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Strongly intermittent chaos and scaling in an earthquake model

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We discuss the relation between scaling laws and dynamical behavior for earthquakes in the framework of a Burridge-Knopoff model. Due to the nontrivial interaction among many degrees of freedom, a new type of strongly intermittent chaos is found. The dynamics is dominated by wild fluctuations, implying exponential tails in the probability distributions. This is caused by very slow relaxation of time correlations, which gives rise to an anomalous behavior for the effective Lyapunov exponent and for the time signal of the earthquake magnitude.

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The comprehension of the fundamental mechanism underlying the presence of scaling invariance in earthquakes has led the introduction of toy models which reproduce the power laws, such as the Gutenberg-Richter law [1] or the Omori law [2], observed in experimental records. The goal is twofold: The understanding of the origin of the scaling laws in earthquakes and the individuation of predictability criteria in the analysis of time records. Both these lines of research have been initiated in different frameworks. The onset of deterministic chaos has been found in a system of two coupled blocks damped by asymmetric frictions [3], suggesting that chaotic phenomena might play an important role in quakes. On the other hand, Carlson and Langer [4] have introduced a simple version of Burridge and Knopoff model [5], consisting of a one-dimensional chain of blocks and springs mimicking the interactions of two faults. The presence of many degrees of freedom makes it possible to obtain the Gutenberg-Richter law.

This Rapid Communication tries to establish a bridge between these viewpoints. Our main result is that the dynamical behavior of the Carlson-Langer model is so strongly intermittent that it qualitatively differs from the one exhibited by systems with few degrees of freedom, such as the two-block model of Huang and Turcotte [3]. In fact, scaling appears as a consequence of strong time correlations in the response to a perturbation of the dynamical state of the system originated by intermittency. For this reason, our tool of investigation, as usual in dynamical systems, is the effective Lyapunov exponent. Its computation allows us to show that *when the number of degrees of freedom is large enough,* the dynamics is dominated by wild fluctuations, implying exponential tails in the probability distribution and anomalous behaviors for both the effective Lyapunov exponent and the time signal of the earthquake magnitude.

The model of Carlson and Langer is composed by a chain of blocks which slide on a rough plane so that their motions are damped by a homogeneous nonlinear friction law. When a block is stuck the static friction is such that it exactly balances the spring forces up to a certain threshold. The blocks are coupled via nearest-neighbor springs and, through a pulling spring, to a fault moving with constant velocity. The dynamical evolution is thus given by a sequence of stuck periods and quakes, consisting of sliding of groups of blocks. The deterministic differential equations describing the system generate scaling behaviors, such as the Gutenberg-Richter law, although peaks corresponding to energetically very strong earthquakes are much more frequent than expected by an extrapolation of the smaller events.

If the number of blocks is large, the numerical study of the evolution for long times is rather difficult. It is then convenient to use a cellular-automaton version of the model introduced by Nakanishi [6]. Basically it is a deterministic evolution mimicking the continuous (in time) equations of Carlson and Langer. Both models appear to exhibit the same scaling behavior. Let us briefly recall the rules defining the Nakanishi automaton. In absence of friction, the force f_i acting on the ith block on position x_i is

$$f_i = -k_p(x_i - v_p t) + k_c(x_{i-1} + x_{i+1} - 2x_i) , \qquad (1)$$

where k_c is the Hooke constant for the spring connecting a block to its nearest neighbor, and k_p is the Hooke constant of the pulling spring, linked to the fault moving at constant velocity v_p. When a block is stuck, the static friction balances the force as long as it is smaller than a threshold value $f_{\rm th}$. Each f_i increase as $k_p v_p t$ with time, and as soon as one of the forces overwhelms the threshold, the corresponding block slips and relaxes a certain amount of force δf. During this elementary process all the other blocks are assumed to be stuck and one has that if $f_j = f_{\rm th}$ then the strain relaxation between the jth block and its nearest neighbors is given by a change of the forces from f to f'

$$f_j' = f_{\rm th} - \delta f, \quad f_{j\pm1}' = f_{j\pm1} + \tfrac{1}{2}\Delta\delta f , \qquad (2)$$

where $\Delta = 2k_c/(k_p + 2k_c)$ measures the stiffness of the system. If $f_{j\pm1}' < f_{\rm th}$, the process stops until the next seismic event. Otherwise a slip cascade starts. Suppose $f_k' - f_{\rm th} \geq 0$ (with $k = j+1$ or $j-1$), then we have a relaxation from k to its nearest neighbors by the rule

$$f_k' = \Phi(f_k - f_{\rm th}), \quad f_{k\pm1}' = f_{k\pm1} + \tfrac{1}{2}\Delta(f_k - f_k') . \tag{3}$$

This process is repeated until all the forces are below the threshold. The whole cascade of these elementary processes is assumed to happen in zero time. This is sensible since the slipping time is much smaller than the quiescent time between two cascades. The function Φ measures how much force is relaxed when f exceeds $f_{\rm th}$ and one should have $\Phi(0) = f_{\rm th} - \delta f$ and $|\Phi(x)| < f_{\rm th}$ for $x \geq 0$. Assuming $v_p = k_p = f_{\rm th} = 1$, a simple form satisfying these constraints is (see Ref. [6])

$$\Phi(x) = \frac{(2-\delta f)^2/\alpha}{x + [(2-\delta f)/\alpha]} - 1 . \tag{4}$$

We have performed extended numerical computations on this automaton with boundary conditions given by fixing $f_0 = f_{N+1} = 0$. We look at it as a discrete (in time) system, where we indicate by $n=1$ the first event at time t_1, by $n=2$ the second event at time t_2, and so on. Since we cannot write the time evolution in terms of an explicit map, it is not possible to use the standard method involving the linearized dynamics for the computation of the Lyapunov exponents. However, one can easily follow the divergence of nearby trajectories, and thus determine the maximum Lyapunov exponent λ [7]. One considers two trajectories $\mathbf{f}$ and $\mathbf{f}'$ and their difference after the event [8]:

$$\delta f(n) = \left\{ \sum_{i=1}^{N} [f_i(t_n) - f_i'(t_n)]^2 \right\}^{1/2} . \tag{5}$$

In order to have $\mathbf{f}'$ close to $\mathbf{f}$ after the event one takes for $\mathbf{f}'$ the new vector with components

$$f_i'(t_n) - \frac{f_i(t_n) - f_i'(t_n)}{\delta f(n)} \delta_0 , \tag{6}$$

where δ_0 is the distance between $\mathbf{f}$ and $\mathbf{f}'$ at the initial time. λ is defined by

$$\lambda = \lim_{\delta_0 \to 0} \lim_{T \to \infty} \frac{1}{T} \sum_{n=1}^{T} \ln \frac{\delta f(n)}{\delta_0} . \tag{7}$$

The maximum Lyapunov exponent $\tilde{\lambda}$ for the original map can be written in terms of λ and the average time $\langle \tau \rangle = \langle t_{n+1} - t_n \rangle$ between two subsequent events as $\tilde{\lambda} = \lambda / \langle \tau \rangle$. For a very long range of values of parameters, λ is found to be positive, indicating that deterministic chaos is a generic feature of Burridge-Knopoff models. For the parameters $\Delta = 0.95$, $\alpha = 1.0$ (which are used throughout the paper) we obtain $\lambda = 7.22 \times 10^{-3}$. The dependence of λ on the parameters is beyond the purpose of the paper.

At first glance, a positive value of the Lyapunov exponent coexisting with the presence of power laws in a wide range of scales could be seen as rather surprising, since $1/\lambda$ introduces a characteristic time scale in the system. However, $1/\lambda$ is not the only possible characteristic time, because the onset of wild fluctuations of the chaoticity degree introduces a hierarchy of characteristic times.

The model has indeed been considered as an example of self-organized criticality. This is particularly apparent when Δ (which is the measure of the force conservation) becomes small so that the stress transfer is more difficult. Then the maximum Lyapunov exponent vanishes and the Gutenberg-Richter law becomes a pure power law without a pronounced peak for big events. At this point the model is "at the edge of chaos" [9]. It is an open problem to understand whether the strongly intermittent behavior can survive in marginal situations.

A standard method to probe the intermittency is by studying the probability distribution of the effective Lyapunov exponent $\gamma_t(\tau)$ [10] on a time interval τ

$$\gamma_t(\tau) = \frac{1}{\tau} \sum_{n=t+1}^{t+\tau} \ln \left[\frac{\delta f(n)}{\delta_0} \right] . \tag{8}$$

In a large class of intermittent dynamical systems one observes [11] that, for a large value of τ, the probability distribution of $\gamma(\tau)$ has the form

$$P(\gamma(\tau)) \sim \exp[-S(\gamma)\tau] , \tag{9}$$

where $S(\gamma)$ is an entropy function, quite similar to the $f(\alpha)$ spectrum for multifractals [10]. $S(\gamma)$ is zero for $\gamma = \lambda$ and otherwise positive. Equation (9) holds if there are no strong correlations among $\gamma_t(\tau)$ and $\gamma_{t'}(\tau)$ at large value of $|t - t'|$. In this case for large τ the variance

$$\mu(\tau) = \tau [\langle \gamma(\tau)^2 \rangle - \langle \gamma(\tau) \rangle^2] \tag{10}$$

is well defined and $\mu(\tau)$ rapidly tends to an asymptotic finite constant μ^*. In such a case, one can apply standard probabilistic arguments (central limit theorem) to show that the small fluctuations of γ around λ are Gaussian, i.e., $S(\gamma) \simeq (\gamma - \lambda)^2 / 2\mu^*$ for $\gamma \simeq \lambda$.

In the Nakanishi model (for $\Delta = 0.95$, $\alpha = 1.0$) the Gaussian-like intermittency holds for a number of blocks N roughly smaller than 100. At N larger than 100, the asymptotic value of μ diverges as a power of τ

$$\mu \sim \tau^{w} , \tag{11}$$

where we find $w \simeq 0.71$ which does not depend on N or on the observation time T. Figure 1 shows the behavior of μ

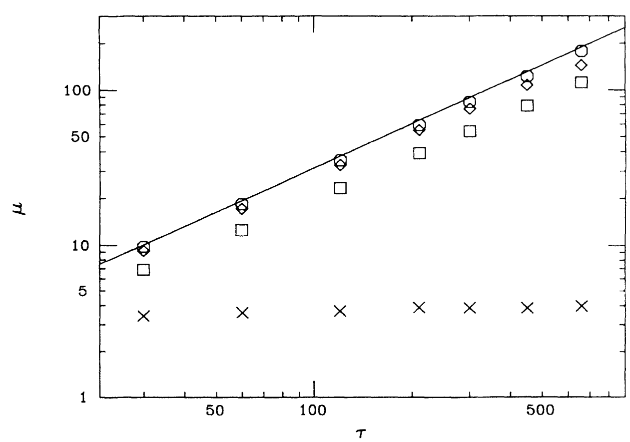

FIG. 1. The scaling behavior, Eq. (11), with μ vs τ plotted for various sizes of the system $N = 25$ (crosses), 200 (squares), 400 (diamonds), and 800 (circles). The line corresponds to $w = 0.71$.

vs τ for different values of N. In our numerical simulations we observe a T-dependent prefactor in (11) which, however, saturates for large values of T. The scaling (11) is incompatible with the form (9) of $P(\gamma(\tau))$. It implies that

$$P(\gamma(\tau)) \sim e^{-G(\gamma)\tau^{1-w}}. \quad (12)$$

The peculiarity of the form (12) shows up by looking at the moments of a response $R(\tau) = \exp[\gamma_t(\tau)\tau]$ to a small perturbation acting at time t, after a delay τ. If (9) holds, the generalized Lyapunov exponents $L(q)$ defined by $\langle R(\tau)^q \rangle \sim \exp[L(q)\tau]$ are related to $S(\gamma)$ by the Legendre transformation: $L(q) = \max_\gamma [q\gamma - S(\gamma)]$. If $S(\gamma)$ increases faster than γ for $\gamma \gg 1$ then $L(q)$ exists for any value of $q > 0$. However, when $S(\gamma) \simeq c\gamma + \text{const}$ for $\gamma \gg 1$, it follows

$$P(R) \sim R^{-(1+c)}, \quad (13)$$

so that $L(q) = \infty$ for $q > c$. This means that a perturbation may grow with exceptionally large γ with relatively large probability during a chaotic burst. Such an event will then dominate the large moments. It is not difficult to see that if (12) holds, then the generalized Lyapunov exponents are not defined, i.e., $L(q) = \infty$ for $q > 0$. For sake of simplicity we discuss the case with $G(\gamma) \simeq a\gamma + \text{const}$. For $\gamma \gg 1$, one has

$$P(R) \sim R^{-[1+c(\tau)]}, \quad (14)$$

so that $L(q) = \infty$ for $q > c(\tau)$, where $c(\tau) = a/\tau^w$ vanishes for $\tau \to \infty$. Scaling (11) has a simple explanation in terms of the correlation of $\gamma_t(1)$. Note that

$$\tau[\langle \gamma(\tau)^2 \rangle - \langle \gamma(\tau) \rangle^2] = 2 \sum_{k=1}^{\tau} \langle [\gamma_{t+k}(1) - \lambda][\gamma_t(1) - \lambda] \rangle$$

so that (11) indicates that $\langle [\gamma_{t+\tau}(1) - \lambda][\gamma_t(1) - \lambda] \rangle$ is not integrable and behaves as $\tau^{-(1-w)} \sim \tau^{-0.29}$. As far as we know such a strong intermittent behavior has not been observed yet in deterministic dynamical systems.

In our context the transition from weak to strong intermittency at $N \sim 100$ seems to be the mechanism that allows the onset of scaling laws, by introducing a hierarchy of characteristic times, different from λ^{-1}. To put in relation the chaotic behavior with quantities of geophysical interest, we study a time record of the magnitude of the seismic events at time n:

$$m(n) = \ln \left| \sum_{i=1}^{N} [x_i(t_n) - x_i(t_{n-1})] \right|, \quad (15)$$

where $x_i(t_n)$ indicates the position of the ith block after the cascade event at time t_n. The displacement $x_i(t_n) - x_i(t_{n-1})$ is proportional to the sum of all the negative increments of f_i in the cascade at time t_n. We thus have computed the probability $\mathcal{P}(m)dm$ that an earthquake has magnitude in the interval $[m, m+dm]$. Figure 2 shows that at $N \sim 100$ there is a transition from an almost flat $\mathcal{P}(m)$ to the Gutenberg-Richter law $\mathcal{P}(m) \sim Ae^{-bm}$ (when m is below the pronounced peak) with $b \approx 0.75$ independent of N. Strong time correlations of $m(n)$ also appear at $N \geq 100$. An even more impressive signature of the connection between the strong intermittency of chaoticity degree and scaling is provided by the fact that the signal $m(n)$ exhibits coarse-grained properties similar to those of $\gamma(\tau)$. Consider the variables

$$M_t(\tau) = \frac{1}{\tau} \sum_{n=t+1}^{t+\tau} m(n). \quad (16)$$

In the absence of strong correlations one expects $\tau[\langle M(\tau)^2 \rangle - \langle M(\tau) \rangle^2] \sim \nu$, with ν constant, and a probability distribution $P(M(\tau)) \sim \exp[-\Sigma(M)\tau]$. This feature holds only for small N, while for large N one finds a situation very close to (11) and (12) for the effective Lyapunov exponent, i.e., $\nu \sim \tau^{w'}$ with $w' \simeq 0.96$ and $P(M(\tau)) \sim \exp[-\Gamma(M)\tau^{1-w'}]$. Here the scaling functions $\Sigma(M)$ and $\Gamma(M)$ play the same role as $S(\gamma)$ and $G(\gamma)$ in (9) and (12), respectively. Moreover, the time correlation of the magnitude $m(t)$ has a very slow decay $\sim \tau^{-(1-w')} \sim \tau^{-0.04}$, as it is for the effective Lyapunov exponent γ_t.

Finally we discuss a rather surprising feature of the correlations between $m(t)$ and $\gamma_t(1)$ and their possible relevance for geophysics. Figure 3 shows that large posi-

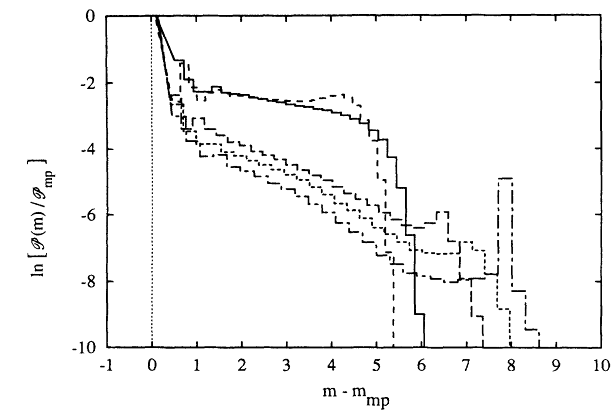

FIG. 2. Transition to the Gutenberg-Richter law: $\ln[\mathcal{P}(m)/\mathcal{P}_{mp}]$ vs $(m - m_{mp})$ for $N = 25$, 50, 200, 400, 800. The observation time is $T = 10^6$, $\mathcal{P}_{mp}$ is the maximum of $\mathcal{P}(m)$ and m_{mp} is the most probable value of the magnitude.

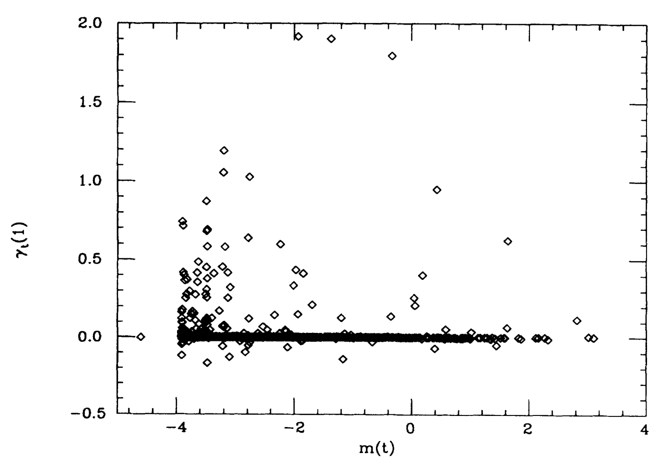

FIG. 3. Correlation between earthquake magnitude and effective Lyapunov exponent in a model with for $N = 400$ blocks: plot of γ_t $(\tau = 1)$ vs $m(t)$ for 2×10^4 events.

tive deviations of the effective Lyapunov exponent from its mean value can occur during seismic events of small magnitude. On the contrary, during big earthquakes, one always has a regular evolution, corresponding to rather small γ_t. Though the heuristic power of Burridge-Knopoff models is questionable [12], the consequence of such a behavior is rather surprising, since the predictability time is roughly proportional to γ^{-1}. For this reason, one is tempted to argue that the search for statistical patterns in long-time records of a seismic signal should not be useful for the prediction of the sudden occurrence of a destructive earthquake. In a dynamical system language, it means that two nearby trajectories in a quiescent normal period of activity can have a rate of divergence γ very large, because of the long tails of the distributions.

In conclusion, we have discussed the chaotic behavior of simple models of earthquakes. We have found that scaling laws in earthquake statistics appear as a nontrivial consequence of the dramatical increasing of intermittency due to a cooperative effect of many interacting degrees of freedom. This is a somewhat exceptional form of deterministic chaos with respect to the usual dynamical systems studied in the literature. Its most spectacular manifestation is that the increasing rates $L(q)$ (the generalized Lyapunov exponents) of the moments of the time response to a small perturbation diverge, though the typical rate λ (the maximum Lyapunov exponent) is well defined. Moreover, this strong intermittency phenomenon is related to very slow relaxation of time correlations (power laws with small exponents), and thus to the absence of a characteristic time scale in the system.

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[1] B. Gutenberg and C. F. Richter, Ann. Geophys. **9**, 1 (1956). The Gutenberg-Richter law is a power law for the seismic moment E, that is the relevant physical quantity. However, one usually considers the probability of occurrence of an earthquake of magnitude $m \sim \ln E$, so that the law is formulated as $\mathcal{P}(m) = Ae^{-bm}$ with $b \approx 1$.

[2] F. Omori, Rep. Earth. Inv. Commun. **2**, 103 (1984).

[3] J. Huang and D. L. Turcotte, Geophys. Res. Lett. **17**, 223 (1990); Nature (London) **348**, 234 (1990).

[4] J. M. Carlson and J. S. Langer, Phys. Rev. Lett. **62**, 2632 (1989); Phys. Rev. A **40**, 6470 (1989); J. M. Carlson, *ibid.* **44**, 6226 (1991).

[5] R. Burridge and L. Knopoff, Bull. Seismol. Soc. Am. **57**, 341 (1967).

[6] H. Nakanishi, Phys. Rev. A **41**, 7086 (1990).

[7] G. Benettin, L. Galgani, and J.-M. Strelcyn, Phys. Rev. A **14**, 2338 (1976).

[8] H. Nakanishi, Phys. Rev. A **43**, 6613 (1991), also studied numerically the divergence of two nearby trajectories for the same model. He sometimes observed lack of scaling, probability because the distance between the trajectories became too large to extract a well-defined Lyapunov exponent.

[9] See the 2d models, K. Chen, P. Bak, and S. P. Obukhov, Phys. Rev. A **43**, 625 (1991); P. Bak and C. Tang, J. Geophys. Res. **94**, 15635 (1989); Z. Olami, H. J. S. Feder, and K. Christensen, Phys. Rev. Lett. **68**, 1244 (1992).

[10] G. Paladin and A. Vulpiani, Phys. Rep. **156**, 147 (1987).

[11] J. P. Eckmann and I. Procaccia, Phys. Rev. A **34**, 659 (1986); G. Paladin, L. Peliti, and A. Vulpiani, J. Phys. A **19**, L991 (1986); Y. Kikuchi and Y. Aizawa, Prog. Theor. Phys. **84**, 563 (1990).

[12] Y. Y. Kagan, Nonlinear Sci. Today **2**, 1 (1992).