RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 46, NUMBER 2 15 JULY 1992

First-passage time, maximum displacement, and Kac's solution of the telegrapher equation

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(Received 7 February 1992)

The distributions of the first-passage time for the Poisson random walk on a straight line (also known as the telegrapher random process) subject to a given number of reversals in the walk are obtained explicitly for both the starting directions. These distributions are then used to obtain, again explicitly, the corresponding distributions of the maximum of the walk, proving the conjecture by Orsingher [Stochastic Process. Appl. **34**, 49 (1990)] for the one started moving to the right. The latter distribution leads to a reinterpretation of Kac's solution of the telegrapher equation.

PACS number(s): 02.50.+s, 03.40.Kf, 05.40.+j

There has been continuing interest in the path-integral solution of the telegrapher equation and the underlying Poisson random walk, since the work of Goldstein [1] and Kac [2]. Recently, DeWitt-Morette and Foong [3], and Foong [4] gave the solution in terms of an ordinary integral. An example solution in Ref. [4] was very recently applied by Mugnai *et al.* [5] to a semiclassical analysis of tunneling time. There are also other applications and extensions; for example, application to polymers [6], and extension to the case when the reversal intensity of the random walk is time-dependent [7–9]. For more references, see Ref. [9, 10]. These developments prompted the current detailed study of the walk, which leads to a reinterpretion of Kac's solution of the telegrapher equation, which is a wave equation with dissipation.

In this connection, it appears that in the literature there is neither an exact evaluation of the distribution of the first-passage time nor a general formula for the distribution of the maximum displacement for this particular walk, even though a related one has been considered [11]. For the distribution of the maximum displacement, Orsingher [10] calculated the cases for which the walker has undergone $N \leq 5$ reversals in a given time interval for both the starting directions, and conjectured a formula for the case of general value of N for only one of the starting directions (to the right). In this Rapid Communication, the distributions of the first-passage time are obtained with the help of Siegert's formula [13] and then used in obtaining the distributions of the maximum of the walk, for *both* the starting directions, thus providing a proof of Orsingher's conjecture.

Let us now define the walk, first on the lattice, where the sites are separated by Δx. Let a particle start at $x = 0$ with a constant velocity v, and let $\Delta t = \Delta x/v$. At each site, it may reverse its direction of travel instantaneously. The probability of a reversal at a site is $a\Delta t$, while the probability of continuing its direction is $1 - a\Delta t$. It is to be understood that the particle is a sample out of an ensemble.

The continuum limit of this lattice random walk, with v held constant, is described by the Poisson distribution [2], namely, the probability $\mathcal{P}$ for a path with k reversals during the time interval $\Delta t \equiv t - t'$ is given by

$$\mathcal{P}(N(t)-N(t') = k) = \begin{cases} e^{-a\Delta t}(a\Delta t)^k/k!\ , & k \geq 0; \\ 0, & k < 0. \end{cases} \tag{1}$$

For convenience, we shall take $N(0) = 0$.

A quantity of interest to the telegrapher equation is $S(t)$ defined by $S(0) = r_0$ and $S(t) = r_0 + V_0 \int_0^t (-1)^{N(\tau)}\, d\tau$. It is the displacement of the particle starting at r_0 with velocity V_0. If $V_0 = 1$ and $r_0 = 0$, it is also the "randomized time" of Kac. We shall now discuss the case where the starting direction of the walk is to the right ($V_0 = 1$), and towards the end quote the results for the other starting direction. Denoting the density distribution of $S(t)$ by $g(t, r)$, its Laplace transform $\mathcal{L}_g$, and with $r_1' \equiv r_1 - r_0$, $u_{1\pm}' \equiv t \pm r_1'$, and $u_1' \equiv \sqrt{t^2 - r_1'^2}$ [where the notations hold if the superscript prime, or subscript 1, or both (in which case $r_0 = 0$) are dropped], then

$$g(t, r'|r_0) = e^{-at}\delta(u_\mp') + \frac{ae^{-at}}{2}\Theta(u'^2)[I_0(au') + \frac{u_\pm'}{u'} I_1(au')] \tag{2}$$

(see, for instance, Refs. [4] and [12]) and [3]

$$\mathcal{L}_g(s, r'|r_0) = \frac{1}{2}[(1 + 2a/s)^{1/2} + 1] \exp[-r'(s^2 + 2as)^{1/2}], \qquad r' > 0 \tag{3}$$

where I_ν are the modified Bessel functions, $\Theta(t) = 1$ for $t \geq 0$, and $\Theta(t) = 0$ for $t < 0$.

Let the probability density that the particle passes the point $r_1 > r_0$ for the first time at time t be $f(r_1, t|r_0)$. Then, for $r > r_1$, Siegert's formula relating f and g is

$$g(t, r|r_0) = \int_0^t f(r_1, t_1|r_0) g(t - t_1, r|r_1) dt_1, \tag{4}$$

which is a convolution of the two densities. By the convolution theorem of the Laplace transform, Eq. (4) gives

$$\mathcal{L}_f(s, r_1|r_0) = \frac{\mathcal{L}_g(s, r|r_0)}{\mathcal{L}_g(s, r|r_1)}\ . \tag{5}$$

It follows from Eqs. (5) and (3) that

$$\mathcal{L}_f(s, r_1|r_0) = \exp[-r_1'(s^2+2as)^{1/2}] ,$$

whose inverse [4] gives the following density of the first-passage time:

$$f(r_1, t|r_0) = e^{-at}\delta(u_{1-}') + e^{-at}\frac{ar_1'}{u_1'}I_1(au_1')\Theta(u_{1-}'). \quad (6)$$

In the remainder of the paper we shall set $r_0 = 0$, and the case for $r_0 \neq 0$ is to be recovered by replacing r_1 by r_1'. Also, unless otherwise stated, Θ denotes $\Theta(u_{1-})$.

We now deduce from Eq. (6) the distribution $f(r_1, t_1, N(t_1) = 2k)$. Since $f(r_1, t, N(t)) = 0$ for $N(t)$ odd, by expanding I_1 we have

$$f(r_1, t) = \sum_{k=0}^{\infty} f(r_1, t, N(t) = 2k)$$
$$= e^{-at}\delta(u_{1-}) + \Theta e^{-at}\frac{a^2 r_1}{2}\sum_{j=0}^{\infty}\frac{(au_1/2)^{2j}}{j!(j+1)!} . \quad (7)$$

Because each reversal associates with a factor a, as we see in Eq. (1), and the factor e^{-at} is independent of the number of reversals, Eq. (7) implies

$$f(r_1, t, N(t) = 2k) = \begin{cases} e^{-at}\delta(u_{1-}), & k = 0; \\ 2e^{-at}(a/2)^{2k} r_1 u_1^{2(k-1)}\Theta/[(k-1)!k!], & k \geq 1 \end{cases} \quad (8)$$

from which the conditional probability is, where B denotes the β function,

$$f(r_1, t|N(t) = 2k) = \begin{cases} \delta(u_{1-}), & k = 0; \\ 2[B(1/2, k)]^{-1}(r_1/t^2)(u_1/t)^{2(k-1)}\Theta, & k \geq 1. \end{cases} \quad (9)$$

This concludes our calculation of first-passage time, and we now turn to the calculation of the distribution of the maximum displacement to be denoted by $S_m(t)$, to the right of the origin, of the walk during the time interval $[0, t]$, namely, $S_m(t) = \max\{S(\tau), 0 \leq \tau \leq t\}$.

We seek the distribution of $S_m(t)$ subject to a given number $N(t)$ of reversals by time t. The probability that the maximum of a path is greater than or equal to r_1 is given by the sum of probabilities that it undergoes $2i$ ($i \geq 0$) reversals by the first-passage time τ, and the remaining $N(t) - 2i$ reversals between τ and t. That is,

$$P(S_m(t) \geq r_1, N(t) = n) = \sum_{i=0}^{[n/2]}\int_0^t f(r_1, \tau, N(\tau) = 2i)P(N(t-\tau) = n - 2i)d\tau, \quad (10)$$

where $[n/2]$ denotes the greatest integer not larger than $n/2$.

For $N(t)$ odd, substituting for $f(r_1, \tau, N(\tau) = 2i)$ given by Eq. (8), and dividing by the probability of $N(t)$ reversals, we have the conditional probability

$$P(S_m(t) \geq r_1|N(t) = 2k+1) = \left(\frac{u_{1-}}{t}\right)^{2k+1}\Theta + \Theta\frac{(2k+1)!r_1}{2t^{2k+1}}\sum_{i=1}^{k}\frac{\int_{r_1}^t d\tau(\tau^2 - r_1^2)^{i-1}(t-\tau)^{1+2(k-i)}}{2^{2(i-1)}(i-1)!i![1+2(k-i)]!} . \quad (11)$$

The derivative with respect to r_1, as outlined in the Appendix, of the complement of this (namely, the conditional probability that the maximum is less than r_1) gives the following density:

$$\rho(S_m(t) = r|N(t) = 2k+1)$$
$$= \Theta\frac{2}{B(\frac{1}{2}, k+1)}\frac{u^{2k}}{t^{2k+1}}, \quad k \geq 0 \quad (12)$$

as conjectured by Orsingher [10]. For the case of $N(t)$ even, we similarly obtained, as remarked in the Appendix, that

$$\rho(S_m(t) = r|N(t) = 2k)$$
$$= \Theta\frac{2}{B(\frac{1}{2}, k)}\frac{u^{2(k-1)}}{t^{2k-1}}$$
$$= \rho(S_m(t) = r|N(t) = 2k-1), \quad k \geq 1 . \quad (13)$$

By integrating the density ρ, the probability of confinement to less than r_1 is easily obtained to be

$$P(S_m(t) < r_1|N(t) = 2k) = P(S_m(t) < r_1|N(t) = 2k-1)$$
$$= \frac{B_{z^2}(1/2, k)}{B(1/2, k)}\Theta \equiv I_{z^2}(1/2, k)\Theta,$$

where $z = r_1/t$, B_{z^2} is the incomplete β function, and I_{z^2} is the regularized β function.

The density of $S_m(t)$ regardless of $N(t)$ for $t \geq r_1$ is given by

$$\rho(S_m(t) = r) = \sum_{k=0}^{\infty}\rho(S_m(t) = r, N(t) = k)$$
$$= e^{-at}\delta(u_-) + \Theta a e^{-at}\left[I_0(au) + \frac{t}{u}I_1(au)\right]. \quad (14)$$

This formula has the same form as the distribution

[3] of $|S(t)|$, if r_1 is taken to be the value of $|S(t)|$ instead. This is remarkable because the two quantities $S_m(t)$ and $|S(t)|$ are obviously different. An immediate implication of this is that a different averaging process could be adopted in Kac's solution to the following wave equation with dissipation:

$$\frac{1}{v^2}\frac{\partial^2 F}{\partial t^2}+\frac{2a}{v^2}\frac{\partial F}{\partial t}-D_{\mathbf{x}}F=0,\quad F(\mathbf{x},0)=\phi(\mathbf{x},0),\quad \left.\frac{\partial F}{\partial t}\right|_{t=0}=0, \tag{15}$$

where $D_{\mathbf{x}}$ is *any* well-behaved, *linear, spatial* operator (for example, ∇^2 or $\boldsymbol{\nabla}\times\boldsymbol{\nabla}$), and $\phi(\mathbf{x},t)$ is the solution of the wave equation without dissipation

$$\frac{1}{v^2}\frac{\partial^2 \phi}{\partial t^2}-D_{\mathbf{x}}\phi=0, \tag{16}$$

that is, the solution $F(\mathbf{x},t)$ can be obtained by

$$F(\mathbf{x},t)=\frac{1}{2}\left\langle\phi(\mathbf{x},S_m(t))\right\rangle+\frac{1}{2}\left\langle\phi(\mathbf{x},-S_m(t))\right\rangle, \tag{17}$$

namely, averaging over $S_m(t)$ or averaging over $|S(t)|$ as in Kac's original solution. In other words, Kac's original prescription is not unique. However, it must be emphasized that the discussion here assumes the boundary condition $\partial F(\mathbf{x},t)/\partial t=0$. Further considerations are needed when this boundary condition is not satisfied.

We end this paper by quoting the corresponding results for the other starting direction, namely to the left, which may be similarly derived [14], except that we do not have the benefit of a conjecture by Orsingher in this case. The distributions of the first-passage time are given by

$$f(r_1,t|V_0=-1)=\Theta\frac{e^{-at}}{u_{1+}}\left[ar_1I_0(au_1)+\left(\frac{u_{1-}}{u_{1+}}\right)^{1/2}I_1(au_1)\right] \tag{18}$$

and

$$f(r_1,t,N_t=2k+1|V_0=-1)=\Theta\frac{D(t,k)}{k!(k+1)!}[t+(2k+1)r_1]u_{1-}u_1^{2(k-1)}, \tag{19}$$

where Eq. (19) follows from Eq. (18), just as Eq. (8) follows from Eq. (7). As for the maximum of the walk, we have of course $P(S_m(t)<r_1|N(t)=0,V_0=-1)=1$, and for $N(t)>1$,

$$P(S_m(t)<r_1|N(t)=2k,V_0=-1)=B(1/2,k)^{-1}\left[B_{z^2}(1/2,k)+\frac{1}{k}(1-z^2)^k\right] \tag{20}$$

and

$$P(S_m(t)<r_1|N(t)=2k+1,V_0=-1)=[(k+1)B(1/2,k+1)]^{-1}\left[kB_{z^2}(1/2,k)+\frac{1}{2}B_{z^2}(1/2,k+1)+(1-z^2)^k\right], \tag{21}$$

where $z=r_1/t$. Notice that unlike the $V_0=1$ case, the probabilities for the $N(t)$ even and odd cases differ. However, as k becomes large, and consequently $B_{z^2}(1/2,k)\sim 1/\sqrt{k}$, we see in these equations that this difference and the difference due to the initial starting direction become smaller, as one would expect intuitively. Orsingher's conjecture can also be proved by a different method to be found in a more detailed and extended version of this Rapid Communication [15].

Notes added

The first-passage time may also be used to evaluate, besides the distribution of the maximum presented here, the distribution of the displacement in the presence of traps [15, 16] which was recently evaluated [17] by a different method, namely, via the differential equations satisfied by the densities.

After this work was completed, I received a copy of unpublished work by J. Masoliver and G. H. Weiss that is closely related to this work. Two other related works, Ref. [18], were also pointed out to me by P. Hänggi.

I thank my colleagues for a supportive environment in which to work, and Professor S. Kanno for his interests in this work.

APPENDIX

We outline the proof of Eqs. (12) and (13). With Eq. (11), the density is given by

$$\begin{aligned}\rho(S_m(t)=r|N(t)=2k+1)&=\frac{\partial}{\partial r}P(S_m(\tau)<r|N(t)=2k+1)\\&=-\frac{\partial}{\partial r}P(S_m(\tau)\geq r|N(t)=2k+1)\\&=\begin{cases}1/t, & k=0;\\(2k+1)\frac{(t-r)^{2k}}{2t^{2k+1}}+(2k+1)k\frac{r(t-r)^{2k-1}}{t^{2k+1}}+\frac{(2k+1)!}{2t^{2k+1}}I(k), & k\geq 1\end{cases}\end{aligned} \tag{A1}$$

where

$$I(k)\equiv\sum_{i=2}^{k}\frac{\int_r^t d\tau[(2i-1)r^2-\tau^2](\tau^2-r^2)^{i-2}(t-\tau)^{1+2(k-i)}}{2^{2(i-1)}(i-1)!i![1+2(k-i)]!}.$$

In other words, recalling Eq. (12), we need to prove for $k \geq 2$

$$I(k,t,r) = \frac{2}{(2k+1)!}\left\{2B(1/2,k+1)^{-1}u^{2k} - \frac{1}{2}(2k+1)(t-r)^{2k-1}[t+(2k-1)r]\right\} \equiv J(k,t,r), \tag{A2}$$

which we will do by induction. It is easily checked that $I(2,t,r) = J(2,t,r)$. In order to show $I(k+1,t,r) = J(k+1,t,r)$, we note the following.

(1) $\partial^2 I(k+1)/\partial t^2 = \partial^2 J(k+1)/\partial t^2$.
(2) $I(k)$ is a $2k$-degree polynomial.
(3) $I(k+1,t,t) = 0 = J(k+1,t,t)$.
(4) The density ρ given by $\partial P/\partial r_1|_{r_1=r}$, Eq. (A1) is normalized for all k, namely $\int_0^t \rho\, dr = 1$, for all k because the probability $P(S_m(\tau) < r_1|N(t) = 2k+1)$ Eq. (11), unlike Eq. (20) or Eq. (21), has no term independent of r_1 by inspection.

Item (1) is established by a straightforward and not long calculation, in which Eq. (A2) is used. The first three items together imply

$$I(k+1,t,r) = J(k+1,t,r) + C(k)r^{2k+1}(t-r). \tag{A3}$$

In order to show $C(k) = 0$, we substitute for $I(k)$ in Eq. (A1) giving

$$\rho(S_m(t) = r|N(t) = 2(k+1)+1)$$

$$= \frac{2}{B(1/2,k+2)}\frac{u^{2(k+1)}}{t^{2k+3}} + \frac{1}{2}(2k+3)!C(k)\frac{r^{2k+1}(t-r)}{t^{2k+3}},$$

which by item (4) implies $C(k) = 0$, and hence the conjecture also holds for $k \to k+1$. Consequently, we have proved that Eq. (12) holds.

The proof for the $N(t) = 2k$ case is similar but simpler, because in this case one can make use of the result for $N(t) = 2k+1$, and consequently only one differentiation with respect to t is needed in order to carry out the proof by induction.

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