phase profile can be also expanded in modes

$$\Theta(x,t)=\sum_{n=0}^{\infty}g_n(t)\cos n\pi x\ , \tag{15}$$

where g_n is the real amplitude of the "phase modes."

Obviously, the laser KS equation has the trivial solution $\Theta=0$, which corresponds to the homogeneous solution of the field equation, Eq. (3), $|E(x)|^2=1$. Linear stability analysis of the trivial solution using Eq. (12) confirms that it becomes unstable when Eq. (8) is satisfied. This indicates that the instability described by Eq. (8) is governed by the behavior of the phase variable. In other words, the inhomogeneous transverse intensity distribution of the field arises from the development of an inhomogeneous transverse phase distribution. Note that the instability arises when there is negative diffusion in the laser KS equation, Eq. (12). (From the optical point of view this condition is that the medium be saturated by strong intensities in a way that causes self-focusing.)

By substituting Eq. (15) into Eq. (12) and truncating after the Nth mode, we obtain a set of ordinary differential equations for the amplitudes of the phase modes g_n, in which the phase Θ can be changed by an arbitrary constant without any consequence to the solution

$$\dot{g}_0=-\tfrac{1}{2}a'\pi^2\sum_{m=1}^{N} g_m^2 m^2(1+a'\Delta\pi^2 m^2)\ ,$$

$$\dot{g}_1=a'\pi^2(\Delta-\tfrac{1}{2}a'\pi^2)g_1-a'\pi^2\sum_{m=1}^{N-1} g_m g_{m+1} m(m+1)[1+\tfrac{1}{2}a'\Delta\pi^2(2m^2+2m+1)]\ ,$$

$$\ldots$$

$$\dot{g}_k=a'\pi^2k^2(\Delta-\tfrac{1}{2}a'\pi^2k^2)g_k-a'\pi^2\sum_{m=1}^{N-k} g_m g_{m+k} m(m+k)[1+\tfrac{1}{2}a'\Delta\pi^2(2m^2+2mk+k^2)]$$
$$+\tfrac{1}{2}a'\pi^2\sum_{m=1}^{k-1} g_m g_{k-m} m(k-m)[1+a'\Delta\pi^2(k-m)^2]\quad (2\le k\le N-1)\ , \tag{16}$$

$$\ldots$$

$$\dot{g}_N=a'\pi^2N^2(\Delta-\tfrac{1}{2}a'\pi^2N^2)g_N+\tfrac{1}{2}a'\pi^2\sum_{m=1}^{N-1} g_m g_{N-m} m(N-m)[1+a'\Delta\pi^2(N-m)^2]\ .$$

Some features of this set of equations are the following. First, the amplitude of the homogeneous mode g_0 never appears in the right-hand side of Eqs. (16), but its derivative dg_0/dt depends on the frequencies of the other modes weighted by their "intensities" g_m^2. The reason is that the value of the amplitude of the homogeneous phase mode is only an arbitrary constant of integration without any physical meaning. It corresponds to the usual arbitrariness of the phase for autonomous laser systems. However, in the case of inhomogeneous stationary solutions, the derivative of the amplitude of the homogeneous part of the phase is just the cooperatively chosen operating frequency

$$\delta=-\dot{g}_0=\tfrac{1}{2}a'\pi^2\sum_{m=1} g_m^2 m^2(1+a'\Delta\pi^2 m^2)\ . \tag{17}$$

Equation (17) may be viewed as the pulling of the frequency of the spatially inhomogeneous solutions away from the frequency of the spatially homogeneous solution as written in "phase language." The pulling can be substantial. And, unlike the more complicated derivation of Eq. (9) (see Ref. [15]), the pulling and locking of the cooperatively chosen frequency can be seen directly from the dynamical equations for the modes of the phase Θ.

Second, according to Eqs. (16), the equation for each inhomogeneous phase-mode amplitude g_n has a factor $(\Delta-0.5a'\pi^2m^2)$ in its linear term, which is the amplification for that mode. We can see that when a particular phase mode has positive gain, the homogeneous stationary solution is unstable [see Eqs. (8) and (16)]. In this sense, we may view amplification of the phase modes as the inherent origin of the instability of the spatially homogeneous solution.

These two features suggest a leading role of the dynamics of the phase in the overall dynamics of the laser field, and more evidence will be seen in the following numerical results.

IV. NUMERICAL RESULTS

As indicated in previous work [31,32], where a larger value for the detuning, $\Delta=0.5$, was chosen, the numerical agreement between the laser field equation and the laser KS equation is not absolute. In this paper we choose a smaller detuning parameter, $\Delta=0.1$, in order to improve the approximation because of two effects [32]: (a) the spatially inhomogeneous field, as a perturbation, is weaker, in comparison with the spatially homogeneous field; and (b) the relaxation time of the amplitude R is much shorter than that of the phase Θ.

We integrate numerically the sets of Eqs. (7) and (16), truncating each to 16 modes. The comparisons of their solutions are as follows.

A. The global bifurcation behavior

As we adiabatically decrease and increase the parameter a', the dynamical equations present almost identical global bifurcation behavior, as shown in Fig. 1. Their bifurcation points coincide quite exactly, as shown in Table I. Three branches, which overlap each other, present rich and complicated bifurcation behaviors, depending mainly on the parameter a', although, in this case, the detuning is small and the homogeneous part of the field

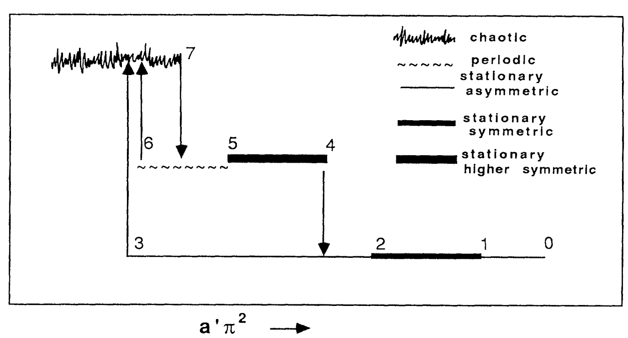


FIG. 1. Global bifurcation behavior for both the laser field equation and the corresponding laser KS phase equation. The detuning parameter $\Delta=0.1$ is held constant throughout this work. For $a'\pi^2>0.2$, only the homogeneous stationary solution is stable. The numbers from 0 to 7 label the bifurcation points for each of which Table I indicates the corresponding value of the bifurcation parameter $a'\pi^2$.

TABLE I. Values of the parameter $a'\pi^2$ in correspondence with the bifurcation points from 0 to 7 shown in Fig. 1.

Bifurcation points	$a'\pi^2$ By KS Eq. (16)	$a'\pi^2$ By field Eq. (7)
0	0.2	0.2
1	0.051	0.051
2	0.0151	0.0151
3	0.0058	0.0058
4	0.0088	0.009
5	0.0068	0.007
6	0.0057	0.0059
7	0.0061	0.0061

dominates the inhomogeneous part of the profile throughout the whole domain, as shown below.

B. Inhomogeneously stationary solutions

The lower branch in Fig. 1 denotes the inhomogeneous stationary solution, which exists for $a'\pi^2 < 0.2$, where the homogeneous solution is unstable with respect to perturbations of mode 1 [see Eq. (8)]. Figure 2 shows the intensities of the field mode of this branch as a function of the parameter $a'\pi^2$: the lines and the symbols correspond to the solutions from the laser field equation and the laser KS equation, respectively. The middle part of the lower branch is a domain of transverse patterns, which are symmetric with respect to the parity transformation $x \to 1-x$ because only even numbered modes contribute.

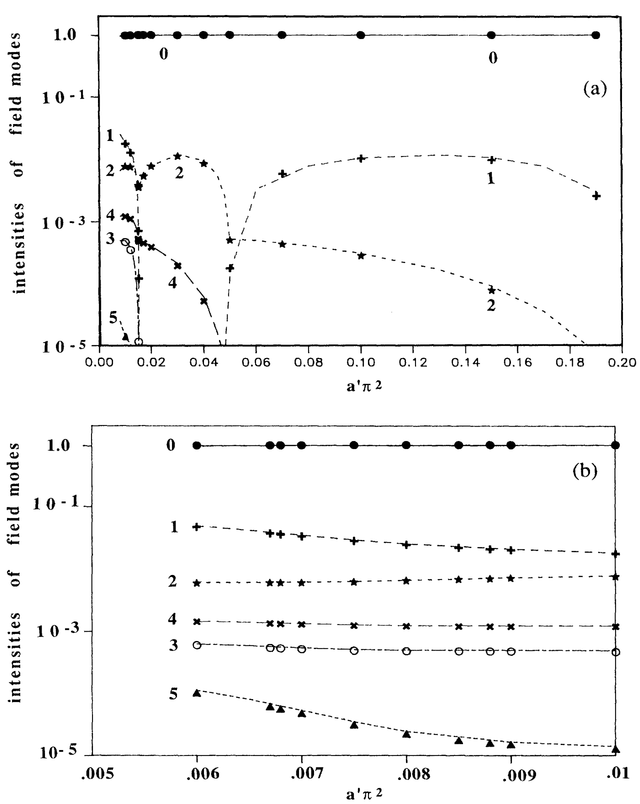


FIG. 2. Intensities of field modes as functions of parameter $a'\pi^2$ for the lower branch of Fig. 1. The lines (solid and dashed) correspond to the solutions of the laser field equation (7); the symbols (● for mode 0, + for mode 1, ★ for mode 2, ○ for mode 3, × for mode 4, and ▲ for mode 5) refer to the solutions of the laser KS equations (12) and (13). Intensities are scaled logarithmically. (b) shows an expanded region of (a) for smaller values of $a'\pi^2$.

As $a'\pi^2$ is decreased from values near 0.2, the nonsymmetric patterns bifurcate to the symmetric ones at $a'\pi^2=0.051$, which is very close to the value, $a'\pi^2=0.05$, at which mode 2 becomes unstable, according to Eq. (8). This is not surprising since in this case the amplitudes of the inhomogeneous modes are very small and the stability of the inhomogeneous patterns is thus likely to be similar to the stability of the homogeneous stationary solution [see Fig. 3(a)]. However, the suppression of mode 1 when mode 2 becomes unstable is a rather surprising evidence of restoration of the parity symmetry.

Figure 3(a) shows the intensity of the homogeneous mode and the average intensity of the transverse field, which is the sum of the intensities for all the transverse modes

$$\int_0^1 E^*(x)E(x)dx = \sum_n |f_n|^2 \qquad (18)$$

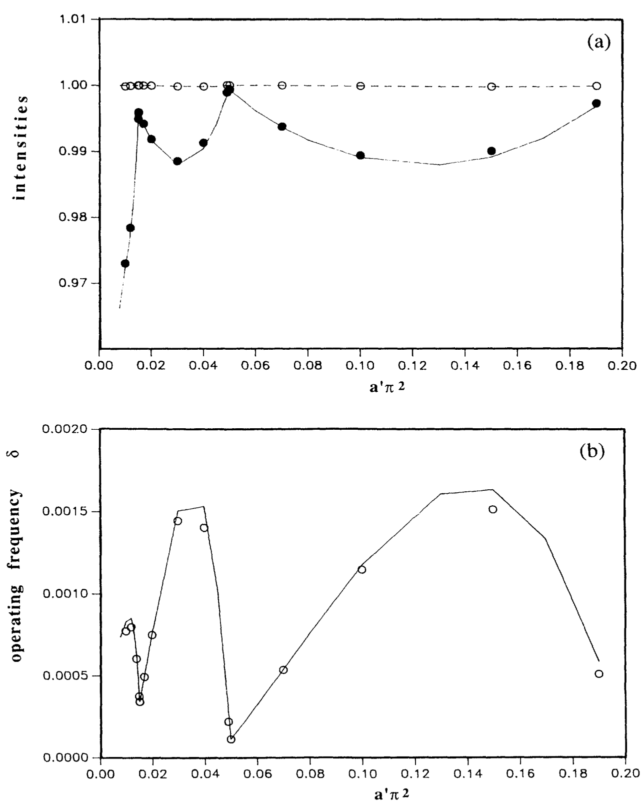


FIG. 3. (a) Intensities of the homogeneous mode (full line and solid circles) and average intensity of the transverse field (dashed line and open circles) and (b) cooperative operating frequencies as functions of parameter $a'\pi^2$. Lines for solutions of the field equation (7), symbols for solutions of laser KS equation (16).

as functions of the parameter $a'\pi^2$. On this scale, the variation of the average intensity (indicated by the dashed line and open circles) is almost indistinguishable (less than 0.1% of the small changes of the intensity of the homogeneous mode). The cooperatively selected operating frequencies of the inhomogeneous stationary solutions are plotted in Fig. 3(b), using Eqs. (9) and (17).

As examples, Figs. 4 and 5 show the profiles of the transverse intensity and phase for symmetric and nonsymmetric patterns, respectively. The differences between solutions of the laser field equation and the laser KS equation are indistinguishable on this scale.

In the middle branch, the right-hand part indicates another kind of stationary solutions, which are also symmetric patterns, but which contain only the 4nth modes ($n=0,1,2,\ldots$), (i.e., the fundamental mode, 4th mode, 8th mode, etc.) as shown in Fig. 6. It is a higher-order symmetric pattern. Under the perturbation of odd-numbered modes, the higher-order symmetric solution becomes unstable at $a'\pi^2 \cong 0.089$, jumping to the nonsymmetric stationary solution; at the other end of its range, at $a'\pi^2 \simeq 0.069$ the higher-order symmetric pattern changes to a time-dependent solution.

These figures exhibit a quantitatively good agreement between the results for the field equation, Eq. (3), and phase equations, Eqs. (12) and (13), for mode intensities, amplitude profiles, phase profiles, and cooperatively chosen operating frequencies. This not only unambiguously demonstrates the leading role that the phase can play in laser dynamics, but also manifests the fact that, in some limits, the phase of the optical field may contain al-

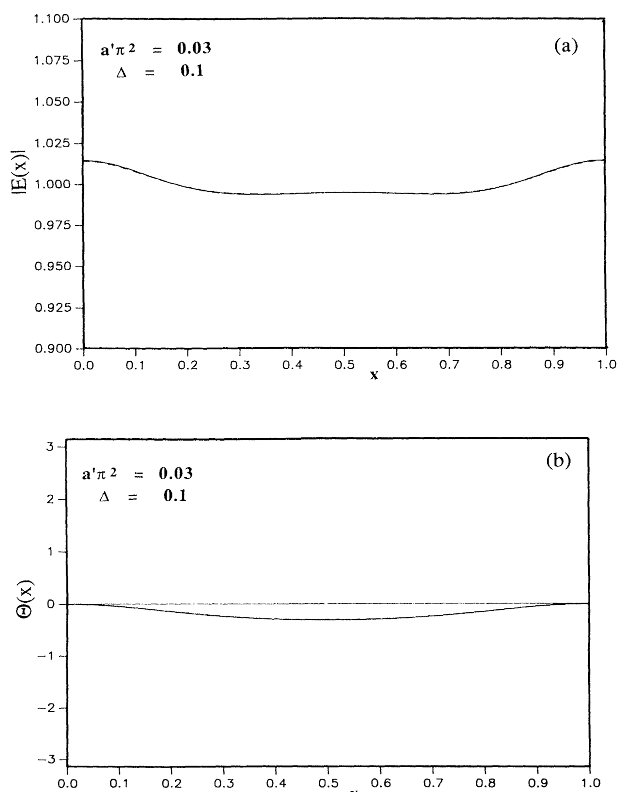


FIG. 4. Transverse pattern of the symmetric solution for $a'\pi^2=0.03$. (a) Modulus profile, (b) phase profile. Solid lines for the solutions of field equation, dashed lines for the laser KS equation are indistinguishable from the solid line.

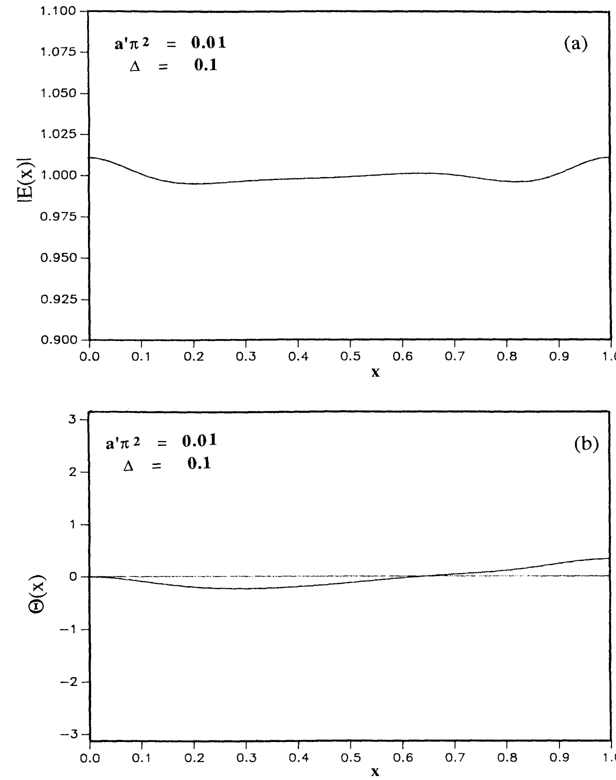


FIG. 5. Transverse pattern of the asymmetric solution for $a'\pi^2=0.01$. Other features as in Fig. 4.

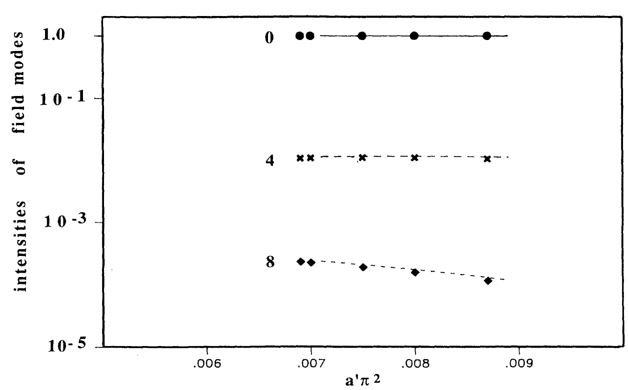


FIG. 6. Intensities of field modes as functions of parameter $a'\pi^2$ for the inhomogeneous stationary solution of higher-order symmetry in the middle branch of Fig. 1. Lines correspond to the solution of the field equation (7); symbols (● for mode 0, × for mode 4, and ♦ for mode 8) correspond to the solution of the laser KS equation (16). In this case the bifurcation points of two equations do not exactly coincide (see points 4 and 5 in Table I).

most the whole information about the field dynamics, as indicated by the fact that the complete complex field amplitude can be accurately reconstructed from the phase pattern.

C. Periodic solutions

The left-hand part of the middle branch in Fig. 1 is a domain of periodic solutions. To the left of point 5 in Fig. 1, the stationary symmetric solutions (which are stable on the right part of this branch) become unstable with respect to perturbations of the odd-numbered modes.

Figure 7 shows the oscillations of the mode intensities and amplitudes: Figs. 7(a) and 7(b) are obtained using the field equations, Eq. (7), whereas Fig. 7(c) is obtained from the phase equations, Eq. (16).

We see in Figs. 7(b) and 7(c) that all of the amplitudes of the odd-numbered phase modes oscillate symmetrically around zero. As the even-numbered modes complete a full period, each odd-numbered mode completes only half its period. In other words, the modulation frequency of the even-numbered modes is twice that of the odd-numbered modes. As a result, the evolution of the transverse phase pattern in the first half period has a parity reversed counterpart in the second half period, with respect to the pattern center ($x=0.5$). Correspondingly, we expect that the same phenomenon will appear in the evolution of the transverse intensity pattern (see Fig. 8). Furthermore, as shown in Figs. 7(a) and 7(b), the modulation of the intensity of individual modes always follows the modulation of the absolute value of the amplitude of the corresponding phase mode. And the modulation frequencies of the intensities of both even- and odd-numbered modes remain equal. The comparison between Figs. 7(b) and 7(c) shows an overall agreement and some quantitative differences in the oscillation amplitudes, and it demonstrates that the phase still governs the evolution, because Fig. 7(c) is obtained from the phase equation.

Figure 8 presents the spatiotemporal oscillation of the intensity profile. Unlike the time-dependent solution in Ref. [33], here there is true spatiotemporal movement of the location of the region of maximum intensity.

In addition to the intensity modulation frequency which is manifest in Fig. 7, these periodic solutions also have an average carrier frequency which is closely similar to the frequency δ that characterizes the inhomogeneous stationary solutions. This carrier frequency, which in the following will be denoted by Ω, becomes manifest if one considers the evolution of any modal variables $f_n(t)$ in the plane $(\mathrm{Re} f_n, \mathrm{Im} f_n)$. As a matter of fact, the trajectory corresponds to a loop which does not close exactly onto itself, but performs a slow rotation around the ori-

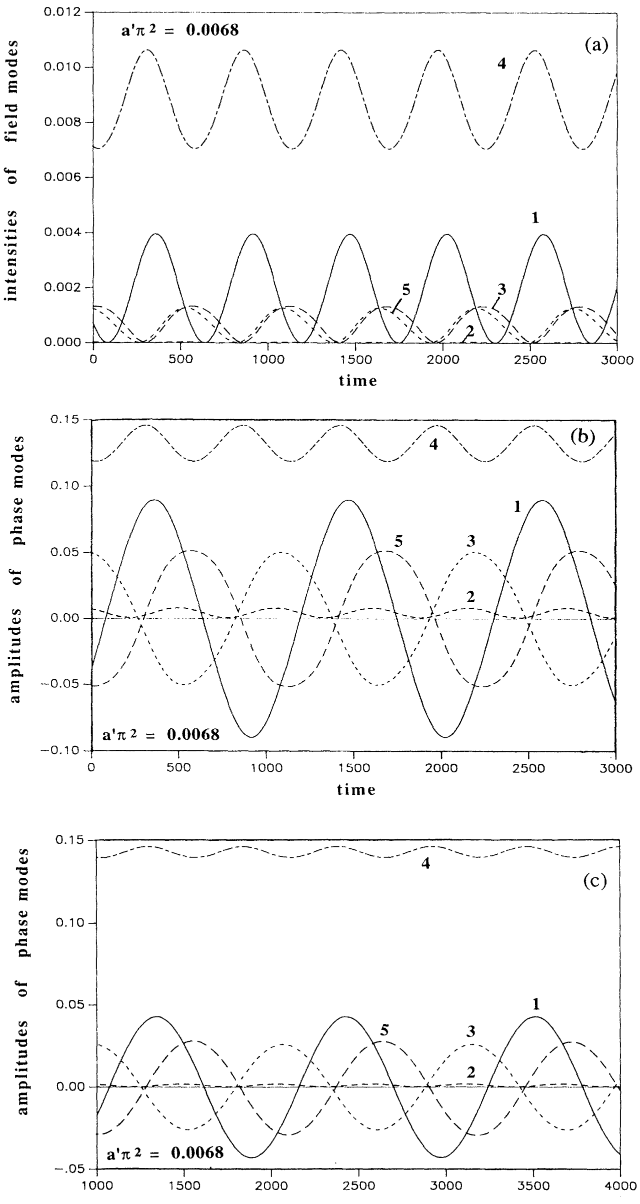


FIG. 7. Time evolution of the periodic solution for $a'\pi^2=0.0068$: (a) intensities of field modes from solution of the field equation (7); (b) amplitudes of phase modes from solution of field equation (7); (c) amplitudes of phase modes from solution of KS equation (16).

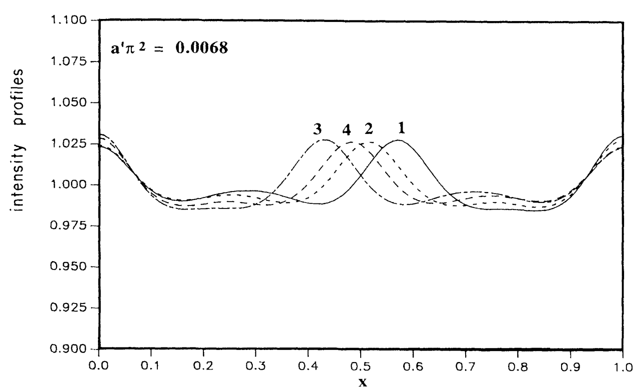
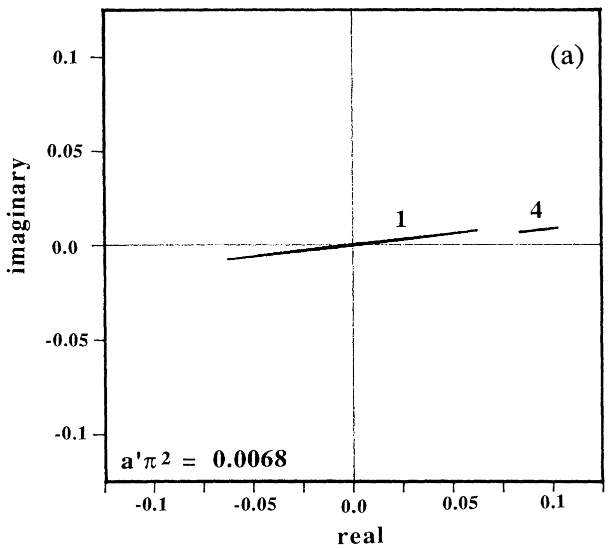


FIG. 8. Transverse intensity patterns at some particular times for $a'\pi^2=0.0068$. Labels: (1) $t_1=0$; (2) $t_2=250$; (3) $t_3=550$; (4) $t_4=800$.

gin of the plane. Thus the carrier frequency is just the slow rotation frequency. We can get rid of this common carrier frequency in two ways: plotting the ratio of the complex amplitudes of the field modes $f_n(t)/f_0(t)$ in the complex plane, or dividing each mode amplitude by $\exp(-i\Omega t)$. The trajectories obtained either of these ways are closed loops. These two methods give almost the same results (an example is shown in Fig. 9) because the modulus of the homogeneous mode is near unity and, its relative modulation is negligible. Figure 9(a) shows the trajectories of the two strongest inhomogeneous modes, mode 1 and mode 4, which are primarily amplitude modulated. In Fig. 9(b) the phase modulation of weaker modes is also evident. Moreover, as expected, the trajectories of all the odd-numbered modes in the complex plane of the complex amplitudes are symmetric with respect to the time translation $[f_n(t+\tau/2)=-f_n(t)$, where τ is the period]. This ensures that the intensity modulation frequency is twice the phase modulation frequency. This situation also offers the possibility of symmetry-breaking bifurcations which may be identified as period-doubling bifurcations of the intensity pulsation. As a matter of fact, at $a'\pi^2 \cong 0.0061$ of the middle branch, period doubling of the intensity pulsations appears. At that bifurcation, from the phase portrait point of view, the even-numbered modes double their period, but the odd-numbered modes do not; instead, they begin to oscillate asymmetrically around zero. However, the intensities of all the field modes seem to show period doubling. This bifurcation is more accurately described as symmetry breaking of the attractor in phase space. (Of course, this symmetry has nothing to do with the spatial parity symmetry that was mentioned earlier.)



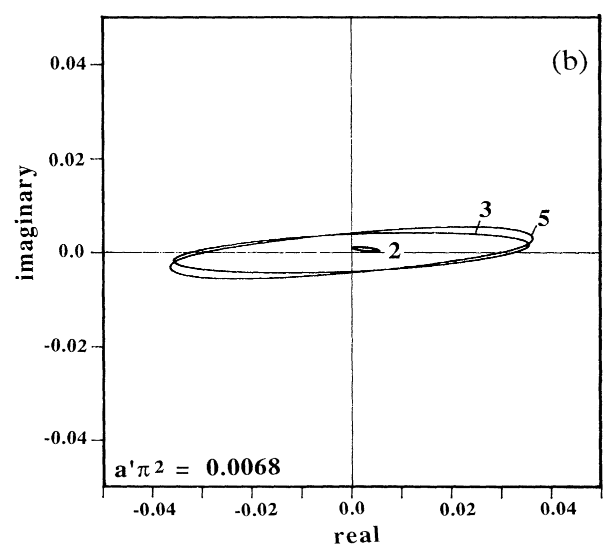


FIG. 9. Trajectories of the complex amplitudes of the inhomogeneous modes in the complex plane for the periodic solution at $a'\pi^2=0.0068$. The common carrier frequency Ω is $2\pi/4979$. (a) Modes 1 and 4; (b) modes 2, 3, and 4.

We have observed these kinds of phase space trajectories of time-dependent solutions in other dynamical systems, where some complex variables possess the symmetry under time translation, such as in the optical parametric oscillator [39] and in degenerate four-wave mixing [40], respectively, but, which are described by models without spatial variables. In contrast, in the present model, the time translation symmetries for the odd-numbered modes are implied in the dynamical partial differential equation.

D. Chaotic solutions

When the parameter a' is decreased further, chaotic solutions are observed. In this case we still can find a common average carrier frequency despite chaotic pulsations of the moduli. Furthermore, the average intensity remains almost unchanged in the evolution of the time-dependent solutions. This suggests an energy conservation of the total transverse field, independent of the time and the parameter a', as long as the pump is fixed. (Here the field E has been normalized to the square root of the pump parameter.)

In order to compare the laser field equation and laser KS equation for the chaotic regime, we plot maps for the moduli of individual field modes $|f_n|$ and for the amplitude of the corresponding phase mode g_n, for example, $g_n(t)$ vs $g_n(t+T)$, where T is a delay time. Figs. 10–12 are the maps for the inhomogeneous modes 1, 2 and 3, respectively; figures (a) refer to the moduli of the field modes $|f_n|$, whereas figures (b) and (c) refer to the amplitudes of phase modes g_n. Both (a) and (b) are obtained from the solution of the field equation, whereas (c) is obtained from the solution of the phase equation.

For each mode, the maps (a), (b), and (c) present astonishingly similar structures, except in the case of mode 3, for which the maps of the modulus and of the phase mode amplitudes differ. The similar structure of the maps for the modulus and phase amplitudes for a particular mode is caused by fact that the oscillation of the modulus always follows the oscillation of the absolute value of the corresponding phase-mode amplitude, as it has been shown for the periodic solutions. Precisely, the

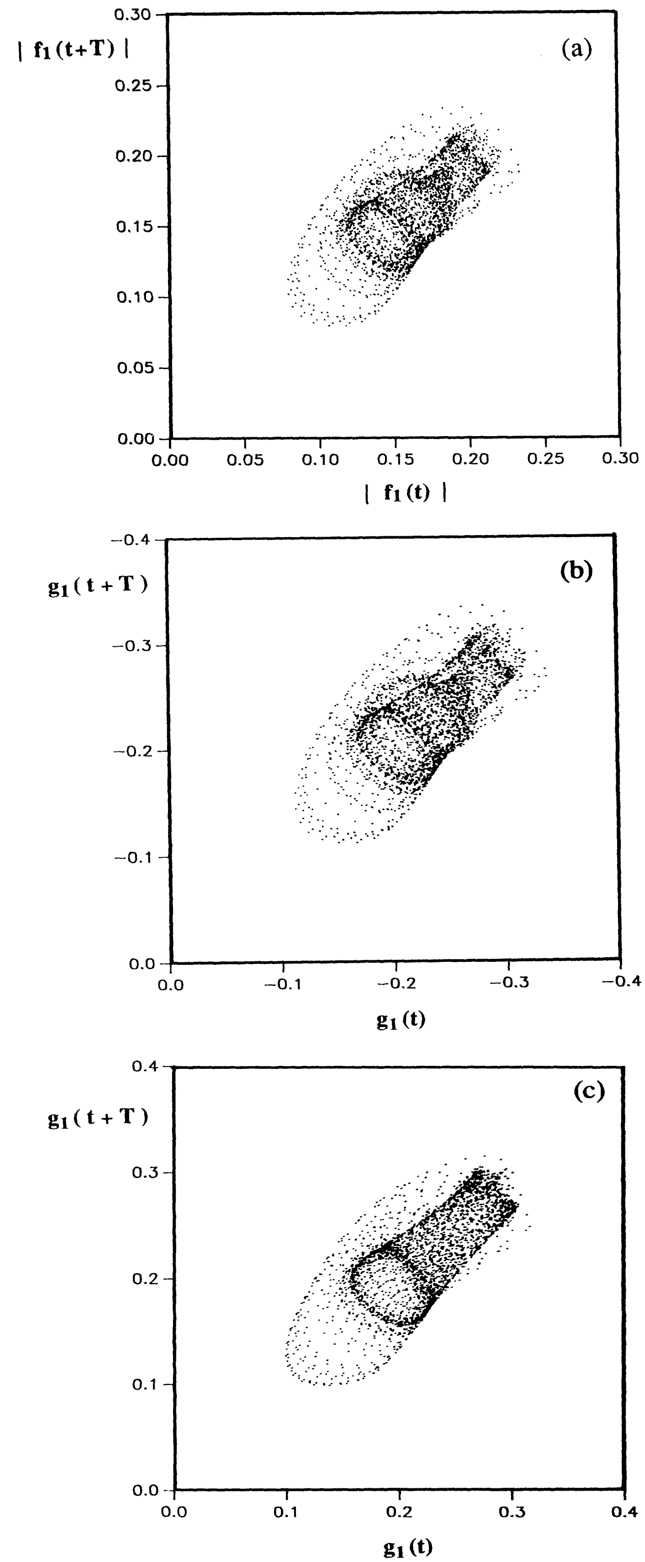


FIG. 10. Time delay maps for mode 1, for $a'\pi^2=0.006$ and delay time $T=100$. (a) Maps for moduli of field modes $|f_n|$. (b) Maps for amplitudes of phase modes g_n. Both (a) and (b) from the solution of the laser field Eq. (7). (c) Maps for amplitudes of phase modes g_n, but from solution of the laser KS equation (16).

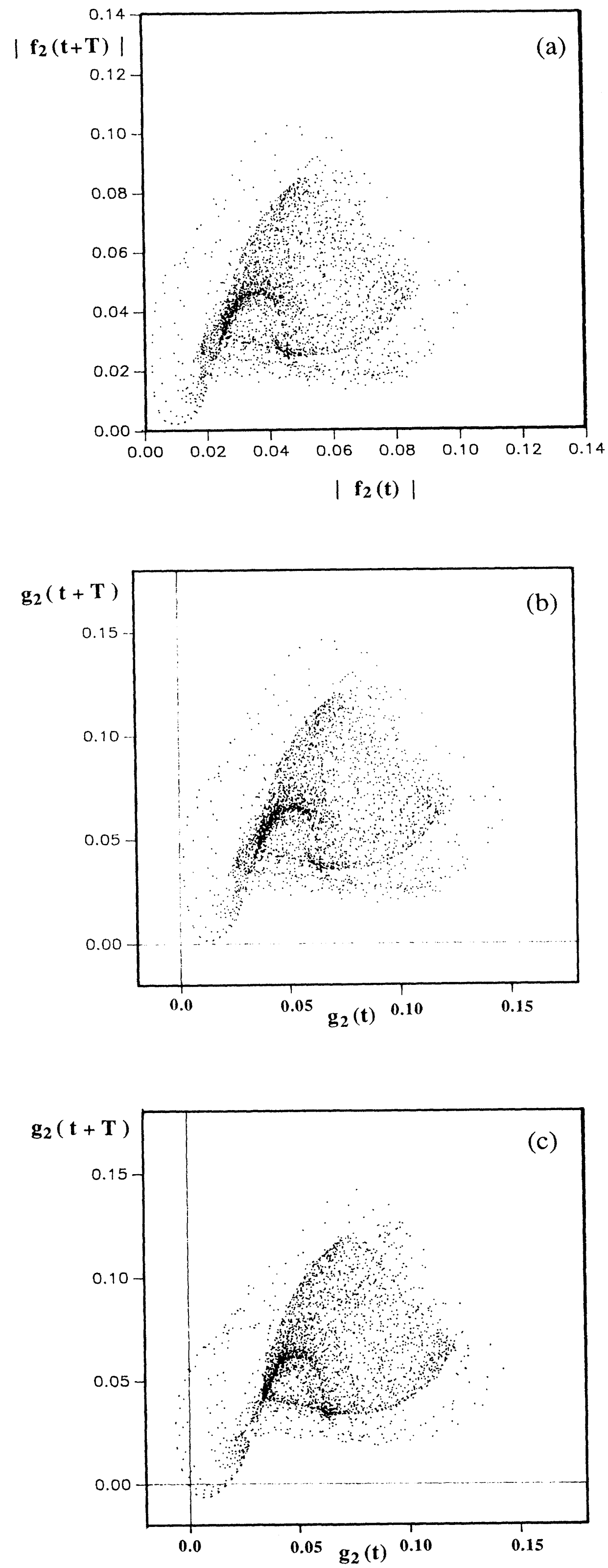


FIG. 11. Time delay maps as in Fig. 10, but for mode 2.

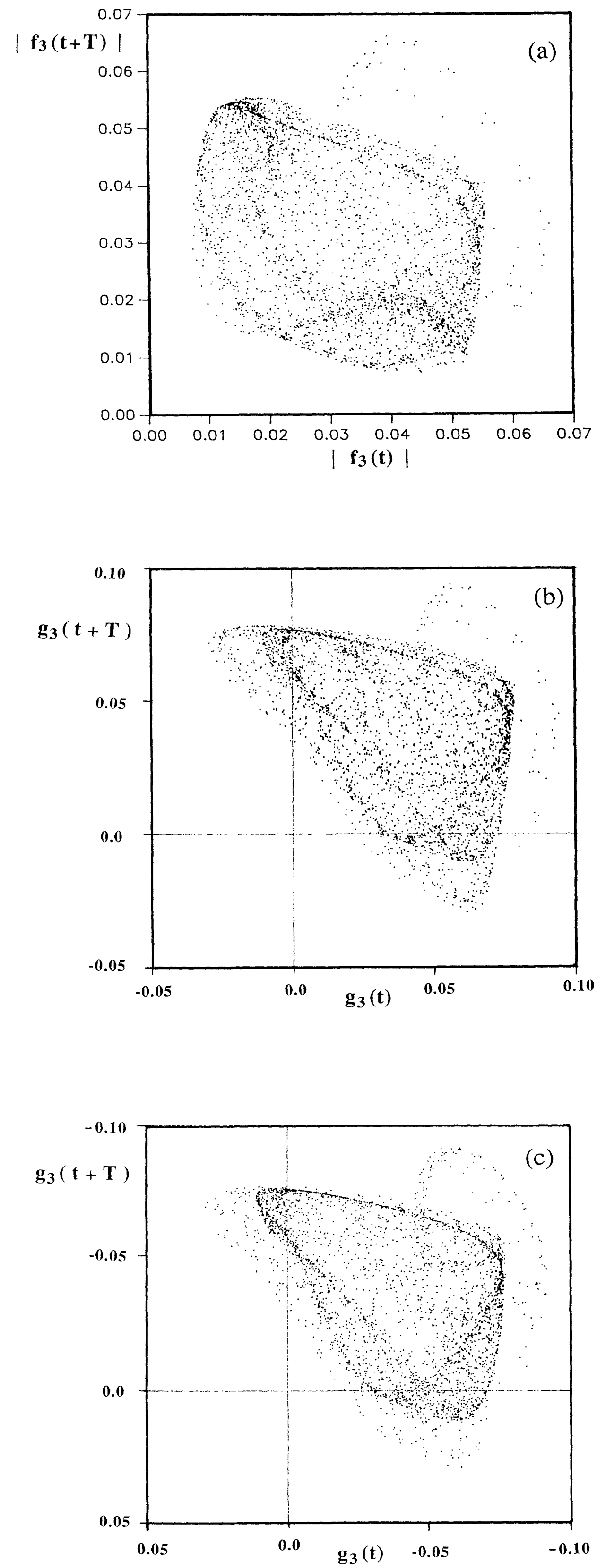


FIG. 12. Time delay maps as in Fig. 10, but for mode 3.

increase in the absolute value of a phase-mode amplitude is accompanied by the increase in the modulus of the corresponding field mode amplitude. When the phase-mode amplitude reaches zero and changes its sign, the corresponding modulus takes on its minimum value. When the evolution of a particular phase-mode amplitude keeps the same sign, the match of the oscillation of the phase-mode amplitude with the corresponding modulus of the field mode works well, such as for mode 1 and mode 2, where we observe a similar portrait. In contrast, in the evolution of the amplitude of a phase-mode amplitude which changes sign, such as for mode 3, the match of modulation is partially destroyed, so that the maps are not similar.

The match of the modulation in evolution and the resulting similarity of maps for the corresponding moduli and phase components, which are taken from the same field equation, shows clear evidence of the leading role of the phase in the dynamics. Moreover, the similarity of the maps for phase-mode amplitudes from different dynamical equations demonstrates that this leading role is expressed by the self-contained KS equation. Hence in these limits the phase dynamics governs the laser dynamics in the formation of the transverse patterns.

V. CONCLUSION

An extensive investigation of the global bifurcation behavior, stationary patterns and spatiotemporal dynamics of solutions of the equation for the laser field in the cubic approximation has been performed for small detuning. We find a common operating frequency which appears not only in the inhomogeneous stationary solutions but also in the time-dependent solutions. We have shown both analytical and numerical evidence which demonstrate unambiguously that the phase of the electric field leads the dynamical behavior in the formation of transverse patterns.

Maps have been used to compare different dynamical variables for the chaotic regime. We find similar structures of maps for different variables which do not have a functional relation, and which are even from different dynamical equations. The similarity of the maps in the chaotic regime reveals the correlation and statistical consistency of different physical variables and may suggest some inherent properties of dynamical systems of this type, such as low dimensions of their strange attractors.

ACKNOWLEDGMENTS

We are grateful to A. M. Albano, R. Lefever, and P. Mandel for many useful suggestions and discussions. This work has been supported in part by the ESPRIT Basic Research Action 3260 TOPP (Transverse Optical Patterns) and by a grant of supercomputing time from the National Science Foundation through the National Center for Supercomputing Applications.

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[1] Special issues on transverse effects in nonlinear optical system, edited by N. B. Abraham and W. J. Firth, J. Opt. Soc. Am. B **7**, (6,7) (1990). See also major sections of *Nonlinear Dynamics in Optical Systems,* edited by N. B. Abraham, E. Garmire, and P. Mandel (Optical Society of America, Washington, DC, 1991).

[2] N. B. Abraham, L. A. Lugiato, and L. M. Narducci, J. Opt. Soc. Am. B **2**, 7 (1985); N. B. Abraham, P. Mandel, and L. M. Narducci, in *Progress in Optics,* edited by E. Wolf (Elsevier Science, Amsterdam, 1988), Vol. XXV, p. 3.

[3] *Instabilities and Chaos in Quantum Optics,* edited by F. T. Arecchi and R. Harrison (Springer-Verlag, Berlin, 1987); *Instabilities and Chaos in Quantum Optics II,* edited by N. B. Abraham, F. T. Arecchi, and L. A. Lugiato (Plenum, New York, 1988).

[4] D. W. McLaughlin, J. V. Moloney, and A. C. Newell, Phys. Rev. Lett. **54**, 681 (1985).

[5] J. V. Moloney, IEEE J. Quantum Electron. **QE-21**, 1393 (1985).

[6] J. V. Moloney, H. Adachihara, D. W. McLaughlin, and A. C. Newell, in *Chaos, Noise and Fractals,* edited by E. R. Pike and L. A. Lugiato (Hilger, London, 1988), p. 137.

[7] M. LeBerre, E. Ressayre, and A. Tallet, Phys. Rev. A **25**, 1604 (1982).

[8] G. Grynberg, E. Le Bihan, P. Verkerk, P. Simonean, J. R. R. Leite, D. Bloch, S. Le Boiteaux, and M. Duclay, Opt. Commun. **67**, 3631 (1988).

[9] G. Giusfredi, J. F. Valley, R. Pon, G. Khitrova, and H. M. Gibbs, J. Opt. Soc. Am. B **5**, 1181 (1988).

[10] S. A. Arkhmanov, M. A. Vorontsov, and V. Yu. Ivanov, Pis'ma Zh. Eksp. Teor. Fiz. **47**, 611 (1988) [Sov. Phys.—JETP Lett. **47**, 707 (1988)]; S. A. Akhmanov, M. A. Vorontsov, V. Yu. Ivanov, A. V. Larichev, and N. I. Zheleznykh, J. Opt. Soc. Am. B **9**, 78 (1992); M. A. Vorontsov and K. V. Shishakov, *ibid.* **9**, 71 (1992).

[11] L. A. Lugiato and R. Lefever, Phys. Rev. Lett. **58**, 2209 (1987).

[12] L. A. Lugiato and C. Oldano, Phys. Rev. A **37**, 3896 (1988).

[13] W. J. Firth, J. Mod. Opt. **37**, 151 (1990).

[14] F. Hollinger and Chr. Jung, J. Opt. Soc. Am. B **2**, 218 (1985).

[15] L. A. Lugiato, C. Oldano, and L. M. Narducci, J. Opt. Soc. Am. B **5**, 879 (1988).

[16] L. A. Lugiato, F. Prati, L. M. Narducci, and G.-L. Oppo, Opt. Commun. **69**, 387 (1989).

[17] J. R. Tredicce, E. J. Quel, A. M. Ghazzawi, C. Green, M. A. Perrigo, L. M. Narducci, and L. A. Lugiato, Phys. Rev. Lett. **62**, 1274 (1989).

[18] P. Coullet, L. Gil, and F. Rocca, Opt. Commun. **73**, 403 (1989).

[19] F. T. Arecchi, R. Meucci, and L. Pezzati, Phys. Rev. A **42**, 5791 (1990).

[20] F. T. Arecchi, G. Giacomelli, P. L. Ramazza, and S. Residori, Phys. Rev. Lett. **65**, 2531 (1990).

[21] F. T. Arecchi, G. Giacomelli, P. L. Ramazza, and S. Residori, Phys. Rev. Lett. **67**, 3749 (1991).

[22] See the articles by W. J. Firth; M. Le Berre, E. Ressayre, and A. Tallet; F. T. Arecchi; and M. Brambilla, M. Cattaneo, L. A. Lugiato, and F. Prati, in *Proceedings of ECOOSA 90–Quantum Optics,* edited by M. Bertolotti and E. R. Pike (Hilger, Bristol, 1992).

[23] Y. Kuramoto and T. Tsuzuki, Prog. Theor. Phys. **55**, 356 (1976).

[24] T. Yamada and Y. Kuramoto, Prog. Theor. Phys. **56**, 681 (1976).

[25] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **57**, 734 (1976).

[26] Y. Kuramoto, Suppl. Prog. Theor. Phys. **64**, 346 (1978).

[27] G. I. Sivashinsky and D. M. Michelson, Prog. Theor. Phys. **63**, 2112 (1980).

[28] J. M. Hyman and B. Nicolaenko, Physica D **18**, 113 (1986).

[29] G. I. Sivashinsky, Acta Astronaut **4**, 1177 (1977).

[30] Y. Kuramoto, Prog. Theor. Phys. Suppl. No. **99**, 244 (1989).

[31] R. Lefever, L. A. Lugiato, Wang Kaige, N. B. Abraham, and P. Mandel, Phys. Lett. A **135**, 254 (1989).

[32] R. Lefever, L. A. Lugiato, Wang Kaige, N. B. Abraham, and P. Mandel, in *Dynamics of Non-linear Optical Systems,* edited by L. Pesquera and F. J. Bermejo (World Scientific, Singapore, 1988), p. 175.

[33] H. Lin and N. B. Abraham, Opt. Commun. **79**, 476 (1990); J. Opt. Soc. Am. B **8**, 2429 (1991).

[34] A. F. Suchkov, Zh. Eksp. Teor. Fiz. **49**, 1495 (1965) [Sov. Phys.—JETP **22**, 1026 (1966)]; R. G. Allakhverdyan, A. N. Oraevsky, and A. F. Suchkov, Fiz. Tekh. Poluprovodn. **4**, 341 (1970) [Sov. Phys.—Semicond. **4**, 377 (1970)].

[35] P. K. Cheo, *Fiber Optics, Devices and Systems* (Prentice-Hall, Englewood Cliffs, NJ 1985), pp. 180–191.

[36] N. Yu, R. K. DeFreez, D. J. Bosert, G. A. Wilson, R. A. Elliot, S.-S. Wang, and H. G. Winful, Appl. Opt. **30**, 2503 (1991); see also, J. Buus, IEEE J. Quantum Electron. **QE-15**, 734 (1979); G. L. Abbas, S. Yang, V. W. S. Chan, and J. G. Fujimoto, *ibid.* **24**, 609 (1988); and papers by R. K. DeFreez *et al.* and H. G. Winful *et al.* in *Nonlinear Dynamics in Optical Systems,* edited by N. B. Abraham, E. Garmire, and P. Mandel (Optical Society of America, Washington, DC, 1991), pp. 106–123.

[37] N. Maronvitz, *Waveguide Handbook* (McGraw-Hill, New York, 1951), p. 64.

[38] G.-L. Oppo, L. Gil, G. D'Allessandro, and W. J. Firth, in *Proceedings of ECOSSA* (Ref. [22]), p. 191; G.-L. Oppo, G. D'Allessandro, and W. J. Firth, Phys. Rev. A **44**, 4712 (1991); L. Gil, K. Emilsson, and G.-L. Oppo, *ibid.* **45**, R567 (1992).

[39] L. A. Lugiato, C. Oldano, C. Fabrè, E. Giacobino, and R. J. Horowicz, Nuovo Cimento **10D**, 959 (1988).

[40] Wang Kaige, L. A. Lugiato, and N. B. Abraham (unpublished).