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Tempered diffusion: A transport process with propagating fronts and inertial delay

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The speed of sound is the highest admissible free velocity in a medium. This property is lost in the classical transport theory that predicts the unphysical divergence of the flux with gradients. Keeping the acoustic speed provides the means to control the growth of the flux and enables us to derive a better transport theory; flux saturates as the gradients became unbounded. Initial discontinuities persist for a finite time and diffusion fronts are convected with a finite speed. Various applications are considered (heat or mass transfer, plasma diffusion, and hydrodynamics).

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I. INTRODUCTION

On the level of a paradigm, the transport of a physical quantity Θ in a continuum medium is given by the Fokker-Planck equation (FPE)

$$\Theta_t = [D_0 \Theta_x]_x, \quad D_0 > 0 . \tag{1}$$

Let q be the associated flux, then Eq. (1) can be written as

$$\Theta_t + q_x = 0 , \tag{2a}$$

$$q + D_0 \Theta_x = 0 . \tag{2b}$$

The FPE is the continuum limit of many complex processes. It misses some of the properties of its antecedents but brings some of its own. One difficulty, though more of conceptual rather than of practical importance, is the lack of propagating fronts. The second deficiency of the FPE is of practical importance; it is the divergence of the flux with gradients; it causes any imposed discontinuity to be dissolved immediately. This subject is the *focus of the present work*. To overcome these shortcomings the FPE must be extended to include some of the nonlocal features of the original problem lost on the road to the continuum. The nonlocality may be due to the discrete nature of interactions or due to inertial delay.

As the simplest example of spatial nonlocality consider the probability distribution for time-continuous Markovian process, $u(x,t)$ that obeys the Kramers-Moyal expansion (KME)

$$u(x,t) = \sum_n (-1)^n D^{(n)}(x) u(x,t) , \tag{3a}$$

where $D^{(n)}(x)$ is the nth expansion coefficient. The FPE is obtained by truncating the KME after the second term. Naive truncation at any higher order leads to either an ill-posed problem or negative transition probability. This trap—Pavula's Paradox—can be circumvented if instead of truncating expansion (3a), one introduces the nonlocal approximation

$$u_t = \partial_x^2 D^{(2)} [1 - (1/D^{(2)})\, \partial_x^2 D^{(4)}]^{-1} u , \tag{3b}$$

where the inverse of the bracketed term is taken in the operator sense. The original process (3a) is nonlocal and so is approximation (3b) but the length scale of nonlocality is limited to the interlattice spacing h. In fact, when D_2 and D_4 are constants, I rewrite (3b) as

$$u_t = D^* \otimes u_{xx} , \tag{3c}$$

where

$$D^*(\mathbf{k}) = 1/(1+\sigma^2 \mathbf{k}^2), \quad \sigma^2 = D^{(4)}/D^{(2)} ,$$

and D^* is a Lorentzian in a Fourier convolution with u_{xx}. The flux gradient relations now become

$$\mathbf{q} = D^* \otimes u_x \tag{3d}$$

with an upper bound on the flux. Consequently, a source placed initially at the origin yields the response

$$u(x,t) = \exp(-D^{(2)} t/\sigma)\delta(x) + (\text{analytical part}), \quad t > 0$$

(see Ref. [1] for more details). Thus the δ function persists for a finite time. This property is completely missed by the FPE and in fact is often cited as its main shortcoming [2]. A proper response to discontinuity is the ultimate test of a system behavior at large gradients.

The second extension of the FPE is concerned with the temporal nonlocality due to inertia and is associated with the speed of sound C_0. The simplest process which yields an inertial extension of the FPE is that of a persisting random walker [3]. At the continuum level it leads to the telegraphers' equation (TE),

$$\Theta_t = [D_0 \Theta_x]_x - \tau \Theta_{tt} , \tag{4a}$$

which predicts a finite propagation speed and a delayed resolution of high gradients. Here the flux-gradients relations take the nonlocal form

$$\tau q_t + q + D_0 \Theta_x = 0 , \tag{4b}$$

$$\tau = \text{const} . \tag{4c}$$

Note that the TE is free from the shortcomings of the FPE and predicts that all discontinuities travel with the speed of sound. Now it is our intention to use nonlinearity in order to simulate the nonlocal inertial effect. For the linear TE this does not make sense as its solution is available in closed form but this is an exception. In more complex transport processes both analytical and good approximate solutions are hard to come by.

We impose the acoustic speed C as an upper bound on the permitted propagation speed in a medium. This provides the global means to control the rate at which the flux—in response to changes in gradients—is allowed to grow. We do not expect from this approach to reproduce a detailed behavior of short wavelengths on the characteristics ("light cones"). We are merely looking for a sensible transport theory without trying to track the detailed hyperbolic aspects of the original process. Consistent with this approach, different wave operators, having the same characteristic (acoustic) speeds and the same diffusive limit, will be approximated in the same way.

II. ANALYSIS

To proceed we associate Θ and the flux q through velocity v defined as

$$q=\Theta v\,. \tag{5a}$$

Equation (2a) then becomes a continuity relation for Θ:

$$\Theta_t+(\Theta v)_x=0\,, \tag{5b}$$

and the flux $-\Theta$ relation (2b) becomes

$$R\equiv D_0\Theta_x/\Theta=-v\,. \tag{6}$$

According to (6) if $|\Theta_x/\Theta|\nearrow\infty$ so will v. However, inertia imposes a macroscopic upper bound on the allowed free speed, namely, the acoustic speed C [here equal to $(D_0/\tau)^{1/2}$]. To this end we modify (6) as follows:

$$R\equiv\frac{D_0\Theta_x}{\Theta}=\frac{-v}{(1-v^2/C^2)^{1/2}}\,. \tag{7}$$

The postulate (7) forces v to stay in the subsonic regime. The sonic limit is approached only if $|\Theta_x/\Theta|\nearrow\infty$. Thus, inasmuch as (6) gives a universal expression valid in the small gradient limit, modification (7) augments it with an additional universal limit—the acoustic speed—which applies at the other end, the infinitely large gradient domain. Between these two limits Eq. (7) provides a smooth interpolation. Solve (7) for v:

$$q\equiv v\Theta=-D_0\Theta_x/\beta,\ \ \beta\equiv(1+R^2/C^2)^{1/2}\,, \tag{8}$$

and use the new flux function (8) in (5) to obtain

$$\Theta_t=\left[\frac{D_0\Theta_x}{(1+R^2/C^2)^{1/2}}\right]_x\,. \tag{9}$$

Equation (9) is our main result. Let us demonstrate that it has the desired features. First, for small gradients, the deviation of the radical from unity may be disregarded yielding back the FPE. Next assume that $|R|\gg1$. This is possible, either if the initial gradient is large or if the initial support of Θ is finite, on the front line where Θ vanishes. Expanding the radical in R^{-1} yields in the limit

$$\Theta_t=[C\Theta\Theta_x/|\Theta_x|]_x\,. \tag{10}$$

Equation (10) is a convection equation with a *sense of direction* that governs the propagation of a front line. On the right (left) side $\Theta_x/|\Theta_x|=-1$ $(+1)$. Thus the right (left) front propagates in the right (left) direction. These fronts persist at all times. In other words, on the front Eq. (9) degenerates into a one-sided wave equation with a novel twist; it has a sense of direction. However, reduction of (10) into (9) depends crucially on the initial Θ having compact support. Otherwise the front will diffuse.

We now turn to the behavior at large gradients. The ultimate test is the response to the presence of discontinuity carried by the TE along the characteristics. In our equation for $|\Theta_x|\nearrow\infty$

$$|q|=C\Theta[1-C^2R^2/2+\cdots] \tag{11}$$

and the flux saturates. While saturation is necessary in order to have "inertial delay" in the resolution of discontinuities, *it is not sufficient*. Indeed, consider another saturating flux:

$$q=-2D_0\Theta_x/\beta_m,\ \ \beta_m=1+(1+4R^2/C^2)^{1/2}\,. \tag{12a}$$

As in (8) it leads to a front propagating with a finite speed. Equation (12a) follows if instead of (7) one takes

$$R=-v/M,\ \ M=1-v^2/C^2\,. \tag{12b}$$

The saturating flux (12a) may compactify the diffusion wave, yet an initial discontinuity in a medium endowed with such a flux will be resolved immediately. The rate of resolution is slower than in the classical, linear, case but it occurs immediately. This conclusion follows from our recent work [4,5] where we have shown that given Eq. (2a) with a flux function $q(\Theta_x)$, an initial discontinuity is resolved within a *finite time* if and only if its asymptotic form for large gradients is given as

$$|q(\Theta_x)|\sim q_0+q_1|\Theta_x|^{-\gamma}+\cdots,\ \ \gamma>1\,,$$

since for flux (12) $\gamma=1$, initial gradients cannot be sustained. For the flux adopted in the present work, $\gamma=2$ [see (11)] and the delay takes place. Thus the rate of saturation is a crucial parameter. Alternatively, let me introduce another variable u. Thus (9) becomes

$$u_\tau=\frac{u_{xx}}{(1+u_x^2)^{3/2}}+\alpha\frac{u_x^2}{(1+u_x^2)^{1/2}},$$
$$\alpha u=\ln\Theta,\ \ \alpha=C/D_0,\ \ \tau=tD_0\,. \tag{13}$$

Consider first Eq. (13) without the α convection, and take

$$u(0,x)=\begin{cases}u_1, & \text{for } x<0,\\ 0, & \text{for } x>0\,,\end{cases} \tag{14a}$$

then the initial evolution follows the self-similar pattern [4,5]

$$u(0,x)=\begin{cases}u_1-\sqrt{t}f(z), & \text{for } x<0,\\ \sqrt{t}f(z), & \text{for } x>0,\ z=x/\sqrt{t}\,,\end{cases} \tag{14b}$$

and $f(z)$ is found via the solution of a boundary-value problem [4,5] in z. The similarity solution sustaining a sharp jump pertains until the height of the discontinuity $h(t)=u_1-2f(0)\sqrt{t}$ has decayed to zero (at $t=t_0\equiv[u_1/2f(0)]^2$). Thereafter, the smooth solution pattern is very much like the classical one. The presence of the α term superimposes convection on diffusion. On a sharp front like (14a) this amounts to $-\alpha u_x$. In a traveling frame of reference $[(x,t)\rightarrow(x-\alpha t,t)]$, the sharp front (during its duration) appears to be perfectly stationary. Elsewhere, the α term cannot balance the apparent convection. Consequently, on the left (right) side of the discontinuity, convection enhances (inhibits) diffusion. A self-explanatory numerical example is shown in Fig. 1.

However, while the present discussion provides a lower bound on the saturation rate, we still lack a selection rule that will make this choice unique. Taking any $\gamma>1$ one can construct another approximation with solutions that are insensitive to the actual choice of γ and are very similar to the one shown in Fig. 1 [6]. We would also like to point out that a delayed diffusion was observed in different density polymers [7].

Generalizations. In three dimensions Eq. (9) reads

$$\Theta_t=\nabla\cdot[D_0\nabla\Theta/\beta],\ \mathbf{R}\equiv\frac{-D_0\nabla\Theta}{\Theta}. \tag{15}$$

Equation (15) remains intact when $D_0(\Theta)$, $C_0(\Theta)$, are *functions* of Θ. Let Θ be the temperature; then Eq. (15) describes diffusion of heat, in a neutral gas $C^2\sim\Theta$, $D_0\sim\Theta^\alpha$, and $\alpha\sim0.5$. (In a fully ionized, unmagnetized, plasma $\alpha=2.5$.) Now Eq. (15) reads

$$\Theta_t=\nabla\cdot\left[\frac{D_1\Theta^\alpha\nabla\Theta}{[1+\delta_1^2\Theta^\omega(\nabla\Theta)^2]^{1/2}}\right],$$
$$\omega=2\alpha-3,\ D_1,\delta_1=\text{const}. \tag{16}$$

Note that the classical counterpart of Eq. (16) also allows for a localized heat wave with a sharp front that in the case of a δ-heat source in v dimensions propagates as $x_f=t^{1/N}$, $N=2+\alpha(v+1)$. Thus the velocity of the front $\partial_t x_f$, which initially is very large, slows quickly. But for Eq. (16), the actual front is determined not by this parabolic degeneracy, but by the hyperbolic issues. Indeed, if the radical is unimportant then local analysis near the front gives $\Theta\sim\xi^{1/\alpha}$, $\xi=|x_f-x|\ll1$, and

$$\Theta^{2\alpha}(\nabla\Theta)^\alpha\sim\xi^{1/\alpha}\rightarrow\infty \text{ as } \xi\searrow0.$$

Hence contrary to the assumption, the radical diverges: A contradiction. Again, the behavior on the front is found by extracting the dominant part from the radical

$$\Theta_t=\nabla\cdot[C_0\Theta^{1.5}\nabla\Theta/|\nabla\Theta|],\ C_0=\text{const}. \tag{17}$$

The acoustic speed is now a function of the temperature and causes the front to be convected nonlinearly. Note that the underlying hyperbolic TE is now nonlinear and under sufficiently large perturbations, the available dissipation is insufficient to arrest the formation of shocks. Here the Telegrapers' model does not describe well the problem on very short scales. On the other hand, without the inertia, the resulting nonlinear diffusion equation never collapses and neither does our model, which in this respect is closer to the parabolic equation. It preserves initial discontinuities and fronts, but unlike the TE it will not collapse.

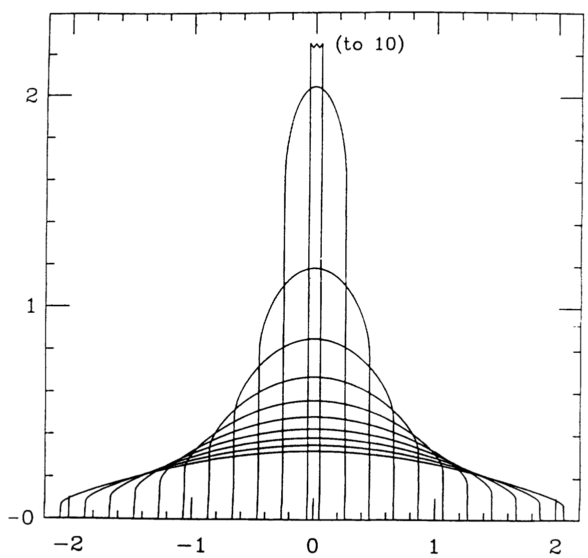

FIG. 1. The evolution of an initially square pulse at intervals of $\Delta t=0.2$. Note the persistence of sharp fronts that propagate with a finite speed, and the formation of a diffusive zone in between. Note also that $x\rightarrow cx$ and $t\rightarrow c^2t$ scales out the acoustic speed c.

III. APPLICATIONS

Magnetically confined plasmas are an example of a system where plasma inertia is neglected on purpose to elucidate its transport properties. So is the ambipolar transport in a partially ionized medium or the flow in a porous medium. In all these cases, momentum balance amounts to

$$\nabla\mathbf{P}=\mathbf{F}_b=-\alpha\mathbf{v}, \tag{18}$$

where $\mathbf{F}_b$ is a body force that in transport theory depends linearly on velocity.

Equation (18) augmented with the continuity relation for the density leads to the classical transport equation. In porous media, α is identified with the effective viscosity μ (that depends on the permeability and geometry [8]), which is a linear function of density.

To set an upper bound on transport relations, we introduce C as the effective speed of sound in a porous medium. *The extended Darcy law* is then

$$\nabla P=\frac{-\mu\mathbf{v}}{(1-v^2/C^2)^{1/2}}. \tag{19a}$$

Together with continuity for fluid density ρ it yields

$$\rho_t=\nabla\cdot\left[\frac{\rho\nabla P(\rho)}{[\mu^2+(\nabla P)^2/C^2]^{1/2}}\right] \tag{19b}$$

and the pressure was assumed to be a function of density. Independent of the extent to which C is also a function of density, Eq. (20) predicts propagation of the type described by Eq. (16). For ambipolar diffusion, the resulting transport takes the form (9) with C being again the sonic speed, and D_0 the coefficient of ambipolar diffusion [9] D_a. For plasma transported across the magnetic field

[9,10] one identifies C with the magnetosonic speed.

In our last application we consider the Navier-Stokes equations (NSE). We aim to bridge these equations with their hyperbolic uplift given by Grad [11] (the 13-moment equations) or in an alternative version by Khonkin [12]. As a prototype for the NSE, we consider the Burgers [13] equation,

$$u_t + u u_x = \mu u_{xx} , \qquad (20)$$

which can be considered a one-dimensional incompressible constant-pressure fluid. Its hyperbolic counterpart can be extracted from Khonkin [12]. It reads

$$(\rho/\mu)(u_t + u u_x) = -[u_{tt} + (u^2)_{xt}] + [(s^2 - u^2) u_x]_x . \qquad (21)$$

In this form, the right-hand side of (21) describes a stress wave propagating with a speed $c_\mu = u \pm s$ where $s = (4p/3\rho)^{0.5}$. Recall that in the Burgers equation, the stress diffuses with an infinite speed. Our approach leads to

$$u_t = u u_x = \left[\frac{\eta u_x}{(1 + R^2/S^2)^{1/2}} \right]_x , \qquad R = -\eta u_x / u , \quad \eta = 4\mu/3\rho . \qquad (22)$$

Note the non-Newtonian nature of stress-strain relations at high gradients.

We now compare the traveling kink structures, $u(x - \lambda t)$, of these equations. The Eulerian left-hand side tends to propel the initial data toward formation of discontinuity. In the Burgers-NS version, the viscous stress can always counterbalance this effect to form a shock layer. On the other hand, in both (21) and in our model, the width of the shock layer depends critically on the strength of the kink. Its width decreases with λ and for $\lambda = s$ it shrinks to a line. This is the largest smooth kink sustainable by the system. Above that speed, part of the upstream-downstream transit must be made via a discontinuous jump [6].

IV. SUMMARY

The standard transport theory is based on linear gradient-flux relations. In the present work we use inertia as a vehicle to control the growth of fluxes with gradients over the whole range of wavelengths. This is accomplished by imposing the acoustic speed as an upper bound on propagation velocity. It forces the flux to move toward the correct "optical" limit where the standard diffusion approximation is meaningless. The persistence of discontinuities along characteristics (light cones) was used to deduce a minimal rate of flux saturation that sustains these discontinuities.

Our results were attained by force de tour and not through a deductive derivation and cannot be considered as a definite statement on the subject but rather as a modest starting point for a theory which blends the infrared and the ultraviolet limits better.

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[1] C. R. Doering, P. S. Hagan, and P. Rosenau, Phys. Rev. A **36**, 985 (1987).

[2] H. R. Hoare, in *The Linear Gas*, edited by S. A. Rice, Advances in Chemical Physics, Ser. 20 (Academic, New York, 1971), p. 135.

[3] D. Joseph, Rev. Mod. Phys. **61**, 41 (1989).

[4] P. Rosenau, Phys. Rev. A **41**, 2227 (1990).

[5] P. Rosenau, P. S. Hagan, R. L. Northcutt, and D. S. Cohen, Phys. Lett. **A142**, 26 (1989).

[6] R. W. Cox and P. Rosenau (unpublished).

[7] D. Andelman (private communication).

[8] A. Scheidegger, *The Physics of Flow Through Porous Media* (University of Toronto Press, Toronto, 1976).

[9] F. F. Chen, *Introduction to Plasma Physics and Controlled Fusion* (Plenum, New York, 1984).

[10] H. Grad and J. Hogan, Phys. Rev. Lett. **24**, 1337 (1970).

[11] H. Grad, in *Principles of the Kinetic Theory of Gases*, in Handbuch der Physik 12: Thermodynamics of Gases, edited by S. Flugge (Springer, Berlin, 1958), p. 205.

[12] A. D. Khonkin, Dokl. Akad. Nauk SSSR [Sov. Phys. Dokl. **18**, 357 (1974)].

[13] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).