Propagator of the general driven time-dependent oscillator

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In this paper we generalize Wang's approach [J. Phys. A **20**, 5041 (1987)] and investigate the algebraic structure of the Schrödinger equation associated with a general driven time-dependent oscillator. Using the Lie-algebraic technique we obtain an exact form of the time-evolution operator which, in turn, enables us to derive the propagator of the system readily. Since the propagator is for the most general time-dependent oscillator, results for any special case can be easily deduced from it. These results will be useful for future studies in quantum optics as well as in atomic and molecular physics.

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In a recent communication Wang [1] investigated the algebraic structure of the Schrödinger equation associated with the Hamiltonian of a general driven time-independent harmonic oscillator, and took advantage of Lie algebra to derive the kernels of the equation. The method is simple and gives the same results as those obtained by the path-integral approach. In this paper we shall extend the method to the case of a general driven time-dependent oscillator whose Hamiltonian takes the form [2]

$$H(t) = \frac{p^2}{2m(t)} + \tfrac{1}{2}m(t)\omega(t)^2x^2 - m(t)f(t)x, \qquad (1)$$

where the mass parameter is taken as

$$m(t) = m_0 \exp\left[2\int \gamma(t)dt\right] \qquad (2)$$

and $\omega(t)$, $f(t)$, and $\gamma(t)$ are arbitrary functions of time. The study of problems involving the time-dependent oscillator has long been a research area of considerable interest. Apart from its intrinsic mathematical interest, these problems have invoked much attention because of their connections with many other problems belonging to different areas of physics, such as molecular physics, quantum chemistry, quantum optics, plasma physics, gravitation, quantum field theory, etc. For instance, Oh *et al.* [3] investigated a molecular system absorbed on a dielectric solid surface, modeled as a damped harmonic oscillator driven by a time-dependent external force. Colegrave and Abdalla [4] studied the harmonic oscillator with a constant frequency and a time-dependent mass in order to describe the electromagnetic-field intensities in a Fabry-Pérot cavity. Also, Lemos and Natividade [5] have solved the harmonic oscillator with a time-dependent frequency and a constant mass in an expanding universe.

To begin with let us consider the Hamiltonian $H_0(t)$

$$H_0(t) = \frac{p^2}{2m(t)} + \tfrac{1}{2}m(t)\omega(t)^2x^2 \; . \qquad (3)$$

It is well known that $H_0(t)$ can be rewritten in terms of the su$(1,1)$ generators as follows [6]:

$$H_0(t) = a_1(t)J_+ + a_2(t)J_0 + a_3(t)J_-, \qquad (4)$$

where

$$J_+ = \frac{i}{2\hbar}x^2, \qquad (5)$$

$$J_- = \frac{i}{2\hbar}p^2, \qquad (6)$$

$$J_0 = \frac{i}{4\hbar}(px+xp) \qquad (7)$$

and

$$a_1(t) = -i\hbar m(t)\omega(t)^2, \qquad (8)$$

$$a_2(t) = 0, \qquad (9)$$

$$a_3(t) = -\frac{i\hbar}{m(t)} \; . \qquad (10)$$

The operators J_+, J_0, and J_- form the su$(1,1)$ Lie algebra

$$[J_+, J_-] = -2J_0, \qquad (11)$$

$$[J_0, J_\pm] = \pm J_\pm \; . \qquad (12)$$

The corresponding Schrödinger equation is

$$H_0(t)|\Psi_0(t)\rangle = i\hbar\frac{\partial}{\partial t}|\Psi_0(t)\rangle \; . \qquad (13)$$

As usual, we will define the evolution operator $U_0(t,0)$ such that

$$|\Psi_0(t)\rangle = U_0(t,0)|\Psi_0(0)\rangle, \qquad (14)$$

where $|\Psi_0(0)\rangle$ is the wave function at time $t=0$. Inserting (14) into (13) yields the evolution equation

$$H_0(t)U_0(t,0) = i\hbar\frac{\partial}{\partial t}U_0(t,0), \qquad (15)$$

$$U_0(0,0) = 1 \; . \qquad (16)$$

Since J_+, J_0, and J_- form a closed Lie algebra su$(1,1)$, the evolution operator can be expressed in the following form:

$$U_0(t,0) = \exp[c_1(t)J_+]\exp[c_2(t)J_0]\exp[c_3(t)J_-], \quad (17)$$

where $c_i(t)$ are to be determined. Then by direct differentiation with respect to time, we obtain

$$\frac{\partial}{\partial t}U_0(t,0) = [h_+(t)J_+ + h_0(t)J_0 + h_-(t)J_-]U_0(t,0) \quad (18)$$

with

$$h_+(t) = \frac{dc_1}{dt} - c_1\frac{dc_2}{dt} + c_1^2\exp(-c_2)\frac{dc_3}{dt}, \quad (19)$$

$$h_0(t) = \frac{dc_2}{dt} - 2c_1\exp(-c_2)\frac{dc_3}{dt}, \quad (20)$$

$$h_-(t) = \exp(-c_2)\frac{dc_3}{dt} . \quad (21)$$

Substituting (4), (17), and (18) into (15), and comparing the two sides, we obtain after simplication

$$c_1(t) = m(t)\frac{\partial}{\partial t}\ln[F(t)], \quad c_1(0) = 0 \quad (22)$$

$$c_2(t) = -2\ \ln\left|\frac{F(t)}{F(0)}\right|, \quad (23)$$

$$c_3(t) = -F(0)^2\int_0^t \frac{du}{m(u)F(u)^2}, \quad (24)$$

where $F(t)$ satisfies the differential equation

$$\frac{d^2F(t)}{dt^2} + \xi(t)\frac{dF(t)}{dt} + \omega(t)^2F(t) = 0 \quad (25)$$

and

$$\xi(t) = \frac{\partial}{\partial t}\ln[m(t)] . \quad (26)$$

The second-order differential equation can be cast in the standard form such that

$$\left\{\frac{d^2}{dt^2} + \lambda(t)\right\}G(t) = 0, \quad (27)$$

with

$$G(t) = g(t)F(t), \quad (28)$$

$$\lambda(t) = \omega(t)^2 - h(t), \quad (29)$$

$$h(t) = \frac{1}{g(t)}\frac{d^2g(t)}{dt^2}, \quad (30)$$

$$g(t) = \sqrt{m(t)}. \quad (31)$$

Infeld and Hull have noted that most of the analytically solvable second-order differential equations, involving a single variable, which are of interest in electromagnetic and quantum theory can be transformed into this standard form [7]. Since $U_0(t,0)$ is known, the evolution operator U describing the whole system will be given by

$$U(t,0) = U_0(t,0)U_I(t,0) , \quad (32)$$

where $U_I(t,0)$ satisfies the evolution equation

$$H_I(t)U_I(t,0) = i\hbar\frac{\partial}{\partial t}U_I(t,0), \quad (33)$$

$$U_I(0,0) = 1 , \quad (34)$$

with $H_I(t)$ being defined by

$$H_I(t) = U_0^\dagger(t,0)\{-m(t)f(t)x\}U_0(t,0) . \quad (35)$$

By straightforward evaluation of (35) we obtain

$$H_I(t,0) = -m(t)f(t)\exp\left[-\frac{c_2(t)}{2}\right]\{x - c_3(t)p\} . \quad (36)$$

In terms of the generators of the Heisenberg-Weyl algebra, $H_I(t)$ can be written as [8]

$$H_I(t) = b_1(t)e_1 + b_2(t)e_2 + b_3(t)e_3 , \quad (37)$$

where

$$e_1(t) = \frac{i}{\sqrt{\hbar}}p , \quad (38)$$

$$e_2(t) = \frac{i}{\sqrt{\hbar}}x , \quad (39)$$

$$e_3(t) = i \quad (40)$$

and

$$b_1(t) = \frac{\sqrt{\hbar}}{i}m(t)f(t)c_3(t)\exp\left[-\frac{c_2(t)}{2}\right] , \quad (41)$$

$$b_2(t) = i\sqrt{\hbar}m(t)f(t)\exp\left[-\frac{c_2(t)}{2}\right] , \quad (42)$$

$$b_3(t) = 0 . \quad (43)$$

The operators e_i form the Heisenberg-Weyl Lie algebra

$$[e_1, e_2] = e_3 , \quad (44)$$

$$[e_1, e_3] = [e_2, e_3] = 0 . \quad (45)$$

Following a similar procedure as shown above, the evolution operator $U_I(t,0)$ is found to be

$$U_I(t,0) = \exp[d_1(t)e_1]\exp[d_2(t)e_2]\exp[d_3(t)e_3] \quad (46)$$

with

$$d_1(t) = \frac{1}{i\hbar}\int_0^t b_1(u)du , \quad (47)$$

$$d_2(t) = \frac{1}{i\hbar}\int_0^t b_2(u)du , \quad (48)$$

$$d_3(t) = -\frac{1}{i\hbar}\int_0^t b_2(u)d_1(u)du . \quad (49)$$

Hence, we have obtained an exact form of the time-evolution operator $U(t,0)$ of the general driven time-dependent oscillator.

Next using the well-known relations

$$e^{\alpha\partial_x} f(x) = f(x+a), \tag{50}$$

$$e^{\alpha x\partial_x} f(x) = f(e^{\alpha} x), \tag{51}$$

as well as the formula [9]

$$e^{\alpha\partial_{xx}} f(x) = \frac{1}{\sqrt{4\pi\alpha}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-y)^2}{4\alpha}\right\} f(y)dy \ , \tag{52}$$

we can readily derive the formal solution of the time-dependent Schrödinger equation associated with the general driven time-dependent oscillator,

$$\Psi(x,t) = U(t,0)\Psi(x,0)$$

$$= \sqrt{\frac{i}{2\pi\hbar c_3}} \int_{-\infty}^{\infty} dy \exp\left\{\frac{(xe^{c_2/2} + \sqrt{\hbar} d_1 - y)^2}{2i\hbar c_3} + \frac{ic_1}{2\hbar}x^2 + \frac{id_2}{\sqrt{\hbar}}y + id_3 + \frac{c_2}{4}\right\} \Psi(y,0) \ . \tag{53}$$

Since the propagator of the system satisfies the relation

$$\Psi(x,t) = \int_{-\infty}^{\infty} dy\, K(x,t;y,0)\Psi(y,0) \ , \tag{54}$$

by comparing (53) and (54) we have

$$K(x,t;y,0) = \sqrt{\frac{i}{2\pi\hbar c_3}} \exp\left\{\frac{(xe^{c_2/2} + \sqrt{\hbar} d_1 - y)^2}{2i\hbar c_3} + \frac{ic_1}{2\hbar}x^2 + \frac{id_2}{\sqrt{\hbar}}y + id_3 + \frac{c_2}{4}\right\}. \tag{55}$$

This propagator is for the most general time-dependent oscillator, and thus results for any special case can be easily deduced from it. In the following we shall present some examples to illustrate the validity of our method.

(i) A free particle — $m(t) = m_0$ and $\omega(t) = f(t) = 0$ [1]

$$K(x,t;y,0) = \sqrt{\frac{m_0}{2\pi i\hbar t}} \exp\left\{-\frac{m_0(x-y)^2}{2i\hbar t}\right\} \ . \tag{56}$$

(ii) A harmonic oscillator in a gravitational field — $m(t) = m_0$, $\omega(t) = \omega_0$, and $f(t) = -g$ with m_0, ω_0, and g being constant [1]

$$K(x,t;y,0) = \sqrt{\frac{m_0\omega_0}{2\pi i\hbar \sin(\omega_0 t)}} \exp\left\{ \frac{im_0\omega_0}{2\hbar \sin(\omega_0 t)}[(x^2+y^2)\cos(\omega_0 t) - 2xy] - \frac{im_0 g}{\hbar\omega_0}\tan\left(\frac{\omega_0 t}{2}\right)(x+y) + \frac{im_0 g^2}{\hbar\omega_0^2}\left[\frac{t}{2} - \frac{1}{\omega_0}\tan\left(\frac{\omega_0 t}{2}\right)\right]\right\} \ . \tag{57}$$

(iii) An oscillator with periodic mass — $m(t) = m_0\cos^2(\delta t)$, $\omega(t) = \omega_0$, and $f(t) = 0$ [10]

$$K(x,t;y,0) = \sqrt{\frac{m_0\Omega\cos(\delta t)}{2\pi i\hbar \sin(\Omega t)}} \exp\left\{\frac{im_0\Omega}{\hbar\sin(2\Omega t)}[x\cos(\delta t) - y\cos(\Omega t)]^2 + \frac{im(t)}{2\hbar}[\delta\tan(\delta t) - \Omega\tan(\Omega t)]x^2\right\} \ , \tag{58}$$

where $\Omega = \sqrt{\omega_0^2 + \delta^2}$.

(iv) A damped oscillator — $m(t) = m_0\exp(-2\delta t)$, $\omega(t) = \omega_0$, and $f(t) = 0$ with $\omega_0 > \delta > 0$ [2]

$$K(x,t;y,0) = \sqrt{\frac{m_0\Omega}{2\pi i\hbar\sin(\Omega t)}} \exp(-\delta t/2) \exp\left\{\frac{im(t)\sin(\Omega t)}{2\hbar\eta(t)}\left[\left(\frac{x\Omega - y\eta(t)\exp(\delta t)}{\sin(\Omega t)}\right)^2 - (\omega_0 x)^2\right]\right\} \ , \tag{59}$$

where $\eta(t) = \Omega\cos(\Omega t) - \delta\sin(\Omega t)$ and $\Omega = \sqrt{\omega_0^2 - \delta^2}$.

In summary, we have investigated the algebraic structure of the Schrödinger equation associated with a general driven time-dependent oscillator. Using the Lie-algebraic technique we have obtained an exact form of the time-evolution operator which, in turn, enables us to derive the propagator of the system readily. Since the propagator is for the most general time-dependent oscillator, results for any special case can be easily deduced from it, and these results will be useful for future studies in quantum optics as well as in atomic and molecular physics.

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