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Statistical signatures of self-organization

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A simple method for testing whether broadband noise comes from driven dynamical effects (rather than equilibrium fluctuations) is described. Asymmetrical higher-order time correlation functions can violate detailed balance, providing a key signature of dynamical effects. Simulations of a one-dimensional sandpile model are used for an initial illustration. Also, a hitherto unreported phenomenon—growth of the non-Gaussian nature of the statistics with system size—is found and attributed to the presence of correlations on the scale of the whole system.

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Nature is replete with examples of phenomena that show extensive scaling behavior as a function of time (e.g., $1/f$ noise) [1,2] or of distance (e.g., fractal coastlines) [3]. Often such scaling behavior is found under conditions in which it is not an obvious consequence of underlying laws. The possibility has been raised that many such scaling phenomena might be the result of "self-organized criticality" (SOC) [4]. SOC is, loosely, the tendency of certain nonlinear driven systems, over a range of parameters, to reach marginally stable states [5] which can exhibit both spatial and temporal scaling analogous to the scaling found at critical points in equilibrium statistical mechanics.

In many cases it has been assumed that temporal signals exhibiting a scaling regime are *prima facie* examples of SOC (e.g., [6]). However, the best studied such temporal phenomenon, $1/f$ noise in condensed matter, ordinarily lacks the necessary spatial scaling to be connected with SOC and usually has a $1/f$ form due to a relatively trivial spread of activation energies in materials with many defects in quasiequilibrium [1,2]. In this paper we present techniques designed to answer a much simpler question than many asked about SOC. We investigate how to analyze a random-looking time series to see if some nonequilibrium self-organization is needed to understand its origin.

We shall illustrate in particular the distinction between behavior that might be dubbed SOC and generic scaling phenomena (see, e.g., [7,8]). The latter occur whenever some random nonequilibrium force drives some linear, or linearizable, dissipative system. Generic scaling reflects no organization whatever (let alone self-organized criticality) but simply results from the power-law form for the Green's functions of elementary transport laws such as the diffusion equation [7,8].

We emphasize that while much of the motivation for this work stems from interest in SOC, we will not present here techniques designed to distinguish SOC from other far-from-equilibrium driven dynamical systems which exhibit scaling properties. In particular, the distinction between turbulence and SOC will either require a more subtle exploration of the scaling properties of the statistical features described here, or else simple prior knowledge of the system of interest—e.g., whether it is driven weakly (as in standard SOC) or very strongly. Thus, for our present purposes, conventional turbulence would be considered self-organized in that its scaling laws arise from the response to driving and are not present in equilibrium.

In the case of equilibrium $1/f$ noise sources, it has now been established that although physically dissimilar models can give very similar spectra, analyses of higher-order spectra (i.e., greater than two-point correlation functions) can show qualitative differences between, for example, hierarchical kinetics and superpositions of independent two-state systems [1,2]. In the case of those driven systems in which individual events can be unambiguously picked out from the time record, statistics of individual event sizes and times can provide more important evidence for or against the relevance of SOC than averaged spectra can provide. Here we shall show that extension of the statistical techniques developed for equilibrium systems can provide generic diagnostics for the possible presence of SOC effects in a random time series without requiring the identification of individual events.

In particular, we argue, and shall illustrate by simulations, that a key feature of self-organization leads to a property that cannot arise in equilibrium systems. The general description of nontrivial self-organization requires that events on one temporal scale set the stage for events on another scale. (The clearest statement of this principle, in the context of SOC, lies in the earthquake models, in which nonlinear friction causes large events to produce inhomogeneities on all smaller scales, setting the stage for small events, while collections of small events create the uniformity which sets the stage for large events [9].) It is precisely such flow of activity from one scale to another that can produce scaling behavior.

Barring accidental symmetry such a flow between scales will violate detailed balance. Any systematic measure of such violations of detailed balance will provide an indication that self-organization may be present. A particular example of such effects can be found in the small precursor quakes seen in simulations of earthquake models [10]. (In a related technique, systematic flows in Poincaré maps have recently been shown to help distinguish deterministic chaos from ordinary stochastic processes [11].) We shall also illustrate that in a very simple generic scaling model this signature is absent, although there is no rigorous argument that it is absent from all such models.

The principal statistical tools that we shall use to identify such nonequilibrium flow between different scales include a collection of fourth-order moments which have been dubbed "second spectra" [1,12]. Second spectra are obtained by *repeatedly* measuring the ordinary spectrum $S(f)$ of segments of the time series $V(t)$ by standard discrete transform techniques over a series of time intervals. $S(f)$ for each measurement is binned into convenient bands (e.g., octaves for broadband noise). Thus one obtains a *time series of noise powers* in a set of several frequency bands. From the time series for any one such band, centered at frequency f, Fourier transforming and squaring gives a second spectrum, $S_2(f_2,f)$. Here f_2, with $f_2<f$, is the frequency at which $S(f)$ is fluctuating. From the time series for any two bands, centered at f_a and f_b, one obtains two Fourier transforms. By multiplying one by the complex conjugate of the other, one obtains a cross second spectrum, $S_2(f_2,f_a,f_b)$. It is convenient to normalize these second spectra by dividing them by the product of the mean powers in the octaves used, so that f_2S_2 is dimensionless.

In addition, we introduce here a collection of third-order moments which we call the three-halves (1.5) spectra. These are the cross spectra of the fluctuating variable itself and the time series for the band powers. Here we normalize by dividing by the mean of the relevant octave power and by the standard deviation of the time record of the variable.

The Fourier transform of a cross second spectrum is the cross correlation function for the two time series of noise powers in the bands. In equilibrium systems the expectation value of any such cross correlation function between variables (noise powers, here) which are even under time reversal must itself be an even function of time, by time-reversal symmetry (i.e., detailed balance). Equivalently, the expected cross second spectra must be purely real. Thus any systematic imaginary cross second spectra indicate nonequilibrium effects of the general type expected to justify the designation "self-organized." Since the imaginary cross second spectrum is a direct measure of any self-organizing flow between temporal scales, we anticipate that it should ultimately provide a more sensitive measure of scaling properties of that flow than can be inferred from ordinary spectra or event-size distributions.

Likewise the Fourier transform of a 1.5 spectrum is the cross correlation function for the variable and its own noise power (at a higher frequency band.) If the measured variable is even under time reversal, the 1.5 spectra have no imaginary component. If the variable is odd under time reversal (e.g., if it is the time derivative of some even quantity) the real component of the 1.5 spectrum vanishes. Thus the 1.5 spectra are not as generically useful as the cross second spectra in finding time asymmetries, since one needs some prior knowledge of the symmetry properties the measured variable would show in equilibrium.

The magnitude and form of these time asymmetries for a typical model taken as an example of SOC are not obvious *a priori*. We chose the simplest one-dimensional model (limited local slope) described in Kadanoff *et al.* for simulation, since its low-order temporal scaling properties have been established [7,13] and since it can be simulated inexpensively.

This model obeys the following simple dynamical rules [13]. A pile of size L is characterized by a height function h, an L-component vector with integer components. The integer h_i represents the number of "grains" at the ith site. The pile is periodically perturbed at a random location, i, by the addition of a grain, which increases h_i by 1. The pile is described as locally unstable if $h_{i+1}-h_i>3$. If the pile satisfies this criterion then a local rearrangement is triggered, subtracting 2 from h_{i+1} and adding 2 to h_i. One Monte Carlo step (MCS) is defined as an update of the entire vector. The periodic input rate, in units of MCS^{-1}, is called J_I. The signal $V(t)$ that we initially analyzed was the output current, of grains falling off the end at $i=0$, at which the boundary condition $h_0=0$ is maintained.

$S(f)$ is presented in Fig. 1. We shall not dwell on describing $S(f)$, which agrees with previous simulations [7,13]. Three different regimes can be identified in $S(f)$. The highest is associated with weakly interacting avalanches, the lowest with negatively correlated avalanches reflecting the overall system memory, and the intermediate regime represents the interesting behavior of highly interacting avalanches [7]. The finite-size scaling behavior also agrees with previous results [7].

Very large non-Gaussian effects were found, again as expected given that the noise mostly comes from intermittent avalanches. An indication of the unusual behavior of this model SOC system, sharply contrasting with typical $1/f$ noise sources [1,2], is that the fractional variance of $S(f)$ *grew* as the system size was increased (see Fig. 2). For a fixed dynamical correlation length smaller than the system size, adding spectra from uncorrelated regions always reduces the non-Gaussian effects, which approach zero in the limit of large systems. Such behavior requires that there be correlation lengths at least as large as the size of the simulated system, supporting the analogy with critical phenomena. The growth of the non-Gaussian fractional variations in $S(f)$ with system size also indicates that statistical analysis of these non-Gaussian effects is likely to be generally useful for SOC systems, not con-

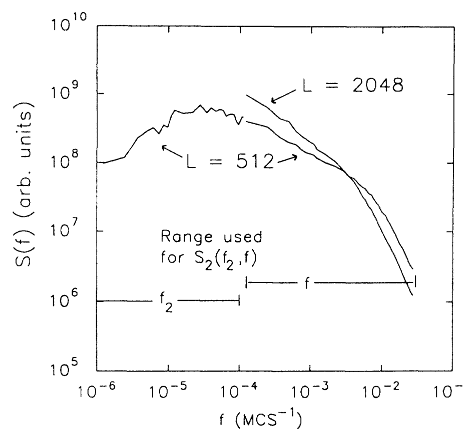

FIG. 1. First spectra $S(f)$ taken for two different array sizes with $J_I=\frac{1}{32}$ are shown. The ranges of frequencies f and f_2 used in calculating $S_2(f_2,f)$ are illustrated on the same scale.

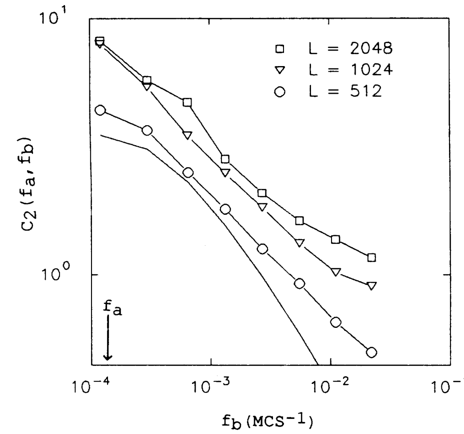

FIG. 2. The fractional covariance, $C_2(f_a,f_b)$, between the power in two octaves centered at f_a and f_b is shown for different lattice sizes. The fractional covariance exceeds one because the noise is intermittent. The unmarked curve is proportional to $\ln(f_a/f_b)/\sinh[\ln(f_a/f_b)]$, which gives the cross correlation for noise made up of independent intermittent processes each of which has a Lorentzian form. In this and subsequent figures the data are from the average of 12 runs, each consisting of a set of 512 first spectra, each taken from a 512 point transform, from a simulation with $L=512$, at a sampling rate of $\frac{1}{32}$ MCS^{-1}. Thus about 10^8 MCS were required.

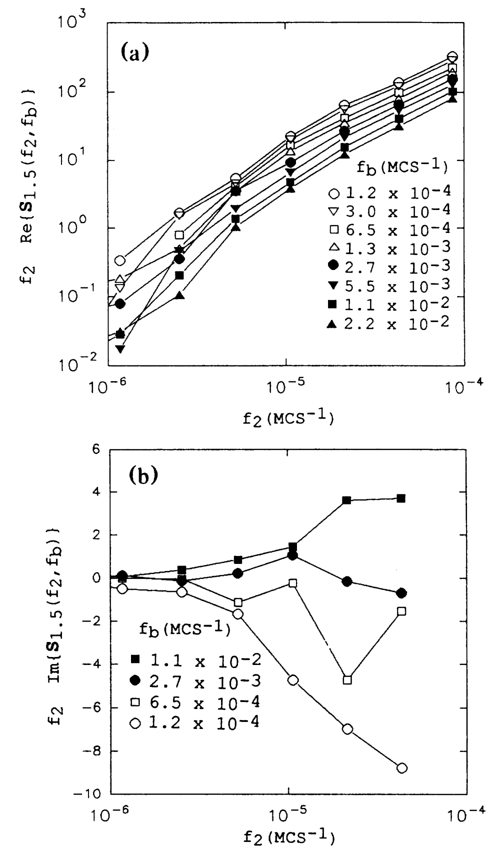

FIG. 3. (a) The real part and (b) the imaginary part of some of the 1.5 spectra of the output current are shown.

strained to specially selected mesoscopic systems. However, we have not yet checked whether such behavior can be found in higher dimensions.

As expected, the correlations in the fluctuations of the noise power in different octaves fell off for widely separated octaves (Fig. 2), but not as rapidly as would be found for a collection of independent intermittent two-state systems. Such behavior is typical of systems with complicated coupled degrees of freedom, including equilibrium systems, and thus cannot be used to identify SOC effects [1,12].

Striking time asymmetries appear in the 1.5 spectra of the output current—a variable that would be odd under time reversal in an equilibrium system (see Fig. 3). The real part is positive, meaning (given the sign convention of our program) that when the output current is high, the noise level is high. The imaginary part is positive. Thus when the output current is high, noisy events are expected to follow, but are not likely to have recently occurred. We checked that the corresponding statistics for the occupation number—an even variable under time reversal in an equilibrium system—show the same behavior but with the roles of the imaginary and real 1.5 spectra reversed. The interpretation is almost trivial; the occupation builds up slowly and quietly due to the steady influx of new grains, but discharges in large, noisy bursts.

The real second spectra [Fig. 4(a)] are approximately white at frequencies above and below the characteristic finite-size frequency found in the spectrum. The real second cross spectra fall off with increasing separation between the frequency bands, as can be most easily seen in Fig. 2 which shows, in effect, the integral of the real cross second spectra.

The imaginary cross second spectra, seen in Fig. 4(b) are much smaller and less simple than the real second spectra. Nevertheless, they are not random, since similar results were found on many simulations.

At low frequencies, i.e., below the scaling regime, these imaginary cross second spectra are very close to zero. Thus the long-time integral of the cross correlation function for the noise in two separated bands must have equal contributions from positive and negative time delays. Knowing that there is noise in some given frequency range conveys no information on whether, over time scales comparable to the longest characteristic time deduced from the first spectrum, there is more likely to be or to have been noise in another frequency range.

However, at frequencies f_2 near the lower end of the intermediate regime, a distinct nonrandom imaginary cross second spectrum emerged. Its sign indicates that the high-frequency noise tends to precede the low-frequency noise. Thus this technique systematically finds precursor effects in noisy data in which such effects are not apparent by inspection.

In contrast, simulations of a purely diffusive one-dimensional sandpile, also constrained to have $h=0$ at the edges and driven by a random rain of grains, found no hint of a nonrandom imaginary cross second spectrum. The absence of the imaginary cross second spectrum is necessary, since the time course of the occupation number

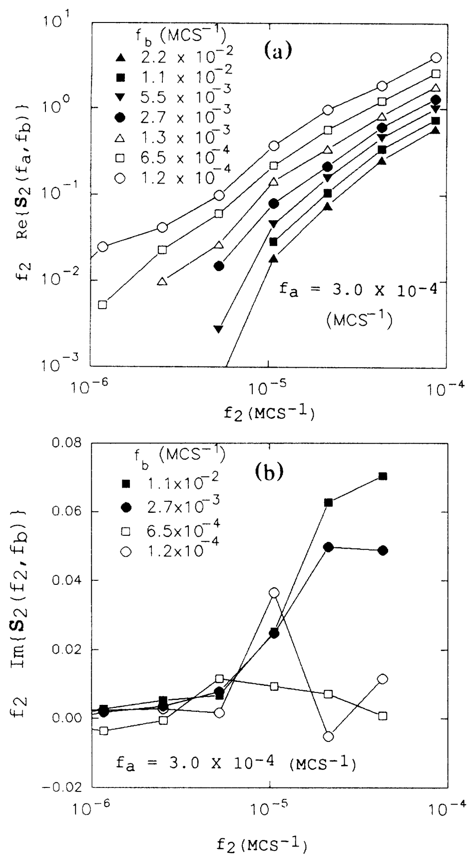

FIG. 4. (a) The real and (b) imaginary parts of some of the cross second spectra are shown. The sign of the curve with the open circles in (b) has been reversed in order to maintain a consistent order for the higher and lower f_a and f_b in the picture. Note that for $f_a \approx f_b$ the imaginary part is small, as expected since it is identically zero for $f_a = f_b$.

of such a system consists of a superposition of independent steplike pulses of varying lengths corresponding to single grains. Each pulse is symmetric under time reversal and, since they are independent, so is their sum.

This simple diffusion model with generic scaling helps to clarify the ingredients needed to give violations of detailed balance in the observable flow. If the input drops consisted of many grains, which subsequently diffused independently, or of drops of a continuum diffusing fluid, the independent events produced by each drop—an initial sharp increase in occupation number followed by a slow decay with a power-law tail—would each manifestly be asymmetric under time reversal. $V(t)$ would show violations of time-reversal symmetry because the initially correlated behavior of the constituents of a drop is lost as they diffuse independently after landing. This simple model also shows a flow of scale of organization, although only an uninteresting flow from organized to disorganized. Thus we do not claim that all time records of broadband noise which show violations of time-reversal symmetry, as manifest in the imaginary cross second spectra, show self-organization even in a broad sense. We do argue that without such effects it is hard to make a case for self-organization.

In conclusion, we have illustrated and presented arguments for the use of time asymmetries in non-Gaussian statistics to determine whether a generic time series with a broadband spectrum shows effects expected for interesting driven dynamical systems or obeys detailed balance, as required for equilibrium systems. This technique does not require that the system be deterministic. The technique is applicable without any algorithm for identifying events and without any model-dependent decisions about how to map a one-dimensional variable onto a multidimensional phase space [11]. The computational overhead required to calculate the second spectra and 1.5 spectra is negligible compared with the computing required to generate the random time series, and no reprogramming is required to adapt these higher spectra to different noise sources. Therefore we suggest computing these spectra would be useful for anyone making extensive simulations of driven dynamical systems with broadband noise, particularly since many serious simulators generate much more extensive data, and on more interesting models, than we used for the illustrative calculations here.

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