Signal-pump entanglement in quantum k-photon down-conversion

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We study the quantum dynamics of the k-photon down-conversion process. We concentrate our attention on the entanglement between the pump and the signal modes. We show that the degree of the entanglement between the pump and the signal depends on the initial statistics of the light field in both modes. Moreover, the higher the nonlinear process the stronger the entanglement is. The entanglement between the modes is related to the marginal entropy in the pump (signal) mode; i.e., the larger the entanglement, the larger the entropy is. In the quantum k-photon down-conversion the signal mode at $t>0$ is not generally in a pure state which restricts applicability of the parametric approximation.

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I. INTRODUCTION

There has recently been considerable interest in the production of nonclassical states of light in various nonlinear processes. In particular, it has been shown that squeezed states of light (i.e., the states of light with reduced fluctuations in one quadrature below the level associated with the vacuum state [1]) can be generated using a degenerate parametric amplifier [2]. The degenerate parametric amplifier is a device which provides a nonlinear coupling between two modes of the radiation field. The first, the pump mode, has a frequency 2ω, while the second, the signal mode, has a frequency ω. It was Takahashi [3] who realized that the degenerate parametric amplifier decreases quantum-mechanical fluctuations in the signal mode due to phase correlations of emitted photons. Kimble and co-workers [4], using the parametric amplifier, have experimentally generated squeezed light exhibiting high degree of noise reduction.

If both the pump and the signal modes are quantized, then the Hamiltonian describing the dynamics of the degenerate parametric amplifier in the rotating-wave approximation has the form (in what follows we assume $\hbar=1$)

$$\hat{H}_2=\omega\hat{a}^\dagger\hat{a}+2\omega b^\dagger b+\lambda_2[(\hat{a}^\dagger)^2 b+\hat{a}^2 b^\dagger] , \tag{1.1}$$

where $\hat{a},\hat{a}^\dagger$ ($\hat{b},\hat{b}^\dagger$) are the annihilation and the creation operators of the signal (pump) mode, respectively, and λ_2 is a coupling constant which is proportional to the second-order nonlinear polarizability coefficient of the medium. Here exact two-photon resonance between modes is assumed. Generally it is supposed that if the pump mode is initially in a highly excited coherent state then the parametric approximation can be adopted and the pump mode can be treated as a classical field, so that the operator $\hat{b}$ in Eq. (1.1) is replaced by the classical c number $\beta e^{-2i\omega t}$, where β is the amplitude of the pump mode (for simplicity we can assume β to be real). In the parametric approximation the Hamiltonian (1.1) reads

$$\hat{H}_2^{(P)}=\omega\hat{a}^\dagger\hat{a}+\lambda_2\beta[(\hat{a}^\dagger)^2 e^{-2i\omega t}+\hat{a}^2 e^{2i\omega t}] , \tag{1.2a}$$

and in the interaction picture it takes the following form:

$$\hat{H}_2^{(I,P)}=\lambda_2\beta[(\hat{a}^\dagger)^2+\hat{a}^2] . \tag{1.2b}$$

The Hamiltonian (1.2) has served recently as a description of the dynamics of the parametric amplifier and statistical properties of the signal mode (see, for instance, Ref. [5]). An evolution operator $\hat{U}_2^{(I,P)}\equiv\exp(-i\hat{H}_2^{(I,P)}t)$ in the interaction picture is equivalent to the "squeeze" operator $\hat{S}(\zeta)$ [6],

$$\hat{S}(\zeta)=\exp[\zeta(\hat{a}^\dagger)^2-\zeta^*\hat{a}^2] , \tag{1.3}$$

describing the Bogoliubov transformation by which the initial vacuum state or a coherent state [i.e., the minimum uncertainty states (MUS's)] are transformed into the squeezed state, which is a MUS with reduced quadrature fluctuations [1].

It is generally stressed that the parametric approximation is accompanied by the neglect of two effects. First, it neglects quantum fluctuations in the pump mode. Second, by treating the mode as a classical mode the depletion of the pump is ignored. The influence of quantum fluctuations of the pump mode on the reduction of the degree of squeezing in the process of parametric amplification has recently been studied extensively by many authors [7]. It has been shown that the parametric approximation described above can be adopted only in the case of sufficiently large pump powers and only for interaction times smaller than $\sim 1/(\lambda_2\beta)$, where β is the amplitude of the pump field. In this case the effect of the pump quantization on the degree of squeezing in the signal mode is negligible.

The degenerate parametric amplifier described by the Hamiltonian (1.1) represents a particular case of a more general nonlinear device, a k-photon down-convertor. The dynamics of the k-photon down-convertor is governed by the Hamiltonian

$$\hat{H}_k=\omega\hat{a}^\dagger\hat{a}+k\omega\hat{b}^\dagger\hat{b}+\lambda_k[(\hat{a}^\dagger)^k\hat{b}+\hat{a}^k\hat{b}^\dagger] , \tag{1.4a}$$

where λ_k is a coupling constant that is proportional to the kth-order nonlinear polarizability coefficient of the crystal. We should note here that, depending on initial conditions, the Hamiltonian (1.4a) describes two processes. First, the k-photon down-conversion, and second, the kth harmonic generation. If mode b is initially excited and mode a is in the vacuum state, the Hamiltonian (1.4a) describes the k-photon down-conversion. On the contrary, if mode a is initially excited and mode b is in the vacuum state, (1.4a) describes the kth harmonic generation. In the simplest case with $k=1$ the Hamiltonian (1.4a) corresponds to a linear directional coupler [8] in which two modes are linearly coupled.

In the parametric approximation using the interaction picture we find the Hamiltonian (1.4a) in the form

$$\hat{H}_k^{(I,P)}=\lambda_k\beta[(\hat{a}^\dagger)^k+\hat{a}^k] , \tag{1.4b}$$

where we assume the amplitude β of the pump field to be real. The evolution operator $\hat{U}_k^{(I,P)}$ corresponding to (1.4b) is given by the relation

$$\hat{U}_k^{(I,P)}=\exp(-i\hat{H}_k^{(I,P)}t)=\exp[\zeta(\hat{a}^\dagger)^k-\zeta^*\hat{a}^k] , \tag{1.5}$$

where $\zeta=-i\lambda_k\beta t$. The above operator is well defined for $k=1$ and $k=2$. For $k=1$ the evolution operator $\hat{U}_1^{(I,P)}$ is equal to the Glauber-Sudarshan displacement operator [9], while $\hat{U}_2^{(I,P)}$ in the case $k=2$ is equal to the "squeeze" operator [6]. The existence of the time-development transformation for $k>2$ was first discussed by Fisher, Nieto, and Sandberg [10]. These authors have found that the vacuum-to-vacuum matrix element of the evolution operator (1.5), i.e., $\langle 0|\hat{U}_k^{(I,P)}|0\rangle$, has for $k>2$ divergent Taylor-series expansion in time for any $t>0$. From this Fisher and co-workers have concluded that it is impossible to define states which result from applying the operator (1.5) for $k>2$ on the vacuum. Braunstein and McLachlan [11] have shown that from the fact that the vacuum state $|0\rangle$ is not an analytical vector of the operator $i\hat{H}_k^{(I,P)}$ it does not follow directly that the evolution operator $\hat{U}_k^{(I,P)}$ does not exist. These authors have used the Padé approximants to show that the Q function (see below) for the states $\hat{U}_k^{(I,P)}|0\rangle$ for $k=3,4$ exists for a *limited* range of time. Recently Hillery [12] has explicitly shown that in the parametric approximation the number of photons in the process described by the evolution operator (1.5) becomes infinite in a *finite* period of time. This unphysical result follows from the fact that in the parametric approximation the pump depletion is neglected. This directly restricts the applicability of the parametric approximation. Namely, the parametric approximation is valid only for a *limited* range of time for which the number of photons in the signal mode is finite. We should note here that even in the case of the linear coupler ($k=1$) and the parametric amplifier ($k=2$) the number of photons is diverging in the parametric approximation. Nevertheless in these two cases the divergence occurs only in the limit $t\rightarrow\infty$ and therefore does not represent a real problem. Higher-order ($k>2$) processes are much more divergent than the process of the parametric amplification [12] and therefore the Hamiltonian (1.4b) can be utilized only for a limited range of time. On the other hand, Hillery has shown [12] that the quantum theory of the k-photon down-conversion with a *quantized* pump in the rotating-wave approximation described by the Hamiltonian (1.4a) is well defined for any $t>0$ (see also recent papers by Tanas and co-workers [13]).

The purpose of this paper is to study quantum correlations between the pump and the signal mode in the quantum k-photon down-conversion. The quantum dynamics described by the Hamiltonian (1.4a) leads to a strong entanglement between the pump and the signal modes. We show that the degree of this entanglement depends on the initial state of the signal-pump system and on the order of the nonlinear process under consideration. The entanglement between the modes leads to the increase of marginal entropies of the pump and the signal modes, respectively (even though that the total entropy is constant and equal to zero, if we assume the system to be initially prepared in a pure state and if we neglect losses in the system). The quantum-mechanical entropy $S_{a(b)}$ (as defined by von Neumann) in mode a (b) [14] (we assume the Boltzmann constant k_B to be equal to unity),

$$S_{a(b)}=-\operatorname{Tr}_{a(b)}(\hat{\rho}_{a(b)}\ln\hat{\rho}_{a(b)}) , \tag{1.6}$$

is defined through the reduced density operator of the signal (pump) mode

$$\hat{\rho}_{a(b)}=\operatorname{Tr}_{b(a)}\hat{\rho} . \tag{1.7}$$

In this paper we assume the pump and the signal modes to be initially prepared in a pure state, that is,

$$S_a|_{t=0}=S_b|_{t=0}=0 . \tag{1.8}$$

From the Araki-Lieb theorem [15], which can be expressed in the form

$$|S_a-S_b|\le S\le S_a+S_b , \tag{1.9}$$

it follows that the total entropy S of the signal-pump system is equal to zero at $t=0$. Due to the fact that we do not take into account losses in our model the total entropy is an integral of motion, i.e., $S=0$ for any $t>0$. Consequently, from (1.9) it follows that

$$S_a=S_b \quad \text{for } t>0 . \tag{1.10}$$

One way to quantify the degree of the entanglement between two modes is to evaluate the index of correlation I_C defined by Barnett and Phoenix [15]:

$$I_C=S_a-S_b-S . \tag{1.11}$$

From the above it follows that in our case the index of correlation is equal to twice the entropy of the signal (pump) mode:

$$I_C=2S_a . \tag{1.12}$$

The stronger the entanglement, the higher the value of the index of correlation and the larger the entropy of the signal (pump) mode.

The increase of the entropy of the signal mode in the k-photon down-conversion reflects the fact that the initially pure state of the signal mode is transformed into a statistical-mixture state. On the other hand, in the parametric approximation described by the Hamiltonian (1.4b) no quantum entanglement between the pump and the signal modes can appear; therefore if the signal mode is initially in the pure state (for instance, in the vacuum state), then the entropy S_a is identically equal to zero for any $t>0$. From here we conclude that parametric approximation leads not only to neglect of the important role of the pump fluctuations and the pump depletion but also to neglect of the signal-pump entanglement. In the remainder of this paper we discuss in detail consequences of this entanglement.

II. QUANTUM DYNAMICS OF THE k-PHOTON DOWN-CONVERTOR

The Hamiltonian (1.4a) governing the quantum k-photon down-conversion in the rotating-wave approximation can be rewritten in terms of two integrals of motion:

$$\hat{H}_k=\hat{H}_{\text{free}}+\hat{H}_{\text{int}} , \tag{2.1}$$

with

$$\hat{H}_{\text{free}}=\omega[\hat{a}^\dagger\hat{a}+k\hat{b}^\dagger\hat{b}]\equiv\omega\hat{C} \tag{2.2a}$$

and

$$\hat{H}_{\text{int}}=\lambda_k[(\hat{a}^\dagger)^k\hat{b}+\hat{a}^k b^\dagger] . \tag{2.2b}$$

The fact that the operator $\hat{C}$ is the integral of the motion in the process under consideration implies that the quantum dynamics of the k-photon down-convertor is well defined [16]. Nevertheless the corresponding dynamical equations cannot be solved analytically except for the case of the linear coupler. Therefore we study the dynamics numerically using the numerical approach based on the diagonalization of the interaction Hamiltonian $\hat{H}_{\text{int}}$ (for details see Ref. [16]).

In this paper we assume the signal mode to be initially prepared in a pure state $|\Phi_1(t=0)\rangle_a$ (i.e., $S_a=0$) and the pump mode to be prepared in the coherent state $|\beta\rangle_b$,

$$|\Phi_2(t=0)\rangle_b=\hat{D}_b(\beta)|0\rangle_b\equiv|\beta\rangle_b , \tag{2.3a}$$

where $\hat{D}_b(\beta)$ is the Glauber-Sudarshan displacement operator,

$$\hat{D}_b(\beta)=\exp(\beta b^\dagger-\beta^*\hat{b}) . \tag{2.3b}$$

The signal-pump system at $t>0$ can be described by a state vector $|\Psi(t)\rangle$ from a two-mode Hilbert space $\mathcal{H}=\mathcal{H}_a\oplus\mathcal{H}_b$ which is a direct sum of the Hilbert spaces $\mathcal{H}_a,\mathcal{H}_b$ of two subsystems a,b. We will show below that at $t>0$ for $k>1$ the state vector $|\Psi(t)\rangle$ cannot be generally written in a factorized form, i.e.,

$$|\Psi(t)\rangle\neq|\Phi_1(t)\rangle_a|\Phi_2(t)\rangle_b , \tag{2.4}$$

which means that both the pump and the signal become entangled at $t>0$. Statistical properties of both subsystems are now given by the reduced density operator $\hat{\rho}_{a(b)}$ given by Eq. (1.7). The entanglement between the pump and the signal modes is reflected by the increase of the marginal entropies S_a and S_b. For simplicity, to measure the degree of the purity of the state we will use in this paper the parameter S_{pur} instead of the entropy. The purity parameter S_{pur} is defined as

$$S_{\text{pur}}=1-\text{Tr}_a[(\hat{\rho}_a)^2]=1-\text{Tr}_b[(\hat{\rho}_b)^2] , \tag{2.5}$$

and is related to the entropy as its lower bound, i.e., $S_{\text{pur}}\leq S_{a(b)}$. If the mode is in a pure state then $S_{\text{pur}}=0$, otherwise for mixed states it takes positive values up to unity. Generally speaking, the higher S_{pur} the higher the entanglement between the pump and the signal mode [see Eq. (1.12), from which it follows that $2S_{\text{pur}}\leq I_c$].

To describe statistical properties of the pump and the signal modes we will consider the Q function corresponding to each of these modes [17],

$$Q_{a(b)}(\alpha)=\frac{1}{\pi}\langle\alpha|\hat{\rho}_{a(b)}|\alpha\rangle , \tag{2.6}$$

and the marginal photon-number distribution

$$P_{a(b)}(n)=\langle n|\hat{\rho}_{a(b)}|n\rangle . \tag{2.7}$$

To analyze the squeezing properties of the modes under consideration we introduce the following quadrature operators for each of the modes,

$$\hat{X}_a=\frac{\hat{a}e^{-i(\pi/4)}+\hat{a}^\dagger e^{i(\pi/4)}}{2} ,\quad \hat{Y}_a=\frac{\hat{a}e^{-i(\pi/4)}-\hat{a}^\dagger e^{i(\pi/4)}}{2i} , \tag{2.8a}$$

$$\hat{X}_b=\frac{\hat{b}+\hat{b}^\dagger}{2},\quad \hat{Y}_b=\frac{\hat{b}-\hat{b}^\dagger}{2i} , \tag{2.8b}$$

with the variances $\langle(\Delta\hat{X}_{a(b)})^2\rangle=\langle\hat{X}^2_{a(b)}\rangle-\langle\hat{X}_{a(b)}\rangle^2$, $\langle(\Delta\hat{Y}_{a(b)})^2\rangle=\langle\hat{Y}^2_{a(b)}\rangle-\langle\hat{Y}_{a(b)}\rangle^2$ related to the squeezing parameters $S^{X(Y)}_{a(b)}$ [18] as follows:

$$S^X_{a(b)}=4\langle(\Delta\hat{X}_{a(b)})^2\rangle-1,\quad S^Y_{a(b)}=4\langle(\Delta\hat{Y}_{a(b)})^2\rangle-1 . \tag{2.9}$$

The squeezing condition in this notation reads $S^{X(Y)}_{a(b)}<0$ and maximum squeezing (100%) corresponds to the value -1.

We will also study whether the modes under consideration are in the minimum uncertainty states or not. To measure the degree of deviation of the state from the MUS we introduce the parameter $u_{a(b)}$, defined as

$$u_{a(b)}=\langle(\Delta\hat{X}_{a(b)})^2\rangle\langle(\Delta\hat{Y}_{a(b)})^2\rangle-\tfrac{1}{16} , \tag{2.10}$$

which for the MUS is equal to zero and otherwise is positive.

III. DYNAMICS OF THE QUANTUM LINEAR COUPLER

In this section we will analyze the simplest realization of the k-photon down-conversion process with $k=1$. In this case the Hamiltonian (1.4a) describes the dynamics of a lossless linear coupler [8] as well as the dynamics of a lossless beam splitter [19]. The dynamics of the quantum system described by the Hamiltonian (1.4a) with $k=1$ is exactly solvable. To show this we rewrite the Hamiltonian $\hat{H}^{(I)}_1$ in terms of generators of the SU(2) Lie algebra $\hat{J}_+$ and $\hat{J}_-$:

$$\hat{H}_1^{(I)}=\lambda_1(\hat{J}_+ + \hat{J}_-) , \quad (3.1)$$

where we use the Schwinger representation [20] for the generators of the SU(2) Lie algebra,

$$\hat{J}_+=\hat{a}^\dagger\hat{b},\quad \hat{J}_-=\hat{a}\hat{b}^\dagger,\quad \hat{J}_3=\frac{\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b}}{2} . \quad (3.2)$$

These generators obey the standard SU(2) commutation relations,

$$[\hat{J}_\pm,\hat{J}_3]=\mp\hat{J}_\pm,\quad [\hat{J}_+,\hat{J}_-]=2\hat{J}_3 . \quad (3.3)$$

Now using the disentangling theorem for the SU(2) Lie algebra (see, for instance, Ref. [21]) we can rewrite the evolution operator $\hat{U}_1^{(I)}$ in the interaction picture as follows:

$$\begin{aligned}\hat{U}_1^{(I)}(t)&=\exp[-i\lambda_1 t(\hat{a}^\dagger b+\hat{a}\hat{b}^\dagger)]\\ &=\exp[-i\tan(\lambda_1 t)\hat{a}^\dagger\hat{b}]\\ &\quad\times\exp\left[\ln\left[\frac{1}{\cos^2\lambda_1 t}\right]\frac{\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b}}{2}\right]\\ &\quad\times\exp[-i\tan(\lambda_1 t)\hat{a}\hat{b}^\dagger] . \end{aligned} \quad (3.4)$$

If we assume the pump mode to be initially prepared in the coherent state (2.3) with the real amplitude β and the signal mode in the vacuum state, i.e., the state vector of the signal-pump system at $t=0$ is

$$|\Psi(t=0)\rangle=|0\rangle_a|\beta\rangle_b , \quad (3.5)$$

then using the explicit expression for the evolution operator (3.4) we find for the state vector at $t>0$ the expression

$$|\Psi(t)\rangle=\hat{U}_1^{(I)}(t)|0\rangle_a|\beta\rangle_b=|\alpha(t)\rangle_a|\beta(t)\rangle_b , \quad (3.6)$$

where $|\alpha(t)\rangle_a$ and $|\beta(t)\rangle_b$ are the coherent states in the signal and the pump modes, respectively, with the amplitudes

$$\alpha(t)=-i\beta\sin\lambda_1 t,\quad \beta(t)=\beta\cos\lambda_1 t . \quad (3.7)$$

From Eq. (3.6) it follows that in the lossless quantum linear coupler with the pump initially in a coherent state and the signal mode in the vacuum state the energy is periodically transferred from the pump to the signal and back. The period of the energy transfer is π/λ_1. At the midpoint of this period the energy is totally transferred from the pump to the signal and the pump is completely depleted. On the other hand, at the initial stages of the time evolution, for which $\lambda_1 t\ll 1$, the pump mode is not affected by the coupling with the signal mode and the amplitude of the pump mode can be assumed to be constant, while the amplitude of the coherent state in the signal increases linearly, i.e., $\alpha(t)\simeq -i\beta\lambda_1 t$. This is the typical situation when the parametric approximation is adopted. The effective Hamiltonian describing the dynamics of the signal mode in the parametric approximation can be written in the form given by Eq. (1.4b) with $k=1$ and the corresponding evolution operator for the signal mode is *equal* to the Glauber-Sudarshan displacement operator.

Here we should emphasize that the parametric approximation in the case of the linear coupler with the pump mode initially in the coherent state and the signal mode in the vacuum state can be performed safely because the pump and the signal modes are not entangled at $t>0$, i.e., both modes are in a pure state with the entropy equal to zero. This is also true when not only the pump mode but also the signal mode is initially in a coherent state. In this case the initial state vector of the signal-pump system reads

$$|\Psi(t=0)\rangle=|\alpha\rangle_a|\beta\rangle_b , \quad (3.8)$$

and at $t>0$ we find that $|\Psi(t)\rangle$ is given as a product of two coherent states,

$$|\Psi(t)\rangle=\hat{D}_a(\alpha(t))\hat{D}_b(\beta(t))|0\rangle_a|0\rangle_b , \quad (3.9)$$

with the time-dependent amplitudes

$$\alpha(t)=\alpha\cos\lambda_1 t-i\beta\sin\lambda_1 t , \quad (3.10a)$$

$$\beta(t)=\beta\cos\lambda_1 t-i\alpha\sin\lambda_1 t . \quad (3.10b)$$

From Eqs. (3.9) and (3.10) it follows that with the initial condition (3.8) the parametric approximation can only be adopted in the case of a weak signal amplitude (i.e., $\alpha\ll\beta$); otherwise the amplitude of the pump mode even for a short range of interaction time cannot be assumed constant. Simultaneously, we should stress once more that with the initial coherent states in the pump and the signal modes the state vector $|\Psi(t)\rangle$ at $t>0$ is factorized and both modes are in a pure state for any interaction time. With other initial states the situation can be different and the pump and the signal modes can become entangled due to the quantum dynamics. For instance, if we assume the signal mode to be initially prepared in the Fock (number) state $|n\rangle_a$ and the pump mode in the coherent state $|\beta\rangle_b$, then at $t>0$ the signal-pump state vector at $t>0$ cannot be factorized as a product of two state vectors but takes the form [8]

$$\begin{aligned}|\Psi(t)\rangle&=\hat{U}_1^{(I)}|n\rangle_a|\beta\rangle_b\\ &=\sum_{m=0}^{\infty}\frac{\beta^m}{\sqrt{m!}}(\cos\lambda_1 t)^{n+m}\sum_{k=0}^{n}\sum_{l=0}^{m}(-i\tan\lambda_1 t)^{m+k+l}\\ &\qquad\times\frac{[m!n!(n+m-k-l)!(k+l)!]^{1/2}}{(n-k)!(m-l)!k!l!}|n+m-k-l\rangle_a|k+l\rangle_b . \end{aligned} \quad (3.11)$$

Even though in the case under consideration the pump and the signal modes are entangled at $t>0$ (i.e., $S_a=S_b>0$), it can be shown that for small interaction times ($\lambda_1 t\ll 1$) and for $\beta^2\gg n$ the entropy of the signal (pump) mode is small and the state vector (3.11) can be approximately written in the form [8]

$$|\Psi(t)\rangle\simeq\hat{D}_a(\alpha(t))|n\rangle_a|\beta\rangle_b , \quad (3.12)$$

with the time-dependent amplitude $\alpha(t)=-i\beta\lambda_1 t$ and β constant. The state $\hat{D}_a(\alpha(t))|n\rangle_a$, which is generated in the signal mode, is called the displaced number state [22] and it exhibits interesting nonclassical properties. From the above we can conclude that if the pump mode is initially in the highly excited coherent state with the number of photons much higher than in the signal mode, then for a limited range of time one can adopt the parametric approximation even though the pump and the signal modes become entangled due to the quantum dynamics (i.e., the entropy of the signal mode is very small for a limited range of time). On the other hand, if the pump mode is not initially in a coherent state the situation can be quite different and the application of the parametric approximation can be questionable even in the case when the signal mode is initially in the vacuum state. To illustrate this we present two examples.

First, if the pump mode is initially in the squeezed vacuum state $|\zeta\rangle_b$ [1] defined as

$$|\zeta\rangle_b=\hat{S}_b(\zeta)|0\rangle_b\ ;\ \ \hat{S}_b(\zeta)=\exp[\zeta(\hat{b}^\dagger)^2-\zeta^*\hat{b}^2]\ , \tag{3.13}$$

then the signal-pump state vector at $t>0$ takes the form

$$\begin{aligned}|\Psi(t)\rangle=\exp\{&\cos^2\lambda_1 t[\zeta(\hat{b}^\dagger)^2-\zeta^*\hat{b}^2]\\&+2i\sin\lambda_1 t\cos\lambda_1 t(\zeta\hat{a}^\dagger b^\dagger+\zeta^*\hat{a}\hat{b})\\&-\sin^2\lambda_1 t[\zeta(\hat{a}^\dagger)^2-\zeta^*\hat{a}^2\}|0\rangle_a|0\rangle_b\ .\end{aligned} \tag{3.14}$$

The explicit expression of the state (3.14) in the Fock bases can be found in Ref. [23]. In this state the pump and the signal modes are strongly entangled and the parametric approximation can be justified only during the first instants of the time evolution (see Sec. IV). Here we note only that the pump and the signal modes in the state (3.14) exhibit interesting nonclassical behavior (for details, see Ref. [23]).

Second, if we assume the pump mode initially in the number state $|n\rangle_b$ and the signal mode in the vacuum state, then at $t>0$ the signal-pump system evolves into the SU(2) coherent state [24,8]

$$|\Psi(t)\rangle=(1+|\zeta|^2)^{-n/2}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}^{1/2}\zeta^k|k\rangle_a|n-k\rangle_b\ , \tag{3.15}$$

which is a highly nonclassical state exhibiting a high degree of sub-Poissonian photon statistics (for details, see Ref. [24]). In this case again the pump and the signal modes are strongly entangled and the application of the parametric approximation is questionable even for short time intervals, except of the limit $t\to 0$.

Before we finish this section we should turn our attention to the fact that in the lossless linear coupler there are moments during the time evolution at which the pump and the signal become disentangled (i.e., $S_a=S_b=0$) irrespective of the initial states of the signal ($|\Phi_1\rangle_a$) and the pump ($|\Phi_2\rangle_b$). To show this we utilize the unitary transformation

$$\hat{U}_1\hat{a}\hat{U}_1^\dagger=\hat{a}\cos\lambda_1 t+i\hat{b}\sin\lambda_1 t\ , \tag{3.16a}$$

$$\hat{U}_1\hat{b}\hat{U}_1^\dagger=\hat{b}\cos\lambda_1 t+i\hat{a}\sin\lambda_1 t\ , \tag{3.16b}$$

where $\hat{U}_1$ is the time-evolution operator of the linear coupler in the interaction picture (3.4). Using this transformation it is easy to find that the initial signal-pump state vector

$$|\Psi(t=0)\rangle=|\Phi_1\rangle_a|\Phi_2\rangle_b\equiv\sum_{n,m=0}^{\infty}C_a^nC_b^m|n\rangle_a|m\rangle_b \tag{3.17a}$$

evolves according the relation

$$\begin{aligned}|\Psi(t)\rangle=\sum_{n,m=0}^{\infty}C_a^nC_b^m&\frac{(\hat{a}^\dagger\cos\lambda_1 t-i\hat{b}^\dagger\sin\lambda_1 t)^n}{\sqrt{n!}}\\&\times\frac{(\hat{b}^\dagger\cos\lambda_1 t-ia^\dagger\sin\lambda_1 t)^m}{\sqrt{m!}}|0\rangle_a|0\rangle_b\ .\end{aligned} \tag{3.17b}$$

From Eq. (3.17b) it follows that at $\lambda_1 t=\pi/2$ the signal-pump state vector can be factorized irrespective of its initial state, i.e.,

$$\begin{aligned}|\Psi(t=\pi/2\lambda_1)\rangle=&\sum_{m=0}^{\infty}C_b^m(-i)^m|m\rangle_a\\&\times\sum_{n=0}^{\infty}C_a^n(-i)^n|n\rangle_b\ .\end{aligned} \tag{3.18}$$

From Eq. (3.18) it follows that at $\lambda_1 t=\pi/2$ the pump and the signal up to the phase factors $(-i)^k$ "exchange" their states. In particular, if the signal mode was initially in the vacuum state and the pump mode in the squeezed vacuum then at $\lambda_1 t=\pi/2$ the signal mode is in the squeezed vacuum state (the direction of squeezing of this mode is rotated by $-\pi/4$ with respect to the direction of squeezing of the pump mode). This total transfer of statistical properties from the pump to the signal mode is only possible in the case of the quantum linear coupler. We will see in the next section that in the nonlinear coupler (i.e., down-convertor with $k>2$) the situation is completely different. We should also note that at $t=\pi\lambda_1$ the state vector $|\Psi(t)\rangle$ can be factorized again and the states of the pump and the signal are up to phase factors equal to their initial states. Complete restoration of initial conditions is obtained at $t=2\pi/\lambda_1$. The last is also seen from the expression (3.4) for the evolution operator which at $t=2\pi/\lambda_1$ turns to the identity operator.

IV. SIGNAL-PUMP ENTANGLEMENT IN THE TWO-PHOTON DOWN-CONVERSION

In the preceding section we have shown that for coherent input states the signal-pump state vector of the linear coupler can be factorized for any $t>0$, i.e., the pump and the signal remain in the pure state during the time evolution. This preservation of the purity of the pump and the signal modes is a very exceptional property of the linear coupler. In this section we show that in the two-photon down-conversion process described by the Hamiltonian

$$\hat{H}_2^{(I)}=\lambda_2[(\hat{a}^\dagger)^2\hat{b}+\hat{a}^2\hat{b}^\dagger]\ , \tag{4.1}$$

the situation is different. First of all, we should stress that the dynamics of the quantum-mechanical system corresponding to the Hamiltonian (4.1) cannot be described in an analytically closed form, but has to be studied numerically. For details of our numerical approach we refer the reader to the recent paper by Drobný and Jex [16].

Let us assume the signal mode initially in the vacuum state and the pump in the coherent state $|\beta\rangle_b$. With this initial state the pump and the signal in the linear coupler will remain in the pure state for any $t>0$. In the two-photon down-convertor the pump and the signal become entangled, their purity parameter $S_{\rm pur}$ becomes positive at $t>0$, and the signal mode is not exactly in a pure state anymore. Nevertheless, for a short range of time the purity parameter is still very small and approximately equal to zero. As we show during this time interval the parametric approximation can be adopted. In Fig. 1 we plot the time evolution of the purity parameter for several values of the initial intensity of the pump mode. From this figure it follows that the higher the initial intensity of the pump, the more rapidly the purity parameter increases and the shorter the time interval during which $S_{\rm pur}$ can be approximated by zero.

In Fig. 2 we plot various parameters describing statistical properties of the signal mode obtained via two-photon down-conversion with the pump intensity $\beta^2=9$ and for two values of the interaction time $\lambda_2 t=0.2$ and 0.5. As seen from Fig. 1 the purity parameter at $\lambda_2 t=0.2$ is approximately equal to zero (i.e., the signal mode is in a pure state), while $S_{\rm pur}$ at $\lambda_2 t=0.5$ is significantly greater than zero. In Fig. 2(a) contour plots of the Q function of the signal mode at $\lambda_2 t=0.2$ are shown. We see that the initial circle contours corresponding to the vacuum state are transformed into elliptical (squeezed) contours. The photon-number distribution [see Fig. 2(c)] exhibits significant oscillations. Taking into account that at $t=0.2$ the degree of squeezing in the signal mode is very large [see Fig. 2(e)] and the fact that the signal mode is in a pure state which simultaneously is a MUS [see Fig. 2(f)], we can conclude that under given conditions the squeezed vacuum is produced in the signal mode.

If we analyze the statistical properties of the pump mode in the process under consideration at $\lambda_2 t=0.2$ we find that the pump mode at this time is approximately in a coherent state, i.e., the Q function is represented by circle contours [see Fig. 3(a)], the photon-number distribution is Poissonian [Fig. 3(c)], there is no squeezing exhibited by the pump [Fig. 3(e)], and finally the pump mode is in the MUS [Fig. 3(f)]. On the other hand, at $\lambda_2 t=0.5$ the quantum-statistical properties of the pump mode are significantly changed by the back action of the signal mode. First of all, the pump is not in a pure state at this moment (Fig. 1), the Q function is "deformed" [Fig. 3(b)], and the photon-number distribution exhibits oscillations [Fig. 3(d)]. Moreover, the pump mode exhibits a large degree of squeezing [Fig. 3(e)]. Needless to say the pump mode is not in the MUS at this moment [Fig. 3(f)]. Obviously, under this circumstances one cannot adopt the

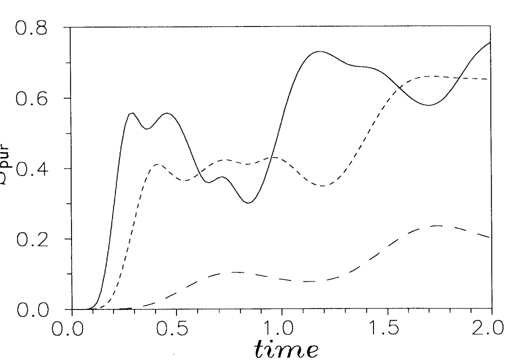

FIG. 1. The time evolution of the purity parameter $S_{\rm pur}$ in the two-photon down-conversion for initial state $|0\rangle_a|\beta\rangle_b$ with $\beta=1$ (long-dashed curve), $\beta=3$ (short-dashed curve), and $\beta=5$ (solid curve). Increase of the intensity of the b mode leads to the stronger entanglement between the pump and the signal. Time scale is given by $\lambda_2 t$.

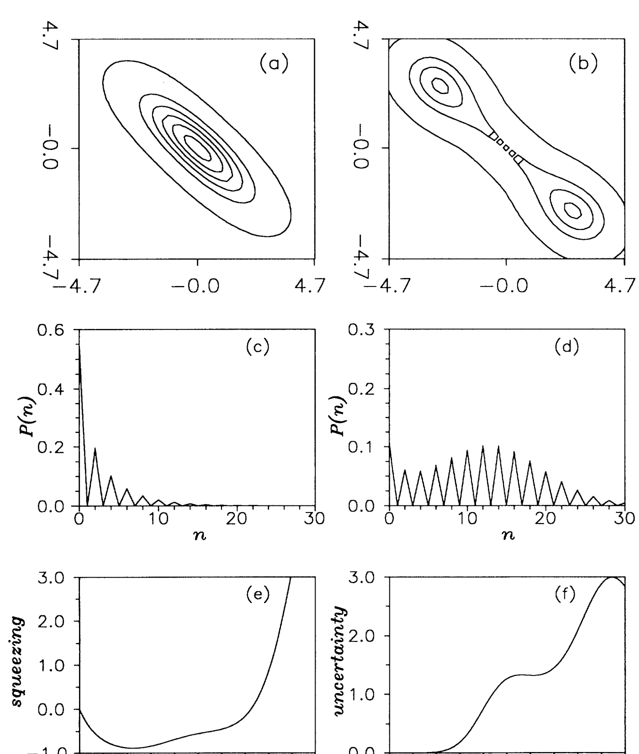

FIG. 2. Parameters of the a mode for two-photon down-conversion ($\beta=3$): (a) Q function for $\lambda_2 t=0.2$; (b) Q function for $\lambda_2 t=0.5$; (c) photon-number distribution for $\lambda_2 t=0.2$; (d) for $\lambda_2 t=0.5$; (e) shows the time evolution of the squeezing parameter S_a^y; and (f) gives the uncertainty function u_a.

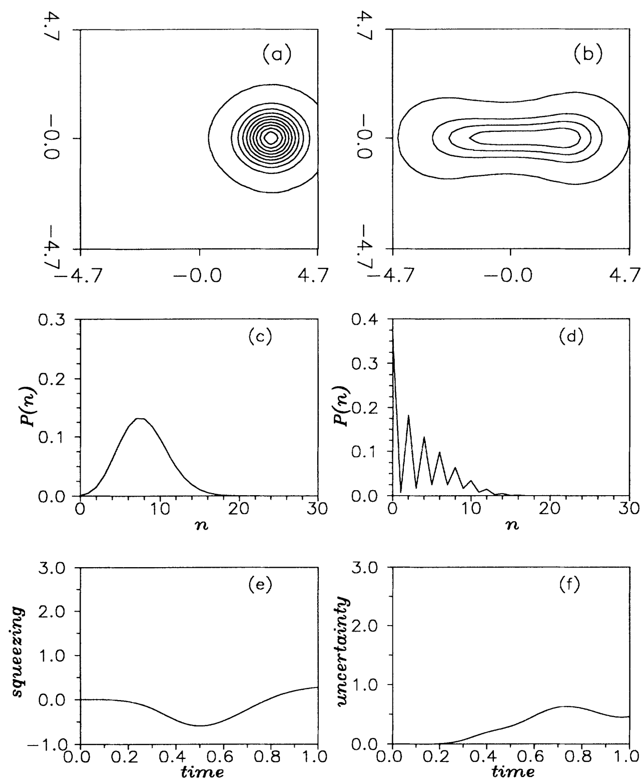

FIG. 3. The same parameters as in Fig. 2 but for the pump mode.

parametric approximation. It is quite interesting to note that the signal mode at this moment also exhibits nontrivial behavior. The Q function of the signal mode is split into two parts [Fig. 2(b)], which means that a sort of superposition [25] state is generated in this mode. Even though this is not a pure state (see Fig. 1) the quantum interference between component states leads to the appearance of nonclassical effects. Namely, one can observe oscillations in the photon-number distribution [Fig. 2(d)] and even some degree of squeezing [Fig. 2(e)]. As seen from Fig. 2(f) the signal mode at $\lambda_2 t=0.5$ is not in a MUS. Moreover, the degree of the deviation from the MUS is much higher for the signal mode than for the pump mode [compare Figs. 2(f) and 3(f) at $\lambda_2 t=0.5$].

From the above it follows that for times small enough ($\lambda_2 t\beta \ll 1$) the signal-pump state vector can be written in a factorized form (2.4) (i.e., $|\Psi(t)\rangle=|\Phi(t)\rangle_a|\Phi(t)\rangle_b$), where

$$|\Phi(t)\rangle_a=\hat{S}_a(-it\beta)|0\rangle_a, \quad |\Phi(t)\rangle_b=\hat{D}_b(\beta)|0\rangle_b , \tag{4.2}$$

with $\hat{S}_a(\xi)=\exp[\xi(\hat{a}^\dagger)^2-\xi^*\hat{a}^2]$ being the squeeze operator [1]. Strictly speaking, the above factorization can take place only for times t smaller than t_c, where

$$t_c=\frac{1}{\sqrt{2}\beta\lambda_2} . \tag{4.3}$$

In this case the evolution operator $\hat{U}_2^{(I)}(t) = \exp(-i\hat{H}_2^{(I)}t)$ can be approximated as

$$\hat{U}_2^{(I)}(t)\simeq\hat{1}-i\hat{H}_2^{(I)}t , \tag{4.4}$$

and the signal-pump state vector takes the form

$$|\Psi(t)\rangle=\hat{U}_2^{(I)}(t)|0\rangle_a|\beta\rangle_b \simeq(|0\rangle_a-it\sqrt{2}\lambda_2\beta|2\rangle_a)|\beta\rangle_b . \tag{4.5}$$

As seen from Eq. (4.5) for times $t \ll t_c$ the signal mode is in a pure state ($S_{\rm pur}=0$, see Fig. 1) and is given as a superposition of two Fock states. The dominant contribution from these states is clearly seen from Fig. 2(c). A significant degree of squeezing in the signal mode in this case can be explained as a direct consequence of the quantum interference between the vacuum state and the two-photon Fock state (for details see Ref. [26]).

If we strictly demand that the parametric approximation can be adopted *only* in the case when the signal mode at the output is in a pure state then this approximation is valid only for times smaller than t_c. For larger times one should take into account the third term $(-it\hat{H}_2^{(I)}t)^2/2!$ in the Taylor-series expansion of the evolution operator [see Eq. (4.4)]. But this term already leads to an entanglement between the pump and the signal.

We should note here that the assumption about the purity of the signal mode in the two-photon down-conversion leads to a peculiar ambiguity in the descrip-

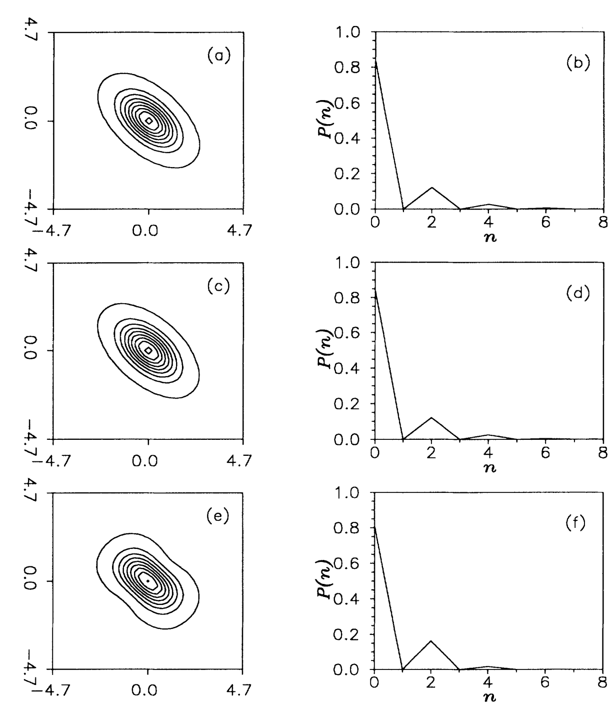

FIG. 4. The Q function and the photon-number distribution of the signal mode in the case of the purely quantized model with $\beta=3$ and $\lambda_2 t=0.1$ [(a) and (b), respectively]; in the case of the parametric approximation with the effective Hamiltonian (4.9) [(c) and (d), respectively]; and in the case of the parametric approximation with the effective Hamiltonian (4.10) [(e) and (f), respectively].

tion of this mode. As we stated above, the signal mode is in a pure state for interaction times $0<t\ll t_c$, when the state vector of the signal mode is given by the expression (4.5). This superposition state is equal to the first two terms in the expansion of the squeezed vacuum state (4.2) into the Fock basis $|n\rangle_a$. Simultaneously, the state $|\Phi_1(t)\rangle_a$ can be described with the same precision as a generalized coherent state given by the relation

$$|\Phi_1(t)\rangle_a=\hat{D}_a^{(2)}(\zeta)|0\rangle_a\ , \tag{4.6}$$

where $\zeta=-i\beta\sqrt{2}\lambda_2 t$, and the generalized displacement operator $\hat{D}_a^{(2)}(\zeta)$ [27,28],

$$\hat{D}_a^{(k)}(\xi)=\exp(\xi\hat{A}_k^\dagger-\xi^*\hat{A}_k)\ , \tag{4.7}$$

is given in terms of the multiphoton Brandt-Greenberg operators [29]

$$\hat{A}_k^\dagger=\left\{\left[\frac{\hat{n}_a}{k}\right]\frac{(\hat{n}_a-k)!}{\hat{n}_a!}\right\}^{1/2}(\hat{a}^\dagger)^k,\quad \hat{A}_k=(\hat{A}_k^\dagger)^\dagger\ . \tag{4.8}$$

Here $[x]$ denotes the integer part of x and the operators $\hat{A}_k,\hat{A}_k^\dagger$ satisfy the Weyl-Heisenberg commutation relations $[\hat{A}_k,\hat{A}_k^\dagger]=1$.

The generalized coherent state (4.6) describes very well the output state in the signal mode for small values of ζ. This can be seen from Fig. 4, where we plot the Q function and the photon-number distribution of the signal mode in three cases: (1) the case of the purely quantized model at the time $\lambda_2 t=0.1$ with the coherent amplitude of the initial pump field equal to 3 [Figs. 4(a) and 4(b)]; (2) the case of the parametric approximation with the effective Hamiltonian

$$\hat{H}_2^{(I,P)}=\lambda_2\beta[(\hat{a}^\dagger)^2+\hat{a}^2] \tag{4.9}$$

[see Figs. 4(c) and 4(d)]; and (3) the case of the parametric approximation with the effective Hamiltonian given in terms of the Brandt-Greenberg multiphoton operators:

$$\hat{H}_2'^{(I,P)}=\lambda_2\beta\sqrt{2}(\hat{A}_2^\dagger+\hat{A}_2) \tag{4.10}$$

[see Figs. 4(e) and 4(f)]. The interaction time in the first case is taken short so that only a fraction of the energy ($\sim 2.5\%$) is transferred from the pump to the signal mode. The signal mode at this time is approximately in a pure state. In the other two cases the interaction time is chosen in such a way that the intensity of the signal mode is equal in all cases. From Fig. 4 we can conclude that both effective Hamiltonians (4.9) and (4.10) describe sufficiently well the signal mode at initial stages of the time evolution. For longer interaction times the parametric approximation with the effective Hamiltonian (4.10) is not as good as the one with the Hamiltonian (4.9). For instance, at $\lambda_2 t=0.2$ when $\sim 12\%$ of the energy is transferred from the pump to the signal mode the effective Hamiltonian (4.9) describes the signal mode better than the Hamiltonian (4.10). This is clearly seen from Fig. 5, from which we can also conclude that the effective Hamiltonian (4.10) gives results which are at least in qualitative agreement with the purely quantum model. We will utilize this observation in the next section, in which we will analyze multiphoton down-conversion.

Up to this moment we have assumed the signal mode in the two-photon down-conversion to be initially prepared in the vacuum state $|0\rangle_a$. The pump mode at $t=0$ was assumed to be in the coherent state $|\beta\rangle_b$. The variation of the statistics of the signal mode at $t=0$ can significantly change the character of the output state in this mode. In particular, if we assume the signal mode to be prepared in the Fock state $|n\rangle_a$, then the entanglement between the pump and the signal modes becomes stronger. Generally, the larger the initial number of photons n in the signal mode the stronger the entanglement

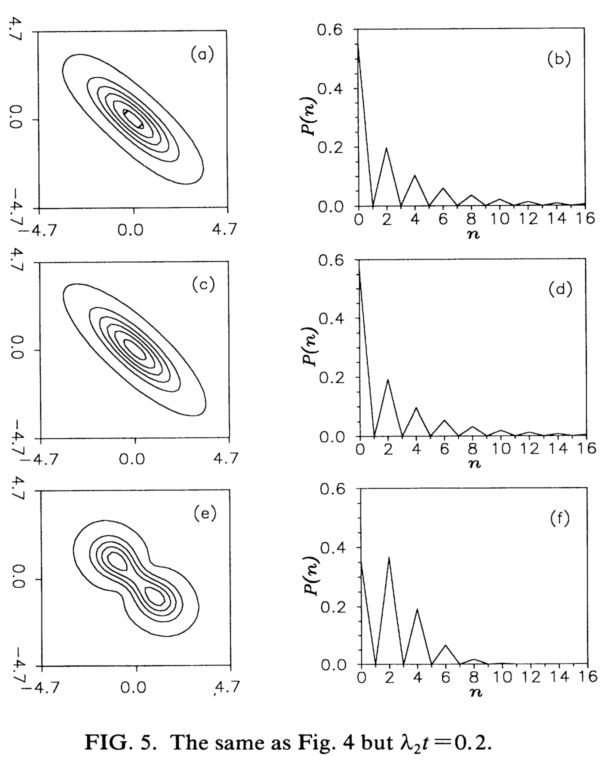

FIG. 5. The same as Fig. 4 but $\lambda_2 t=0.2$.

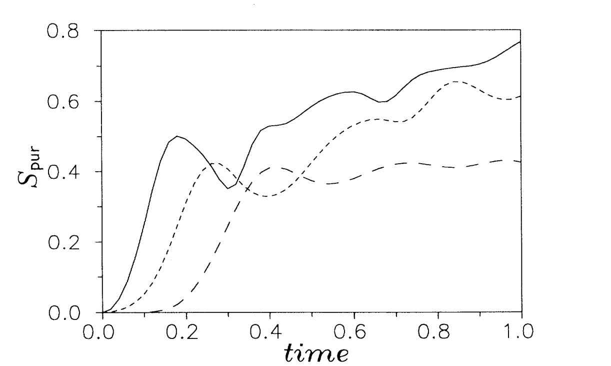

FIG. 6. The time evolution of the purity parameter $S_{\rm pur}$ in the two-photon down-conversion. The b mode is initially in the coherent states with $\beta=3$ and the a mode is in a Fock state $|n\rangle$: $n=0$ (long-dashed curve), $n=2$ (short-dashed curve), $n=4$ (solid curve). The higher the number of photons in the a mode the stronger the entanglement between the modes.

for given β and $\lambda_2 t$ and consequently the larger the value of the purity parameter $S_{\rm pur}$ (see Fig. 6). From here we can conclude that variation in the statistics of the initial signal mode can significantly constrain the applicability of the parametric approximation.

V. k-PHOTON DOWN-CONVERSION

In the k-photon down-conversion process one photon of the pump mode with frequency $k\omega$ is transformed into k photons of the signal mode with frequency ω. If we assume the signal mode initially to be prepared in the vacuum state then at $t>0$ we find

$$\langle \hat{a} \rangle_a = \langle \hat{a}^2 \rangle_a = \cdots = \langle \hat{a}^{k-1} \rangle_a \equiv 0 \ , \tag{5.1}$$

which means that in the k-photon down-conversion with $k>2$ the signal mode does not exhibit quadrature squeezing. Nevertheless, observation of higher-order squeezing [30], or amplitude-squared [31] and amplitude kth-power squeezing [28], is not excluded.

In the k-photon down-conversion the signal and the pump modes become entangled in the same way as in the two-photon down-conversion. Generally speaking, the higher the order of the process (i.e., the higher the k), the stronger the entanglement (at least during the first instants of the time evolution). This is seen from Fig. 7. For initial states of the signal mode other then the vacuum state the entanglement is even stronger.

The signal-pump state vector in the k-photon down-conversion process can be written in the factorized form (4.2a) only during first instants of the time evolution when

$$t < \frac{1}{\sqrt{k!}\beta\lambda_k} \ . \tag{5.2}$$

For times obeying condition (5.2) the evolution operator $\hat{U}_k^{(I)}(t)$ can be approximated as

$$\hat{U}_k^{(I)}(t) = \hat{1} - it\hat{H}_k^{(I)} \ , \tag{5.3}$$

and the signal-pump state vector can be factorized as $|\Psi(t)\rangle = |\Phi(t)\rangle_a |\Phi(t)\rangle_b$, where

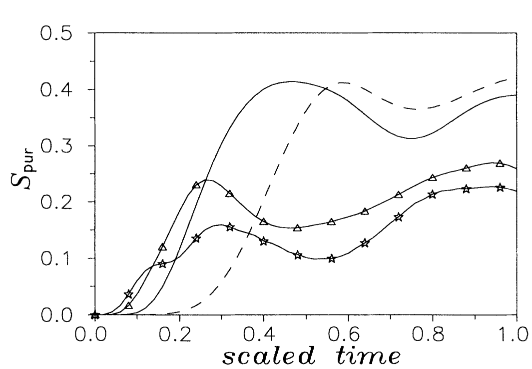

FIG. 7. The time evolution of the purity parameter $S_{\rm pur}$ in the k-photon down-conversions with $\beta=3$: $k=2$ (dashed curve), $k=3$ (solid curve), $k=4$ (curve with △), $k=5$ (curve with ☆). The scaled time is taken to be $\tau = t\sqrt{k!}\lambda_k$.

$$|\Phi(t)\rangle_a = |0\rangle_a - i\lambda_k t\sqrt{k!}\beta|k\rangle_a, \quad |\Phi(t)\rangle_b = |\beta\rangle_b \ . \tag{5.4}$$

The Q function corresponding to the state $|\Phi(t)\rangle_a$ exhibits k-fold rotational symmetry (group of symmetries $\mathcal{C}_k$) and can be written in the form

$$Q(\alpha) \sim e^{-|\alpha|^2}[1 - 2\beta t|\alpha|^k \sin k\Theta + \beta^2 t^2 |\alpha|^{2k}] \ , \tag{5.5}$$

where $\alpha = |\alpha| \exp(i\Theta)$. This Q function, as well as the phase properties of the signal mode in the process under consideration, has recently been analyzed by Tanas and Gantsog [13].

In the Introduction we have shown that the application of the parametric application in the k-photon down-conversion ($k>2$) is questionable because of the divergence of the vacuum-to-vacuum matrix element of the evolution operator (1.5). In other words, it is questionable to describe the signal mode in the k-photon down-conversion at $t>0$ by the state vector

$$|\xi;k\rangle_a = \hat{S}_a(\xi;k)|0\rangle_a = \exp[\xi(\hat{a}^\dagger)^k - \xi^* \hat{a}^k]|0\rangle_a \ , \tag{5.6}$$

even though some numerical calculations can be performed by using the Padé approximants [for instance, Braunstein and McLachlan [11] have shown that for small values of ζ, the Q function of the state (5.6) is equal to the expression (5.5)].

In the preceding section we have shown that for small interaction times the dynamics of the signal mode can be approximated either by the effective Hamiltonian (4.9) given in terms of the operators $\hat{a}^2$ or the Hamiltonian (4.10) given in terms of the multiphoton Brandt-Greenberg operators $\hat{A}_2$. The effective Hamiltonian (4.9) describes the signal mode at $t>0$ better than the Hamiltonian (4.10). Nevertheless, the latter gives qualitatively good results. In the case of the k-photon down-conversion ($k>2$) the application of the effective Hamiltonian (1.4b) given in terms of the operators $\hat{a}^k$ is problematic (see above). On the other hand, the dynamics of the signal mode during the first instants of the time evolution (when the purity parameter is approximately equal to zero) can be described by the effective Hamiltonian

$$\hat{H}_k'^{(I,P)} = \sqrt{k!}\beta\lambda_k(\hat{A}_k^\dagger + \hat{A}_k) \ , \tag{5.7}$$

given in terms of the multiphoton operators $\hat{A}_k, \hat{A}_k^\dagger$. In the time interval in which $S_{\rm pur}$ is approximately equal to zero this evolution describes the signal mode very well. Moreover, divergences of the matrix elements do not occur in this case.

VI. CONCLUDING REMARKS

We have studied the entanglement between the pump and the signal modes in the k-photon down-conversion process. We have shown that the entanglement between the pump and the signal depends on the order of the nonlinear process under consideration and the initial quantum statistics of the signal and the pump modes.

The analysis of the signal-pump entanglement in the k-photon down-conversion presented in this paper can be performed in a more realistic way. Namely, one can consider a model of the k-photon down-conversion when the optical crystal is placed within an optical cavity which is coherently driven at the frequency of the pump. The Hamiltonian describing this device is

$$\hat{H}_k = \omega \hat{a}^\dagger \hat{a} + k\omega \hat{b}^\dagger \hat{b} + \lambda_k [(\hat{a}^\dagger)^k \hat{b} + \hat{a}^k b^\dagger] + (\varepsilon \hat{b}^\dagger + \varepsilon^* \hat{b}) , \qquad (6.1)$$

where ε is the amplitude of a classical driving field multiplied by the coupling constant between the pump and the signal field. In the model described by the Hamiltonian (6.1) the pump depletion is compensated by the action of the classical driving field. The dynamics of this model with a special emphasis on the entanglement between modes are planned to be studied elsewhere. The influence of mode damping will be taken into account as well.

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