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Stability limits of spirals and traveling waves in nonequilibrium media

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We show that the Eckhaus instability for traveling waves is of convective nature and does not directly lead to absolute instability. As a consequence spiral waves remain stable in a larger range than expected previously, transition to defect-mediated turbulence can be delayed beyond the Benjamin-Feir limit, and the occurrence of phase turbulence can be made plausible. We calculate the onset of absolute instability using the complex Ginzburg-Landau equation and verify the results by simulations.

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Point defects in two-dimensionally extended pattern-forming systems (or line defects in three dimensions) have received considerable attention recently. The oscillatory or excitable case is especially interesting. The defects are then the well-known spiral waves in reaction diffusion systems, like, e.g., the Belousov-Zhabotinski reaction [1], or the dislocations in anisotropic systems supporting traveling waves, like, e.g., electrohydrodynamic convection under appropriate conditions (for a review see, e.g., [2]). Recently it has also become clear that analogous defects can be observed in the phase field of multimode lasers [3].

The simplest description of such systems is provided by the complex Ginzburg-Landau equation (we will here mostly consider two space dimensions)

$$\partial_t A = [(1+ib)\nabla^2 + 1 - (1+ic)|A|^2]A \,. \tag{1}$$

It exhibits traveling plane wave solutions

$$A = F\exp[i(\mathbf{Q}\cdot\mathbf{r} - \omega t)] \tag{2}$$

where $F^2 = 1 - Q^2$ and $\omega = c + (b-c)Q^2$. They exist for $Q^2 < 1$. In order to test their stability one usually considers the complex growth rate λ of the modulational modes. Restricting oneselves to the most dangerous longitudinal perturbations proportional to $\exp(\pm i\mathbf{k}\cdot\mathbf{r})$ with $\mathbf{k}\parallel\mathbf{Q}$ one easily finds

$$\lambda(k) = -k^2 - 2iqbk - F^2 \pm [(1+c^2)F^4 - (bk^2 - 2iqk + cF^2)^2]^{1/2} \,. \tag{3}$$

In the long-wavelength limit ($k \to 0$) one may expand (3) leading to

$$\lambda(k) = iv_g k - D_\parallel k^2 + O(k^3) \tag{4}$$

with

$$v_g = 2(c-b)Q \,, \qquad D_\parallel = 1 + bc - 2(1+c^2)Q^2/(1-Q^2) \,. \tag{5}$$

Traveling waves are long-wavelength stable for $D_\parallel(Q) > 0$ which designates the Eckhaus stable range.

Stable defect solutions of Eq. (1), corresponding to simple zeros in A, are spiral waves. An isolated spiral wave has the form

$$A = F(r)\exp(i[-\omega t - m\phi + \psi(r)]) \,. \tag{6}$$

Here (r,ϕ) are polar coordinates, $m = \pm 1$ is the topological charge or circulation, $F(r)$ and $\psi(r)$ are functions with the asymptotic behavior

$$F(0) = \psi(0) = 0, \quad \lim_{r\to\infty} F(r) = (1 - Q_s^2)^{1/2} \,,$$

where $Q_s = \lim_{r\to\infty}\psi'(r)$ is the asymptotic wave number which is a unique function of b and c [4] and $\omega = c + (b-c)Q_s^2$ is the frequency of rotation. Thus the spirals provide a wave-number selection mechanism. They emit asymptotically plane waves and a necessary condition for its stability would seem $D_\parallel(Q_s) > 0$. In Fig. 1 the boundary of this Eckhaus-stable range is shown in the b-c plane [dash-dotted; the plot can be continued to negative c by noting the symmetry $(b,c) \to -(b,c)$]. Also shown in Fig. 1 is the Benjamin-Feir (BF) limit where the solution with $Q = 0$ becomes unstable—all other plane-wave solutions being then unstable already [5].

The present picture for the generic long-time behavior of sufficiently large systems starting from random initial conditions and b,c of order one is roughly as follows: In the BF unstable range one has "defect-mediated turbulence," i.e., a disordered state [6] with random creation, motion, and annihilation of defect pairs and a well-defined average defect density [7]. For this state the dimension of chaos is proportional to the size of the system [8]. In the BF stable but Eckhaus unstable range one has considerable sensitivity on initial conditions and one may have various defect densities up to a maximum with the defects either disordered and sometimes rather motionless or ordered in a lattice [7,9]. Finally, in the Eckaus stable range, the density of defects would usually vanish. However, with appropriate initial conditions defect lattices can be formed [9]. Their minimal spacing appears to diverge when $b - c$ becomes too small.

Here we provide analytic and numerical evidence that spirals, even if they are essentially isolated, remain stable in the Eckhaus and, for appropriate parameters, even in

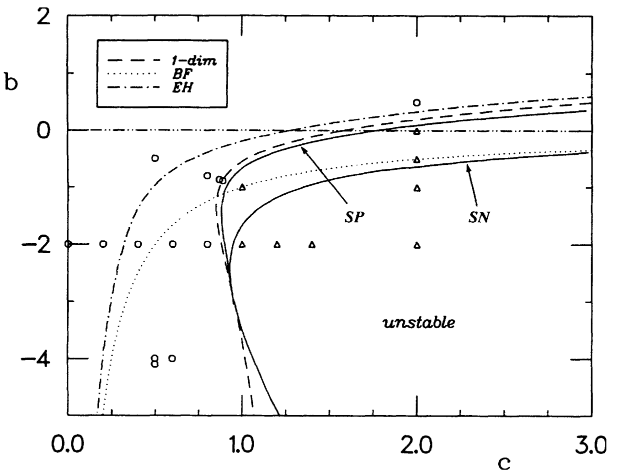

FIG. 1. Stability limits for 2D and 1D spirals in the b-c plane. Presented are B-F limit $1+bc=0$ (dotted line); long wavelength Eckhaus limit $(1+bc)/(3+bc+2c^2)=Q^2$ with $Q(b,c)$ corresponding to 2D spirals (dash-dotted line); absolute instability limits according to (8) for 1D spirals (dashed line) and 2D spirals (solid line SP). Circles (stable defects) and triangles (defect turbulent states) give the results of the 2D numerical simulations of Eq. (1). The solid curve SN gives the limit up to which convectively unstable wave numbers exist. A pseudospectral code was used with usually 256×256 grid points, time step of 0.05 and a system size of 100×100. We used periodic as well as nonperiodic (normal derivative of A equal to zero) boundary conditions.

the BF unstable range. This has a profound influence on the turbulent regime and explains some of the observed features. Some numerical evidence in this direction was already found earlier [9]. In fact, since spirals emit traveling waves with a nonzero group velocity $v_g=\partial_{Q_s}\omega=2(b-c)Q_s$ directed outward the Eckhaus criterion can be taken only as a test for convective instability where a localized initial perturbation $S_0(x)$ of the asymptotic plane wave, although amplified in time drifts away and does not necessarily amplify at a fixed position. [10]. To test for absolute instability one has to consider the time evolution of the localized perturbation which is in the linear range given by

$$S(x,t)=\int_{-\infty}^{\infty}dk(2\pi)^{-1}\hat{S}_0(k)\exp[ikx+\lambda(k)t] \qquad (7)$$

where $\hat{S}_0(k)$ is the Fourier transform of $S_0(x)$. [It can be shown strictly that the destabilization occurs at first for purely longitudinal perturbations ($\mathbf{Q}\|\mathbf{k}$).] The integral can be deformed into the complex k plane. In the limit $t\to\infty$ the integral is dominated by the largest saddle point of $\lambda(k)$ (steepest-descent method [11]) and the test for absolute instability is

$$\mathrm{Re}[\lambda(k_0)]>0 \text{ with } \partial_k\lambda(k_0)=0\,, \qquad \mathrm{Re}[\partial_k^2\lambda(k_0)]<0\,. \qquad (8)$$

The long-wavelength expansion (4) indicates that at the Eckhaus instability, where $D_\|$ becomes negative, the system can remain stable in the above sense. When $D_\|$ vanishes and $Q\neq 0$ the main contribution comes from the linear term that then suppresses instability.

Using the general expression (3) we have the following results in the b-c plane: On the BF unstable side traveling waves with $|Q|$ sufficiently small are absolutely unstable since v_g is small. Sufficiently near to the BF line there exists a "stable" (actually convectively unstable) band of wave numbers $0<Q_{a1}<Q<Q_{a2}$ (we restrict ourselves to $Q>0$ since there is a symmetry $Q\rightleftarrows -Q$). Q_{a1} goes to zero at the BF line as $(1+bc)^{3/2}$. Moving away from the BF line (into the unstable regime) Q_{a1} increases and Q_{a2} decreases until they come together in a saddle node at a value which we denote by Q_{ac}. In Fig. 2 the scenario is demonstrated for four cuts in the b-c-Q space. The saddle-node curve is also plotted in Fig. 1 (solid line SN). Thus convectively unstable waves exist up to this curve.

The absolute stability limit for spirals is now obtained as the intersection of the surfaces $Q_s(b,c)$ and $Q_{a1}(b,c)$ or $Q_{a2}(b,c)$ (solid line SP in Fig. 1). Also included is the corresponding curve for the one-dimensional (1D) analog of spirals, i.e., the stationary 1D solutions of Eq. (1) in the range $0<x<\infty$ with $A=0$ at $x=0$. These solutions are analytically accessible [4,9]. We have tested these results systematically by direct numerical simulations of 1D spirals in a large interval and were able to verify the stability limit. Some points were also checked for the 2D case (see Fig. 1).

Clearly in a finite system the stability can be achieved only if the reflection at the boundaries is not too strong, so in principle one has to expect the system to lose stability before the exact limit is reached. In the complex Ginzberg-Landau (CGL) equation system the interaction of the emitted waves with the boundaries leads to shocks, that are also formed when waves of different spirals collide. These shocks are very strong perturbations to the plane-wave solutions but it turns out that they absorb the incoming perturbations. Also, the convectively unstable

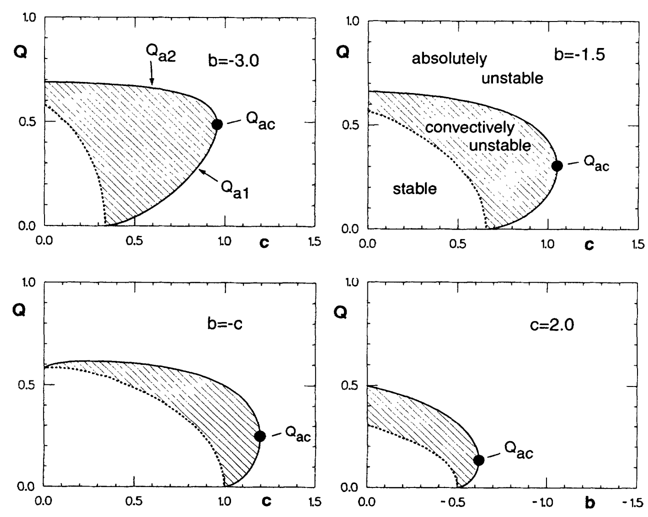

FIG. 2. Four cuts in the b-c-Q plane are shown. The upper bound Q_{a2} for convectively unstable solutions merges with the lower bound Q_{a1} at Q_{ac}. Q_{ac} lies on the curve SN in Fig. 1. Q_{a1} goes to zero at the BF limit.

state is very sensitive to noise, so noise-induced transitions to turbulence should occur.

We note that for $b > -1.2$ the new stability limit is for spirals more restrictive than the BF instability and less restrictive otherwise. In the first region earlier simulations [7] along the line $c=2$ (actually -2 was used, but we make use of the symmetry) exhibited existence of a disordered state until about $b=0$ when coming from the BF unstable regime. This boundary is rather near to the new stability curve so apparently the curve has some significance as a limit of existence of defect-mediated turbulence. Our stability boundary is also consistent with the boundary for persistent turbulence determined numerically for $b=-1$ by the criterion of the Lyapunov coefficient dropping to zero in a discretized version of the CGL system [12]. Of course the region $b < -1.2$ is especially interesting. Results for numerical simulations of Eq. (1) with small random initial conditions along the lines $c=2$ and $b=-2$ are shown in Fig. 1. Initially always many defects are formed. In the convectively unstable range eventually one defect becomes dominant pushing away the other ones which ultimately annihilate. However, with periodic boundary conditions, for topological reasons another defect with opposite circulation and without the far field typical for spirals survives inside the shock regime. In Fig. 3 an example of the final state is shown. Note the shocks where the waves of spirals from neighboring periodicity regions collide. Such defect lattices exist also in the Eckhaus stable regime and were observed before [9]. In fact defect lattices presumably exist also in the (slightly) absolutely unstable range, however, with an upper bound on their spacing. Actually there is some evidence that also stationary, spatially nonperiodic arrangements exist.

We expect the interaction of well-separated spirals to be exponentially screened in the convectively unstable region, as was shown to be the case essentially in the Eckhaus-stable regime [13,14]. The evidence comes from the fact that localized static perturbations decay exponentially towards the spiral core in both regimes. The decay length $2\pi/\mathrm{Im}(k)$ is obtained from Eq. (3) with $\lambda=0$ and it coincides with the interaction length obtained before for $Q^2 \ll 1$ [13,14].

Clearly our results are relevant not only for spirals but for any system where traveling waves are emitted from some source. Recent numerical simulations with the CGL equation in one dimension show that actual phase turbulence without phase slippage exists in a region between the Benjamin-Feir limit (dotted line in Fig. 1) and roughly our saddle node curve SN [15,16]. This indicates that the existence of the convectively unstable wave-number band can prevent the occurrence of phase slips. This seems quite plausible since large-amplitude depressions correlate with large local wave numbers, which presumably get swept away in the convectively unstable range. Apparently such fluctuations are then annihilated by collisions with fluctuations from neighboring regions. We point out that in many cases Eq. (1) has a drift term in the form $\mathbf{s}\cdot\nabla A$ (this is typical when the background solutions are traveling waves and the boundary conditions do not allow to transform this term away). Then our results are changed.

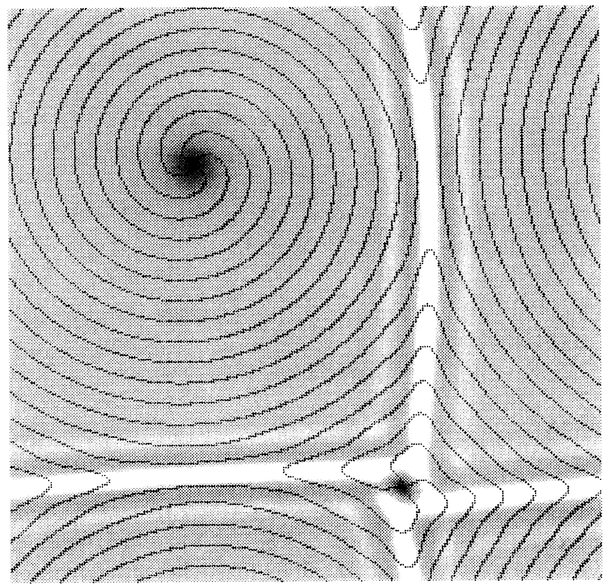

FIG. 3. Final state of a simulation with small random initial conditions and periodic boundary conditions in the convectively unstable range ($b=-2$, $c=0.6$). $|A|^2$ is coded in the grey scale, the lines designate $\mathrm{Re}A=0$ and $\mathrm{Im}A=0$.

We have shown that the Eckhaus instability for traveling waves is of convective nature. It may be worth mentioning that the instability is in some parameter range around the BF limit of forward (supercritical) type and one then has stable modulated solutions which appear to persist also in the BF unstable range [17]. Thus one can expect that the crossing of the stability curve may sometimes lead to such modulated solutions rather than to a turbulent state.

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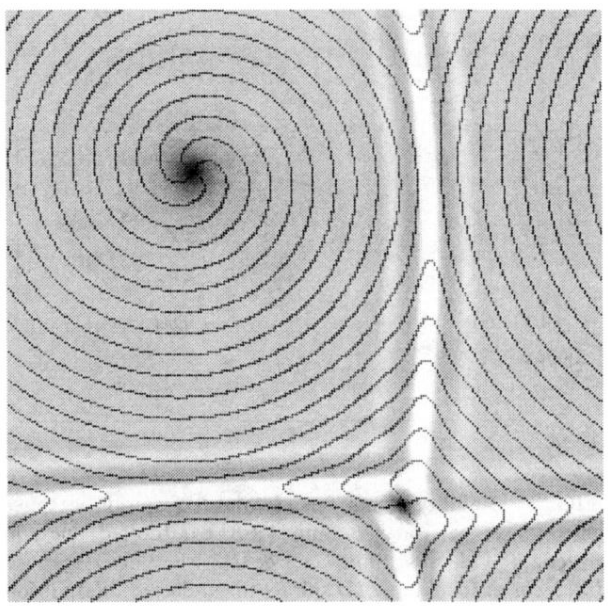

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