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Synchronization of regular and chaotic systems

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We investigate the synchronization between two systems consisting of coupled circle maps that have a common drive, which may be chaotic or regular. We observe several new aspects of chaotic and regular synchronization. In the chaotic regime the transition from synchronization to nonsynchronization corresponds to the transition from one to two positive Liapunov exponents. We find regions in the parameter space with periodic motion where synchronization is always achieved, never achieved, or, depending on the initial conditions, sometimes achieved. The nonsynchronization or synchronization are stable in the presence of a weak chaotic (or noisy) signal.

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This work concerns the study of synchronization of chaotic and nonchaotic systems. Our motivation comes from the recent publications on chaotic synchronization [1-4]. Pecora and Carroll [1] observed that it is possible to synchronize two identical stable systems with a chaotic drive, even if the initial conditions are different for the two systems. They used a dynamical system of the type $\dot{u}=g(u,w)$, $\dot{w}=h(u,w)$, and asserted that a variable $\dot{w}'$ governed by $\dot{w}'=h(u,w')$ will synchronize with w only if the sub-Liapunov exponents of the driven subsystem are all negative. The sub-Liapunov exponents they defined depend on the Jacobian matrix of the w subsystem, taking derivatives with respect to w only. The synchronization condition is also valid for discrete time systems, as was found for the example in [4].

Here we show that the sub-Liapunov exponents as defined in [1] are Liapunov exponents of the global system consisting of driving and driven systems together. In a simple system consisting of coupled sine-circle maps we find that the regime of chaotic synchronization occurs when one of the Liapunov exponents of the global system is negative and the other positive. The synchronization is lost when both exponents become positive, which has been referred to as the hyperchaos regime [5].

In our studies of chaotic synchronization in coupled digital phase-locked loops [4] we found that, depending on the parameters and initial conditions used, chaotic synchronization may sometimes occur, never occur, or always occur between the driving and stable subsystem. This may also be observed when the driving and driven systems are completely stable, i.e., in the periodic or quasiperiodic regime. Here we show that this phenomenon is caused by the lack of symmetry between w and w'. In the driving system there is a feedback between w and u, which does not exist in the driven system. It turns out that w and w' are in fact *different* subsystems, which may have different orbits and distinct stability properties.

In our system of coupled sine-circle maps we will show that synchronization between the driving and driven system is never observed in most of the Arnold tongues, where the systems are completely stable. There are regions of periodic motion where synchronization is always obtained, and in other regions synchronization may or may not occur, depending on the initial conditions used. In the latter case we study the basin of attraction and find a nonfractal structure.

Consider the system of equations

$$\phi_1^{n+1}=\phi_1^n+\Omega+\frac{k}{2\pi}\sin[2\pi(\phi_2^n-\phi_1^n)]\,, \tag{1a}$$

$$\phi_2^{n+1}=\phi_2^n+\Omega'+\frac{k}{2\pi}\sin[2\pi(\phi_1^n-\phi_2^n)]\,, \tag{1b}$$

as the driving system. Now consider a driven subsystem of the above equations, identical to the first equation,

$$\phi_3^{n+1}=\phi_3^n+\Omega+\frac{k}{2\pi}\sin[2\pi(\phi_2^n-\phi_3^n)]\,. \tag{2}$$

The operation modulo 1 is assumed on the right-hand side of the above equations. We show in Fig. 1(a) the phase diagram for any of the variables ϕ_1, ϕ_2, or ϕ_3. The white part represents periodic orbits and the shaded area represents chaotic or quasiperiodic motion. (In all the numerical calculations shown here we have neglected a transient of 3000 iterations.) The Arnold tongues [6] emanating from $k=0$ are evident in the figure. The structure of the phase diagram can be better understood if we make the following change of coordinates: Define $\theta_1^n\equiv\phi_1^n-\phi_2^n$, $\theta_2^n\equiv\phi_1^n+\phi_2^n$, $\theta_3^n\equiv\phi_3^n-\phi_2^n$, $\Omega_-\equiv\Omega-\Omega'$, and $\Omega_+\equiv\Omega+\Omega'$. In the new variables Eqs. (1) and (2) become

$$\theta_1^{n+1}=\theta_1^n+\Omega_--\frac{k}{\pi}\sin(2\pi\theta_1^n)\,, \tag{3a}$$

$$\theta_2^{n+1}=\theta_2^n+\Omega_+\,, \tag{3b}$$

and

$$\theta_3^{n+1}=\theta_3^n+\Omega_--\frac{k}{2\pi}[\sin(2\pi\theta_1^n)+\sin(2\pi\theta_3^n)]\,. \tag{4}$$

Thus the evolution of ϕ_1 and ϕ_2 can be decomposed in two motions: the circle map [Eq. (3a)] and a trivial linear motion [Eq. (3b)]. Equation (4) is a driven circle map. The motion of θ_2 is decoupled from the motion of θ_1 and

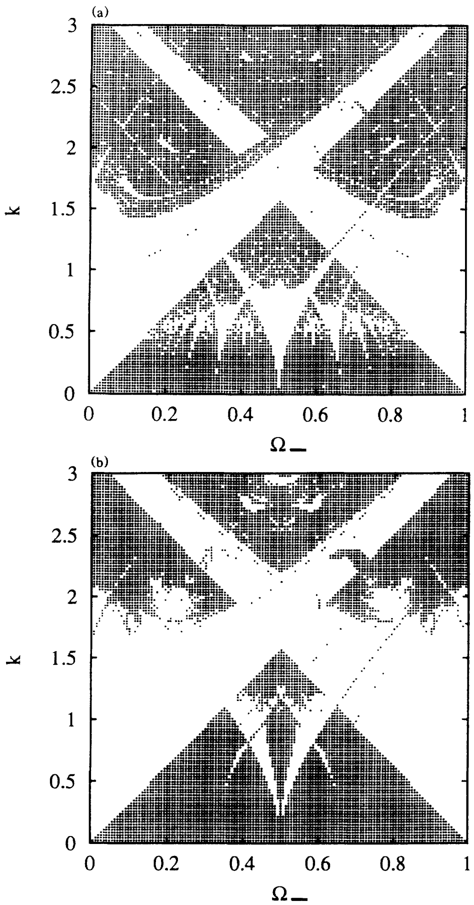

FIG. 1. (a) Regions of periodic motion (white) for any of the variables ϕ_1, ϕ_2, and ϕ_3. We consider the motion periodic if within 1000 iterations the system returns to the initial point within a radius of 10^{-6}. (b) Regions where synchronization (white) and nonsynchronization (shaded) are observed for the initial conditions $\phi_1=0.2$, $\phi_2=0$, and $\phi_3=0.5$. We consider the orbit synchronized if after the transient period (3000 iterations) $|\phi_1-\phi_3|\leq 10^{-6}$.

θ_3. In the new coordinate system the period of a regular orbit is not necessary the same as in the ϕ coordinate system. The border of invertibility for the circle map [Eq. (3a)] is given by $k=0.5$. Below this line chaotic motion does not exist; there are only periodic or quasiperiodic orbits.

We note that synchronization between ϕ_1 and ϕ_3 implies synchronization between θ_1 and θ_3, because the same change of coordinate is made for ϕ_1 and ϕ_3. The concept of synchronization is coordinate independent if and only if one makes the same change of coordinate in both driving and driven systems.

The region where synchronization between ϕ_1 and ϕ_3 (or θ_1 and θ_3) is observed (white) is shown in Fig. 1(b) for the initial conditions $\phi_1^1=0.2$, $\phi_2^1=0.0$, and $\phi_3^1=0.5$. Comparing Figs. 1(a) and 1(b) we see that synchronization is generally not observed when the motion is periodic, with the exception of the period one tongue, nor when the motion is quasiperiodic. In fact, as we will show, synchronization in most of the periodic tongues is never possible. We also see regions where the motion is chaotic (above the $k=0.5$ line) and synchronization is observed as found previously [1,4]. In other chaotic regions synchronization is not found.

All these features can be understood by studying the eigenvalues (or equivalently the Liapunov exponents) of the global system consisting of the driving and driven systems *together*.

The Jacobian matrix of the global system in the θ coordinates is given by

$$J=\begin{pmatrix} 1-2k\cos(2\pi\theta_1^n) & 0 & 0 \\ 0 & 1 & 0 \\ -k\cos(2\pi\theta_1^n) & 0 & 1-k\cos(2\pi\theta_3^n) \end{pmatrix}. \tag{5}$$

Now we calculate the product of the Jacobian matrices in a given orbit of period N and find the eigenvalues of the resulting matrix, which are

$$\lambda_1=\prod_{n=1,N}[1-2k\cos(2\pi\theta_1^n)]\,, \tag{6a}$$

$$\lambda_2=1\,, \tag{6b}$$

$$\lambda_3=\prod_{n=1,N}[1-k\cos(2\pi\theta_3^n)]\,. \tag{6c}$$

[The eigenvalues are, of course, the same if calculated in the ϕ coordinate system.] The Liapunov exponent associated with the eigenvalue λ_i is defined as

$$\Lambda_i=\lim_{N\to\infty}\frac{1}{N}\ln|\lambda_i|\,. \tag{7}$$

In our system one of the Liapunov exponents Λ_2 is zero, reflecting the fact that one of the variables has a trivial motion. We calculate the two other Liapunov exponents Λ_1 and Λ_3 and plot the region where they are positive (shaded area) in Figs. 2(a) and 2(b), respectively. Comparing Figs. 1(b) and 2 we see that synchronization is possible only if Λ_3 is nonpositive. The driven system is more stable than the driving system, and when both Liapunov exponents become positive chaotic synchronization is lost. The presence of more than one positive Liapunov exponent in a given system has been called hyperchaos [5]. Using this nomenclature, it is the hyperchaos regime that determines the region of nonsynchronization when the system is chaotic.

Now we calculate the sub-Liapunov exponent as defined by Pecora and Carroll [1]. The sub-Liapunov exponent $\bar{\Lambda}_3$ for ϕ_3^{n+1} is a function of the Jacobian with respect to ϕ_3^n and is given by

$$\bar{\Lambda}_3=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1,N}\ln|\partial\phi_3^{n+1}/\partial\phi_3^n|$$
$$=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1,N}\ln|1-k\cos[2\pi(\phi_2^n-\phi_3^n)]|\,. \tag{8}$$

It turns out that $\bar{\Lambda}_3$ is in fact Λ_3; that is, the sub-Liapunov exponent of the driven subsystem as defined in [1] is one of the Liapunov exponents of the global system. This occurs because θ_1^{n+1} and θ_2^{n+1} do not depend explicitly on θ_3^n, which makes the elements J_{13} and J_{23} of the Jacobian matrix equal to zero. When one calculates the product of the Jacobian matrices for a given orbit, these elements of

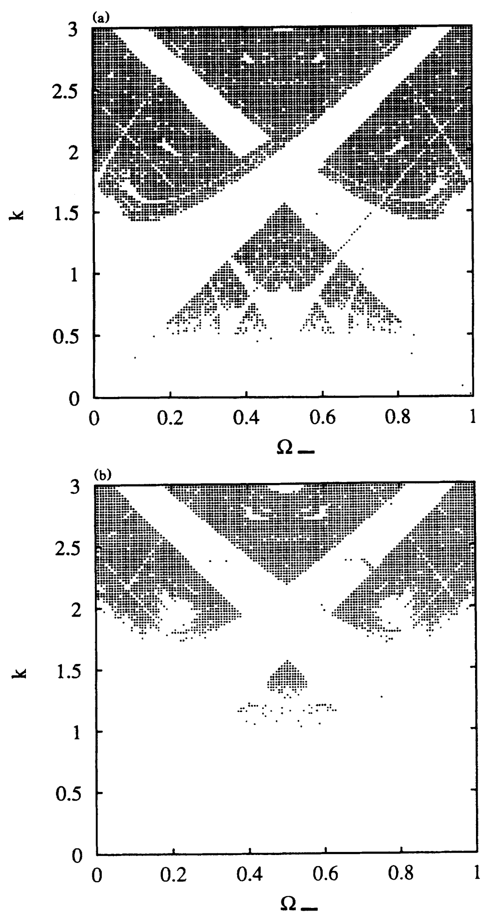

FIG. 2. Regions with positive Liapunov exponents (a) Λ_1 (shaded) and (b) Λ_3 (shaded). We considered the Liapunov exponents positive if $\Lambda_i \gtrsim 10^{-4}$ for $N=30\,000$.

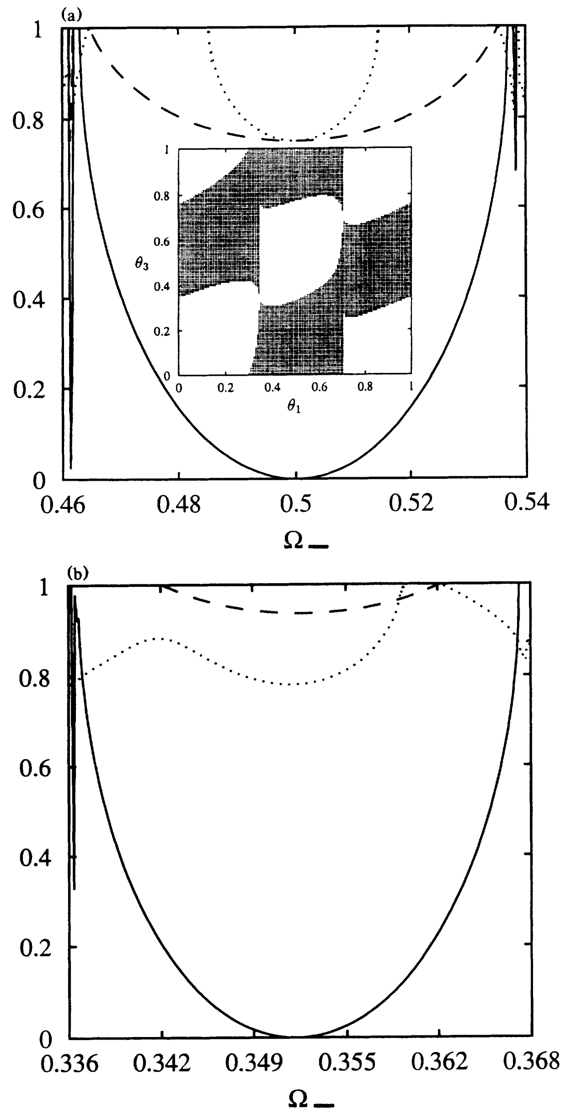

FIG. 3. Eigenvalues λ_1 (solid) and λ_3 (with the dashed and dotted lines corresponding, respectively, to the synchronizing and nonsynchronizing attractors) for $k=0.5$; (a) period-two and (b) period-three orbits. The inset in (a) shows the basins of attraction for the synchronizing (white) and nonsynchronizing attractors (shaded) for a period-two orbit ($k=0.5$, $\Omega_-=0.49$).

the product remain zero. This insures that one of the eigenvalues gives the sub-Liapunov exponent defined by Pecora and Carroll. This result holds also for higher-dimensional systems, in maps as well as in flows. The sub-Liapunov exponents are Liapunov exponents of the global system because also in these cases the Jacobian matrix has the elements originated from the derivative of the driving variable with respect to the driven variables equal to zero. The corresponding elements are also zero in the matrix resulting from the product of the Jacobian matrices.

Now we turn our attention to phenomenon of nonsynchronization in the periodic regions for the system governed by Eqs. (1) and (2). The first case we consider is the period-two tongue (in the θ coordinate system), which is the tongue situated in the middle of Figs. 1 and 2. For $k=0.5$ the period two orbit is stable for $0.464 \lesssim \Omega_- \lesssim 0.535$. There is only one stable attractor for θ_1, whereas for θ_3 we find two attractors, one of them being the same as the attractor for θ_1. We calculate the nontrivial eigenvalues λ_1 and λ_3 according to Eq. (6) with $N=2$. In Fig. 3(a) we show λ_1 as a solid line and λ_3 as dashed and dotted lines for the synchronizing and nonsynchronizing attractors, respectively. For $0.485 \lesssim \Omega_- \lesssim 0.514$, λ_3 is less than one for both synchronizing and nonsynchronizing attractors. Thus synchronization may or may not be observed depending on the initial conditions. Outside this interval the eigenvalue corresponding to the nonsynchronizing orbit (dotted curve) is greater than one, and therefore unstable. This implies that in these regions θ_1 and θ_3 will always synchronize, since the basin of the synchronizing attractor now constitutes the entire phase space.

By analyzing the period-three orbit we identified regions where synchronization never occurs (except if the initial conditions for θ_1 and θ_3 are completely identical). For $k=0.5$ the period-three orbit is stable for $0.336 \lesssim \Omega_- \lesssim 0.367$. In this case we also find one stable attractor for θ_1 and two attractors for θ_3, one of them synchronizing with θ_1. The nontrivial eigenvalue λ_1 is shown as a solid line in Fig. 3(b). The eigenvalues λ_3 for the synchronizing and nonsynchronizing attractors are the dashed and dotted lines, respectively. At $\Omega_- \lesssim 0.342$ or $\Omega_- \gtrsim 0.362$, λ_3 for the synchronizing attractor is greater than one, consequently it is unstable. Therefore, in these parameter ranges, the period-three orbits for the two systems are always different, independent of the initial conditions (when they are not identical). For $0.359 \lesssim \Omega_- \lesssim 0.362$ synchronization is always found, since in this region the nonsynchronizing attractor is unstable.

For periodic tongues with period greater than three synchronization is never obtained. We find that the synchronizing attractor is always unstable (that is, the corresponding λ_3 is always greater than one) for the driven system in these Arnold tongues. Synchronization in the period-one tongue is always observed, because there is only one (identical) stable attractor for both driving and driven systems. In the quasiperiodic regime synchronization is not observed, since there the Liapunov exponents are zero and we start to evolve the system with different initial conditions for θ_1 and θ_3. For the system we study here we did not find regions of chaotic motion where synchronization may or may not occur depending on the initial conditions.

The nonsynchronization we see in the periodic regime is not related to the situation in which ϕ_1 and ϕ_3 have the same attractor, but are out of phase. For our system where ϕ_1 and ϕ_2 are coupled the attractors are always in phase when they are stable and identical.

We studied the basin of attraction where synchronization may or may not occur for the period-two and -three orbits. In the inset in Fig. 3(a) we show the initial conditions, in the θ_3 vs θ_1 plane, which lead to synchronization (white) and nonsynchronization (shaded) for a period-two orbit ($k=0.5$ and $\Omega_-=0.49$). The basins of attraction are regular, and do not show a fractal structure. This implies that the addition of a chaotic signal with small amplitude to the two subsystems does not cause their synchronization, as can be the case if the basins are entirely fractal. For the period-three orbit we also find nonfractal basins of attraction.

We observe that the regions where synchronization is always achieved, never achieved, or sometimes achieved remain with the addition of a weak chaotic (or noisy) signal to both subsystems governed by ϕ_1 and ϕ_3. Also, the regions of positive sub-Liapunov exponent for ϕ_3 do not change. In other words, Figs. 1(b) and 2(b) remain the same. This shows that the necessary condition for chaotic synchronization stated in [1], that is, negative sub-Liapunov exponent for ϕ_3, is not sufficient.

In conclusion, we have observed several new aspects of regular and chaotic synchronization. By considering driving and driven subsystems as a whole system we have shown that the sub-Liapunov exponents defined by Pecora and Carroll are Liapunov exponents of the global system. This result holds for higher-dimensional systems in maps and in flows. Chaotic synchronization is possible when the driven subsystem is more stable than the driving system and the synchronization is lost when the hyperchaos regime appears. We verified that the lack of symmetry between the driving and driven subsystems may result in nonsynchronization even when they are completely regular. We found that the eigenvalues of the global system characterize the regions where regular synchronization is always achieved, never achieved or sometimes achieved depending on the initial conditions.

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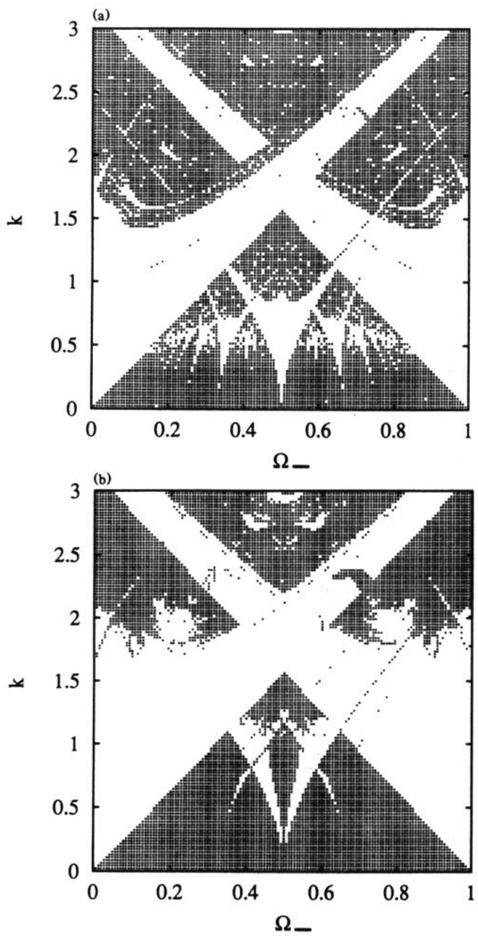
[1] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[2] H. G. Winful and L. Rahman, Phys. Rev. Lett. **65**, 1575 (1990).
[3] J. M. Kowalski, G. Albert, and G. W. Gross, Phys. Rev. A **42**, 6260 (1990).
[4] M. C. de Sousa Vieira, A. J. Lichtenberg, and M. A. Lieberman, Int. J. Bifurcation Chaos **1**, 691 (1991); M. C. de Sousa, P. Khoury, A. J. Lichtenberg, M. A. Lieberman, J. Gullicksen, J. Y. Huang, R. Sherman, and M. Steinberg, *ibid.* (to be published).
[5] O. E. Rössler, Notes Appl. Math. **17**, 141 (1979); O. E. Rössler, Phys. Lett. **71A**, 155 (1979); T. Kapitaniak and W.-H. Steeb, Phys. Lett. A **152**, 33 (1991).
[6] See, for example, A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer, Berlin, 1983).



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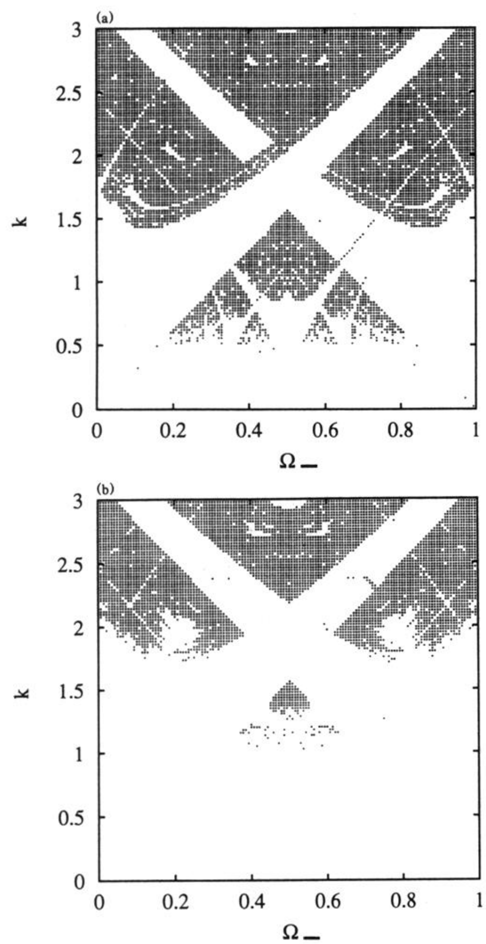

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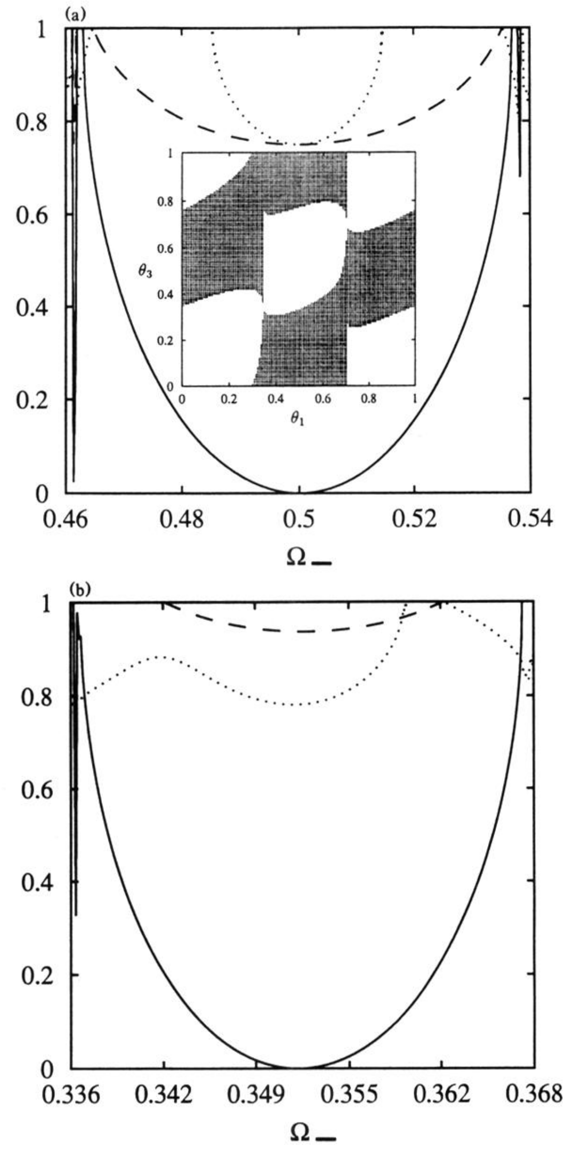

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