RAPID COMMUNICATIONS

Quantum phase distributions and quasidistributions

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We analyze the relationship between the phase distribution for optical fields derived using phase-operator techniques with that derived from the Wigner phase-space quantum quasiprobability. We show that these two approaches agree for field states dominated by Fock states within a narrow distribution, but differ for cases involving widely separated photon-number contributions. Dominance by even Fock states can result in a negative Wigner phase distribution. An example of this is drawn from the Jaynes-Cummings model.

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The phase dependence of quantum noise in squeezed light has provided the motivation for a reanalysis of phase in quantum optics. Nonclassical light fields are described in terms of quasiprobabilities such as the Wigner or Q (Husimi) functions, and the phase dependence of such distribution functions is a useful parametrization of their properties [1]. An alternative is to use the phase operators developed by Pegg and Barnett [2] to construct a phase distribution $P_{\rm PB}(\theta)$, and a number of authors have compared the angular properties of both approaches (especially Tanaś and co-workers [1,2] and Agarwal *et al.* [3]). In much of this work the field states considered have had a large mean occupation number, and semiclassical approximations have been suitable. If, however, states of small mean photon number and states exhibiting substantial quantum interference are considered, such approximation schemes may be invalid and a purely quantum analysis becomes necessary. In some cases the Wigner phase distribution can be negative, in contrast to the positivity of the Pegg-Barnett distribution. In this paper we investigate the relationship between these two approaches to phase distributions. We show that the appropriate matrix elements for the relevant normally ordered characteristic function can be employed to write the Wigner phase distribution function in a number state form identical to that of the Pegg-Barnett distribution function except for the addition of a weight factor [4]. The variation of this weight factor with photon number can then result in a negative Wigner phase distribution. We illustrate the utility of this approach for two fields: a superposition of number states, and the field generated by an atom in the Jaynes-Cummings model. We show that the Wigner phase distribution is essentially identical to the Pegg-Barnett distribution if the field is dominated by a narrow range of Fock states.

The Wigner phase distribution function is written simply as

$$P_W(\theta)=\int_0^\infty r\,dr\,W(r,\theta) \tag{1}$$

if the Wigner function is expressed in polar coordinates. At first it seems that there is little connection with the Pegg-Barnett phase distribution function. The Pegg-Barnett phase distribution is found from the overlap of a quantum-mechanical state with the phase states. For a "physical state" $|\psi\rangle$ we obtain the phase distribution [2]

$$P_{\rm PB}(\theta)=\frac{1}{2\pi}\sum_{m,n=0}^{\infty}a_m^*a_n e^{i(m-n)\theta} \tag{2}$$

using the number basis

$$|\psi\rangle=\sum_{n=0}^{\infty}a_n|n\rangle\,. \tag{3}$$

In the phase distribution $P_{\rm PB}(\theta)$ above, we have set the phase origin to zero for definiteness.

An illustration of the two types of phase distribution has been made with a coherent state $|\alpha\rangle$. The Pegg-Barnett phase distribution has to be found by a numerical summation of Eq. (2). However, for the Wigner phase distribution we may take the Wigner function and integrate radially in the fashion of Eq. (1) to obtain an analytic result [3]. A comparison of the two types of phase distribution then shows that they agree for large $|\alpha|$. We have also seen that they agree for $\alpha\to 0$ where both the distributions become flat. The maximum difference between the distributions is approximately 11% of the peak value when $|\alpha|^2\sim 1.46$.

To determine the Wigner phase distribution for a more general state such as that given in Eq. (3) we write the Wigner characteristic function χ for the state $|\psi\rangle$ as a sum over matrix elements χ_{mn};

$$\chi(\xi)=\sum_{m,n=0}^{\infty}\chi_{mn}(\xi)a_m^*a_n\,, \tag{4}$$

where

$$\chi_{mn}(\xi)=\langle m|e^{\xi a^\dagger}e^{-\xi^* a}|n\rangle\exp(-|\xi|^2/2)\,. \tag{5}$$

The matrix elements χ_{mn} can be expressed in Laguerre polynomials L_n^{m-n} [5]:

$$\chi_{mn}(\xi)=\left(\frac{n!}{m!}\right)^{1/2}\exp(-|\xi|^2/2)\xi^{m-n}L_n^{m-n}(|\xi|^2) \tag{6}$$

for $m\geq n$. The Wigner function follows as the complex Fourier transform of the characteristic function (4). Performing this Fourier transform in polar coordinates we obtain the contribution $a_m^*a_nW_{mn}(r,\theta)$ to the full Wigner function from one term χ_{mn} where

$$W_{mn}(r,\theta)=\frac{2}{\pi}(-1)^n\left[\frac{n!}{m!}\right]^{1/2}e^{i(m-n)\theta}(2r)^{m-n}\times e^{-2r^2}L_n^{m-n}(4r^2) \quad (7)$$

for $m\geq n$ (in agreement with Ref. [6]).

The contribution to the Wigner phase distribution is now given by Eq. (1). With the substitution $y=4r^2$ we may write the radial integral of Eq. (7) as

$$\Delta P_{mn}(\theta)=\frac{1}{4\pi}(-1)^n\left[\frac{n!}{m!}\right]^{1/2}e^{i(m-n)\theta}\times\int_0^\infty dy\, y^{(m-n)/2}\exp(-y/2)L_n^{m-n}(y)\,. \quad (8)$$

We now use a table of integrals [7] to perform the integration over y in terms of a hypergeometric function which in turn can be expressed as a series of gamma functions:

$$\Delta P_{mn}(\theta)=\frac{1}{2\pi}(-1)^n\sqrt{m!n!}e^{i(m-n)\theta}2^{(m-n)/2}\times\sum_{k=0}^{n}\frac{(-2)^k\Gamma(1+k+(m-n)/2)}{(n-k)!k!(m-n+k)!}\,. \quad (9)$$

The Wigner phase distribution for a general state can then be written by analogy with the Pegg-Barnett phase distribution (2) as

$$P_W(\theta)=\frac{1}{2\pi}\sum_{m,n=0}^{\infty}a_m^*a_ne^{i(m-n)\theta}F(m,n)\,. \quad (10)$$

Indeed, Eq. (10) goes over to Eq. (2) if $F(m,n)=1$. By summing the series (9) we find that the factors $F(m,n)$ take the form

$$F(m,n)=\begin{cases}2^{(m-n)/2}\left[\frac{n!}{m!}\right]^{1/2}\frac{\Gamma(1+m/2)}{(n/2)!}, & n\text{ even}\\ 2^{(m-n)/2}\left[\frac{n!}{m!}\right]^{1/2}\frac{\Gamma((m+1)/2)}{[(n-1)/2]!}, & n\text{ odd}\end{cases} \quad (11)$$

provided $m\geq n$. For $m\leq n$ the indices m and n should be exchanged in Eq. (11). These results enable us to calculate the Wigner phase distribution in the same way that we find the Pegg-Barnett phase distribution. This has a numerical advantage if the field state is expressed in terms of Fock states.

We now consider two applications. First, we will look at a superposition of two number states. This is not only of academic interest, but may be realizable experimentally by using techniques for producing number states outlined in Ref. [6]. We choose an equal superposition of states to ensure the maximum modulation of the phase distribution. The Pegg-Barnett phase distribution for $|n\rangle$ and $|n+2\rangle$ immediately follows from Eq. (2) and is

$$P_{PB}(\theta)=\frac{1}{2\pi}[1+\cos2\theta]\,. \quad (12)$$

However, if we use Eqs. (10) and (11) to find the Wigner phase distribution we obtain

$$P_W(\theta)=\begin{cases}\frac{1}{2\pi}\left[1+\left[\frac{n+2}{n+1}\right]^{1/2}\cos2\theta\right], & n\text{ even}\\ \frac{1}{2\pi}\left[1+\left[\frac{n+1}{n+2}\right]^{1/2}\cos2\theta\right], & n\text{ odd}\,.\end{cases} \quad (13)$$

It is clear from Eq. (13) that superpositions of *even* number states can produce a negative Wigner phase distribution, unlike the Pegg-Barnett phase distribution which remains positive. The negativity is worst for the case $n=0$ (i.e., when $|0\rangle$ and $|2\rangle$ are superposed) and decreases as n even increases. For odd n there is no negativity in the Wigner phase distribution. If $n\to\infty$ we obtain the Pegg-Barnett phase distribution (12) from the Wigner phase distribution (13) and there is no negativity for odd or even n.

The vacuum is especially good at producing negativity in superposition. By using Stirling's formula we can show that for large n, $F(0,n)\to(n\pi/2)^{1/4}$ so that an equal superposition of $|0\rangle$ and $|n\rangle$ gives an increasing amount of negativity as n increases. This is not the case for a superposition of the states $|1\rangle$ and $|n\rangle$, for which the modulation of the phase distribution is less than that found in the Pegg-Barnett distribution, and so $P_W(\theta)$ is always positive.

If we have an equal superposition of the states $|m\rangle$ and $|n\rangle$ we find negativity in the Wigner phase distribution only if the smallest of m and n is even. This general rule is illustrated in Table I which shows the asymptotic behavior of $F(m,n)$ when both m and n are large. We can see that $F(m,n)$ takes one of two values depending on whether or not m or n are even or odd. Negativity of the Wigner phase distribution follows if $F(m,n)>1$. However, we also see that if $m\sim n$ then the value of $F(m,n)$ is close to one and the contribution to the phase distribution is similar to the Pegg-Barnett case. In general, the Wigner phase distribution will be close to the Pegg-Barnett phase distribution if there are no significant contributions to the phase distribution from widely differing Fock states (i.e., for peaked photon distributions, the coherent state provides a good example). Parts of the Wigner phase distribution may be negative if even number states dominate.

We turn next to the Wigner phase distribution of the field mode in the Jaynes-Cummings model which underlies recent micromaser experiments [8,9]. We calculate the phase distributions of the cavity field using the Pegg-Barnett method and the Wigner function [in the latter case from Eqs. (10) and (11)]. Results are shown in Figs.

TABLE I. Here we show approximate values of $F(m,n)$ when both m and n are large. If $F(m,n)$ is greater than 1 we will find negativity in the Wigner phase distribution for an equal superposition of $|m\rangle$ and $|n\rangle$.

m \ n	$m\geq n$ even	$m\geq n$ odd	m \ n	$m\leq n$ even	$m\leq n$ odd
even	$(m/n)^{1/4}$	$(n/m)^{1/4}$	even	$(n/m)^{1/4}$	$(n/m)^{1/4}$
odd	$(m/n)^{1/4}$	$(n/m)^{1/4}$	odd	$(m/n)^{1/4}$	$(m/n)^{1/4}$

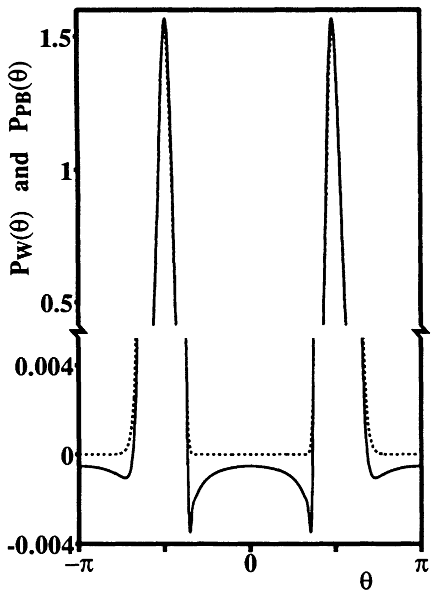

FIG. 1. This figure shows the Wigner phase distribution (solid curve) and the Pegg-Barnett phase distribution (dotted curve) when the resonant Jaynes-Cummings model has evolved for half a revival time. The lower part of the figure is shown on an expanded scale to display the negativity in the Wigner phase distribution clearly. The phase distribution is double peaked at half the revival time because of the splitting of the quasiprobability distribution in phase space. The initial state of the field was a coherent state with a mean photon number of 49. The atom was initially in its excited state.

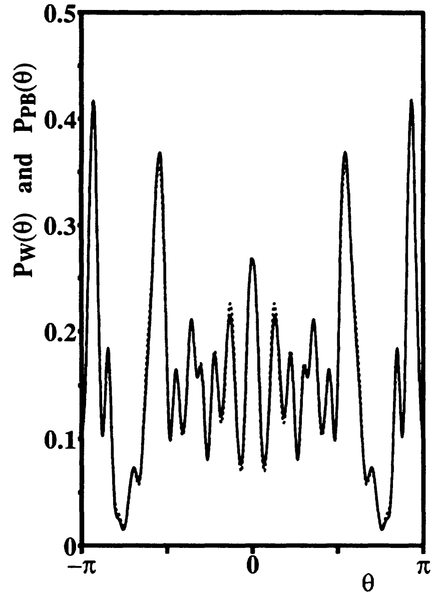

FIG. 2. This shows the two phase distributions for the same parameters as in Fig. 1, but after a long evolution time (50 revival times). The original two-peaked structure is lost, and there is no negativity now. The two phase distributions are quite close to each other.

1 and 2. It is known that if the initial field is a coherent state then during the evolution of the field the Wigner function splits into two parts that rotate in different directions in phase space [10]. At the "revival time" these two pieces overlap again, having traveled through an angle of π. During the collapse region at half the revival time the two pieces have a maximum separation in phase space. We then find a two-peaked phase distribution reflecting the separation of the distribution function into two parts. For the example given in Fig. 1 the Wigner and Pegg-Barnett phase distributions are found to be remarkably close. The most significant difference is, however, the presence of negative regions in the Wigner phase distribution.

The possibility of negativity might have been foreseen in the following way. We know that the field at half of the revival time is nearly a pure state [11] and similar to a superposition of two coherent states. Such superpositions yield a Wigner function with interferences near the origin of phase space. These interferences contain negative regions and it is seen in Fig. 1 that for certain angles the negative interference regions dominate the phase distribution. We have found that if the mean photon number of the initial coherent state is reduced from 49 (as in Fig. 1) to 16 the negativity disappears.

In Fig. 2 we show that phase distributions after a considerable interval of time has elapsed in the evolution of the model. The two pieces we had in phase space have now dispersed resulting in phase distributions that are neither simply peaked nor regular. It is interesting to note that despite the long evolution time the Pegg-Barnett and Wigner phase distributions remain quite close to each other. There is no longer any negativity.

We conclude that we have shown how to calculate the Wigner phase distribution in a way similar to the Pegg-Barnett phase distribution by using the expansion of a state in the Fock basis. Because we do not need to calculate explicitly the Wigner function we believe that this method could have a wide application, especially to states which are found by numerical means. Of course, in dissipative systems it is easy to generalize to mixed density matrices. We have illustrated the use of the method with simple superpositions of number states and with the field in the Jaynes-Cummings model. In both cases we have compared the Wigner phase distribution with the Pegg-Barnett phase distribution. It is significant that in both cases the Wigner phase distribution can take negative values. We regard this as an unphysical property of the Wigner *phase* distribution, although it may be regarded as a nonclassical signature. It also means that it cannot, in general, be possible to construct "Wigner phase states." This is because the overlap of such states with a physical state would yield a Wigner phase distribution with positive values only.

In a future presentation we will report in more detail on this work, and discuss the phase distribution that follows from the Q function as well as the phase distributions of other "cat"-like states such as the odd coherent states (which do not exhibit negative Wigner phase probabilities) and the even coherent states (which do show negative Wigner phase). A study of the displaced number states of Ref. [4] has not revealed any negative Wigner phase probability.

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